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A notion of true dimension theory is defined to which is assigned a dimension function D. We consider those D which have an enhanced Bockstein basis; these include $D = \dim$ and $D = \dim_G$ for any abelian group G. We prove that for each countable polyhedron K, the set of compacta $X \in 2^{\mathbf{Q}}$ with $K \in \mathrm{AE}(\{X\})$ is a G_δ-subspace. We apply this fact to show that the hyperspace of the Hilbert cube $\mathbf{Q}$ consisting of compacta (or continua) X with $D(X) \leq n$ is a G_δ-subspace. Let $D^{\geq n}$ (resp., $D^{\geq n} \cap C(\mathbf{Q})$) denote the space of compacta X (resp., continua) with $D(X) \geq n$. We prove that $\{D^{\geq n}\}_{n=1}^{\infty}$ and $\{D^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty}$ are absorbing sequences for σ-compact spaces. This yields that each $D^{\geq n}$ and $D^{\geq n+1} \cap C(\mathbf{Q})$ ($n \geq 1$) is homeomorphic to the pseudoboundary B of $\mathbf{Q}$; their respective complements are homeomorphic to the pseudointerior of $\mathbf{Q}$; and the intersections $\bigcap_n D^{\geq n}$, $\bigcap_n D^{\geq n} \cap C(\mathbf{Q})$ are homeomorphic to B^∞, the absorbing set for the class of $F_{\sigma\delta}$-sets. Results for the hyperspaces of compacta X for which $D(X) \geq n$ uniformly are also obtained.

1. Introduction. The ultimate objective of this paper is topological identification of certain hyperspaces of compacta related to dimension. Hyperspaces we are dealing with are subspaces of $2^{\mathbf{Q}}$, the hyperspace of all compacta of the Hilbert cube $\mathbf{Q} = [0, 1]^\infty$ endowed with the Vietoris topology (Hausdorff metric) or $C(\mathbf{Q})$, the hyperspace of continua. By dimension we mean true dimension (§2) which includes covering dimension dim and cohomological dimension $\dim_G$ for any abelian group G. The classical result, IV.45.4 of **[Kur]**, states that the hyperspace of compacta with $\dim \leq n$ is an absolute G_δ-set (equivalently: a G_δ-subset of $2^{\mathbf{Q}}$). A natural question arises whether the above statement holds true for $\dim_G$. (If this were not true for $G = \mathbb{Z}$, then it would implicitly yield examples of Dranishnikov type **[Dr1]**.) The question for, $G = \mathbb{Z}$, was brought to our attention in January 1991 by R. Pol. Apparently this had already been considered by the authors of **[DvMM]**. Furthermore, Robert Cauty has communicated to us a proof that this set is coanalytic. We shall prove (Corollary

3.4(a)) that for all abelian groups G, the hyperspace of compacta of $\dim_G \leq n$ is a G_δ-subset.

The second part of the paper is an application of the above G_δ result. It can be viewed as an expansion of the work **[DvMM]** of Dijkstra, van Mill and Mogilski, where methods of infinite-dimensional topology were applied to identify hyperspaces related to dim. We extend the results of **[DvMM]** in two directions: we replace dim by any true dimension function D, where D has an enhanced Bockstein basis (§3), and we settle the case of the hyperspace of continua. We also recover their result concerning uniform dimension in this setting including continua. More precisely, we show that the sequences $\{D^{\geq n}\}_{n=1}^{\infty}$ and $\{D^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty}$, where $D^{\geq n}$ consists of all compacta X with $D(X) \geq n$, are absorbing for the class of σ-compact spaces in $2^{\mathbf{Q}}$ and $C(\mathbf{Q})$, respectively. By the uniqueness theorem for absorbing sequences, we infer that $\{D^{\geq n}\}_{n=1}^{\infty}$ and $\{D^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty}$ are homeomorphic to the unique absorbing model sequence. In particular, each $D^{\geq n}$ and $D^{\geq n+1} \cap C(\mathbf{Q})$ $(n \geq 1)$ is homeomorphic to the pseudoboundary B of the Hilbert cube $\mathbf{Q}$; their complements are homeomorphic to the pseudointerior s of $\mathbf{Q}$; and $\bigcap_n D^{\geq n}$ and $\bigcap_n (D^{\geq n} \cap C(\mathbf{Q}))$ are homeomorphic to B^∞, the absorbing set for the class of absolute $F_{\sigma\delta}$-sets. If we denote by $\overline{D}^{\geq n}$ the hyperspace of compacta X for which $D(X) \geq n$ uniformly, then the sequences $\{\overline{D}^{\geq n}\}_{n=1}^{\infty}$ and $\{\overline{D}^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty}$ are absorbing for the class of absolute $F_{\sigma\delta}$-sets. In particular, $\overline{D}^{\geq n}$ and $\overline{D}^{\geq n+1} \cap C(\mathbf{Q})$ $(n \geq 1)$ are homeomorphic to B^∞; their intersections $\bigcap_n \overline{D}^{\geq n}$ and $\bigcap_n (\overline{D}^{\geq n} \cap C(\mathbf{Q}))$ are also copies of B^∞.

For background in the theory of cohomological dimension, the reader may consult **[Dr2]**, **[Ku]** or the appendix of **[Na]**. The theory of covering dimension can be found in **[En]**, **[HW]**, **[N]**, **[Na]** and many other sources.

Unless otherwise stated in this paper, we shall use the metric

$$d((x_i), (y_i)) = \sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|$$

for the Hilbert cube $\mathbf{Q}$. We shall often treat $I^n = [0, 1]^n$ as $[0, 1]^n \times \{0\} \times \{0\} \times \cdots \subset \mathbf{Q}$ with the metric determined by d. Open balls will be written $B(x, \epsilon)$ and closed balls $\overline{B}(x, \epsilon)$.

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2. Absolute extensors; dimension functions. The theories of covering dimension dim and cohomological dimension $\dim_G$ for metrizable spaces share a common attribute. In order to properly describe this and to effectively generalize it, let us first recall a notation. Let X be a space and K be an absolute neighborhood extensor for the class of metrizable spaces. Then $K \in \mathrm{AE}(\{X\})$ means that K is an absolute extensor for $\{X\}$. We shall also use $X \in \mathscr{E}(K)$ to designate this property.

We leave the proof of the next result to the reader.

2.1. LEMMA (*sum theorem*). *If a metrizable space X is the countable union of closed subsets $A_1, A_2, \ldots$ such that for each i, $A_i \in \mathscr{E}(K)$, then $X \in \mathscr{E}(K)$.* □

The next result concerns limits of compacta. By a polyhedron we shall mean the space $|K|$ of a simplicial complex K with the Whitehead topology. When no confusion is likely, we shall use the symbol K to denote both the polyhedron and a given triangulation of it.

2.2. THEOREM. *Let K be a polyhedron and suppose $\mathbf{X} = (X_a, p_{a,a'}, A)$ is an inverse system of Hausdorff compacta such that for each $a \in A$, $X_a \in \mathscr{E}(K)$. Then $X = \lim \mathbf{X} \in \mathscr{E}(K)$.*

Proof. Let $B \subseteq X$ be closed and let $f: B \to K$ be a map. Choose a compact subpolyhedron P of K with $f(B) \subseteq P$. We take $B = \lim \mathbf{B}$ where $\mathbf{B} = (p_a(B), p_{a,a'}|, A)$. As a consequence of Theorem 8 of [**Ma**], there exist $a \in A$ and a map $g: p_a(B) \to P$ such that $g \circ p_a|B$ and f are homotopic. Then since $X_a \in \mathscr{E}(K)$, there is a map $G: X_a \to K$ extending the map g.

Let P_0 be a compact subpolyhedron of K containing $G(X_a) \cup P$. Since $G \circ p_a: X \to P_0$ restricts to the map $g \circ p_a$ on B, then the homotopy extension theorem yields a map $F: X \to K$ extending f. □

We shall now develop an abstract notion of dimension, the idea being to unify standard ones such as dim and $\dim_G$. Suppose $\mathscr{K} = \{K_0, K_1, \ldots\}$ is a sequence of polyhedra with the property that for each metric space X, $X \in \mathscr{E}(K_n)$ implies that $X \in \mathscr{E}(K_{n+j})$ for all $j \geq 0$. Then we shall say that $\mathscr{K}$ is a *stratum* for a dimension theory. Whenever we have such a $\mathscr{K}$, then we want to define for each metrizable space X, its *dimension* $D(X) \in \{-1, 0, 1, \ldots, \infty\}$. This is done inductively as follows. One says $D(X) = -1$ if and only if $X = \varnothing$; $D(X) = n$ $(0 \le n < \infty)$ if $X \in \mathscr{E}(K_n)$ is true and $D(X) = m$ is false for all $m < n$; and $D(X) = \infty$ otherwise. We shall write $D(X) \ge n$ to mean that $D(X) = k$ is false for all $k < n$. We call D a *dimension function* stratified by $\mathscr{K}$.

For $D = \dim$, one chooses $K_i = S^i$. For $D = \dim_G$, one chooses $K_0 = S^0$ and K_i a polyhedral $K(G, i)$ for $1 \le i < \infty$. In any event, whenever $\mathscr{K}$ is a stratum for a dimension theory and D is defined as above, certain properties are always true.

2.3. THEOREM (*monotonicity*). *For each dimension function D, metrizable space X and subspace Y of X, $D(Y) \le D(X)$.*

Proof. This follows from the next lemma. □

2.4. LEMMA. *Let K be a polyhedron (or any absolute neighborhood extensor for metrizable spaces). If X is metrizable and $X \in \mathscr{E}(K)$, then $Y \in \mathscr{E}(K)$ for all $Y \subseteq X$.*

Proof. The reader may simply adjust the proof of (D_1) and (CD_1) on page 107 of [**Wa**]. The requirement of separability is unnecessary since every open subset of a metrizable space is a countable locally finite union of sets closed in the space. □

2.5. LEMMA. *Let D be a dimension function, X be metrizable, $\mathscr{U}$ be a base for the topology of X, and $0 \le n < \infty$. Then $D(U) \ge n$ for all nonempty $U \in \mathscr{U}$ if and only if $D(U) \ge n$ for all nonempty open subsets U of X.* □

2.6. LEMMA. *For any dimension function D and any nonempty discrete space X, $D(X) = 0$.* □

2.7. COROLLARY. *For any metrizable compactum X with* $\dim(X) = 0$, *$D(X) = 0$.*

Proof. Since X can be written as the inverse limit of a sequence of finite spaces, apply 2.6 and 2.2. □

A dimension function D will be called *true* if $D(R^n) = n$ for all n. For such D, it follows from 2.1 and 2.4 that if P is a compact polyhedron and $\dim P = n$, then $D(P) = n$. If X is a metric compactum and $\dim(X) \le n$, then X can be written as the inverse limit of a sequence of compact polyhedra of dimension $\le n$. Hence from 2.2 we have, 2.8. THEOREM. *If D is a true dimension function and X is a metrizable compactum with* $\dim(X) \leq n$, *then* $D(X) \leq n$. □

3. Certain G_δ-subsets of $2^{\mathbf{Q}}$. We shall now prove a theorem whose corollaries will be vital to later developments in this paper.

3.1. THEOREM. *Let K be a separable metric* ANR *and Δ be a closed subset of a compactum Z. Then the space*

$$S(Z, \Delta) = \{X \in 2^Z \mid X \cap \Delta \in \mathscr{E}(K)\}$$

is a G_δ-subset of 2^Z.

Proof. Choose a countable net $\mathscr{D}$ of closed subsets of Z. By this we mean that for every neighborhood U of X in Z, there exists $D \in \mathscr{D}$ with $X \subset D \subset U$. For each $D \in \mathscr{D}$, let $\mathscr{M}(D)$ be a countable dense set taken from the space of maps of D to K, topologized by the sup metric (here we use separability of K). Let $\mathscr{S} = \bigcup\{\mathscr{M}(D) \mid D \in \mathscr{D}\}$; then $\mathscr{S}$ is a countable set. Let us index $\mathscr{S}$ as $\{f_i : D_i \to K \mid i \in \mathbb{N}\}$, where of course $D_i \in \mathscr{D}$ is the domain of f_i.

Choose an open cover $\mathscr{U}$ of K so that any two $\mathscr{U}$-near maps into K are homotopic. Now define G_i ($i \in \mathbb{N}$) to be

$$\{X \in 2^Z \mid \exists \text{ a map } g : X \cap \Delta \to K \text{ such that on } X \cap \Delta \cap D_i,\ g \text{ and } f_i \text{ are } \mathscr{U}\text{-near}\}.$$

We are going to show that each G_i is open in 2^Z and that $S(Z, \Delta) = \bigcap_{i=1}^\infty G_i$. These facts will establish the truth of our theorem.

To see why G_i is open, let $X \in G_i$, and $g : X \cap \Delta \to K$ be a map satisfying the requirements in the definition of G_i. There exists a neighborhood U of $X \cap \Delta$ in Z and a map $h : U \to K$ which extends g. Making U smaller if necessary, we may assume that on $U \cap \Delta \cap D_i$, h and f_i are $\mathscr{U}$-near. Now there is a neighborhood W of X in 2^Z so that if $Y \in W$ then $Y \cap \Delta \subset U$. Then the map $h|Y \cap \Delta : Y \cap \Delta \to K$ witnesses the property needed to show that $Y \in G_i$.

It follows from the definition that $S(Z, \Delta) \subseteq \bigcap_{i=1}^\infty G_i$, so let us prove the opposite inclusion. Suppose that $X \in \bigcap_{i=1}^\infty G_i$, let A be closed in $X \cap \Delta$ and let $f : A \to K$ be a map. We must show that f extends to a map of $X \cap \Delta$ to K. Extend f to a map h from a neighborhood in Z of A to K. Then find a $D \in \mathscr{D}$ such that $A \subset D$ and h is defined on D. There exists i so that $D_i = D$ and $f_i|D_i \cap \Delta$ is $\mathscr{U}$-near $h|D_i \cap \Delta$. Since $X \in G_i$, there exists a map $g : X \cap \Delta \to K$

such that on $X \cap \Delta \cap D_i$, g and f_i are $\mathscr{U}$-near. Restricting all three maps f, f_i and g to A, one sees that f is homotopic to f_i and f_i is homotopic to g. Since g extends to $X \cap \Delta$, then the homotopy extension property shows that f does also. □

3.2. Corollary. *For the Hilbert cube* $\mathbf{Q}$, $S(\mathbf{Q}, \Delta)$ *is a* G_δ*-set in* $2^{\mathbf{Q}}$ *and hence the complementary subspace* $2^{\mathbf{Q}} \setminus S(\mathbf{Q}, \Delta)$ *is a* σ*-compactum. If* $C(\mathbf{Q})$ *denotes the (closed) subspace of* $2^{\mathbf{Q}}$ *consisting of all subcontinua of* $\mathbf{Q}$, *then* $C(\mathbf{Q}) \cap (2^{\mathbf{Q}} \setminus S(\mathbf{Q}, \Delta))$ *is a* σ*-compactum.* □

Let us translate these two results into one concerning dimension functions.

3.3. Corollary. *For any dimension function* D *with stratum* $\mathscr{K} = \{K_0, K_1, \dots\}$ *consisting of countable polyhedra* K_i, *and for any* $n \geq 0$,

(a) *the space* $D^{\leq n}(\Delta) = \{X \in 2^{\mathbf{Q}} | D(X \cap \Delta) \leq n\}$ *is a* G_δ*-subset of* $2^{\mathbf{Q}}$;

(b) *the space* $D^{\geq n}(\Delta) = \{X \in 2^{\mathbf{Q}} | D(X \cap \Delta) \geq n\}$ *is an* F_σ*-subset of* $2^{\mathbf{Q}}$.

Proof. By [**M-S**, Theorem 10, p. 302] each K_i is homotopy equivalent to a separable, metrizable ANR. □

Given a dimension function D, an *enhanced Bockstein basis* for D is a collection of dimension functions D_i each having a stratum consisting of countable polyhedra and having the property that for all compacta X,

$$D(X) = \max\{D_i(X) | \, i \in \mathbb{N}\}.$$

Observe that if $D = \dim$ or $D = \dim_G$ (for arbitrary abelian group G), then D has an enhanced Bockstein basis. For dim this follows easily. For $\dim_G$ let us refer to [**Ku**, Theorem 6], in which it is shown that for each abelian group G there is a Bockstein basis $\sigma(G)$ consisting of a countable set of countable abelian groups G_i. It is not difficult to show that for each n and i there is a $K(G_i, n)$ which is a countable CW-complex, or indeed, a countable simplicial complex. Hence for each i, $\dim_{G_i}$ has a stratum consisting of countable polyhedra. Therefore $\{\dim_{G_i} | \, i \in \mathbb{N}\}$ is an enhanced Bockstein basis for $\dim_G$. (We let $K(G, 0) = S^0$.)

3.4. Corollary. *Let D be a dimension function having an enhanced Bockstein basis. Then* (a) *and* (b) *of Corollary* 3.3 *are true for D. In particular this is so if $D = \dim$ or $\dim_G$ for any abelian group G.*

Proof. Let $\{D_i\}$ be an enhanced Bockstein basis for D. By Corollary 3.3, and the fact that each D_i has a stratum consisting of countable polyhedra, we have that (a) and (b) are true for each D_i. Since $D^{\leq n}(\Delta) = \bigcap_{i\in\mathbb{N}} D_i^{\leq n}(\Delta)$, (a) follows for D, and (b) is a consequence of (a). □

In §6 we are going to obtain results related to the notion of uniform dimension. The following fact will be used there.

3.5. Corollary. *For any dimension function D having an enhanced Bockstein basis and $n \geq 0$, the set $\overline{D}^{\geq n}(\Delta)$ consisting of all $X \in 2^{\mathbf{Q}}$ such that*

$$X \cap \Delta \neq \varnothing \text{ and for every open } W \subseteq X \cap \Delta,\ W \neq \varnothing,\ D(W) \geq n$$

is an $F_{\sigma\delta}$-subset of $2^{\mathbf{Q}}$. In particular this is true if $D = \dim$ or $\dim_G$ for any abelian group G.

Proof. In case $\Delta = \varnothing$ there is nothing to prove, so assume $\Delta \neq \varnothing$. We need to show that $\overline{D}^{\geq n}(\Delta)$ is a countable intersection of F_σ-sets. Take $\mathscr{U}$ to be a countable base for the topology of Δ with the empty set removed from $\mathscr{U}$. Let $G = \{X \in 2^{\mathbf{Q}} | D(\overline{U} \cap X \cap \Delta) \geq n$ for all $U \in \mathscr{U}\}$ and $H = \{X \in 2^{\mathbf{Q}} | D(U \cap X \cap \Delta) \geq n$ for all $U \in \mathscr{U}\}$. Let $U \in \mathscr{U}$ and denote $\mathscr{D}(U) = \{X \in 2^{\mathbf{Q}} | D(\overline{U} \cap X \cap \Delta) \geq n\}$. From Corollary 3.4(b) we conclude that $\mathscr{D}(U)$ is an F_σ-set. Thus $G = \bigcap\{\mathscr{D}(U) | U \in \mathscr{U}\}$ is a countable intersection of F_σ-sets.

We claim that $G = H$. Clearly, $H \subseteq G$. To show the reverse inclusion, suppose $X \in 2^{\mathbf{Q}}$ and $D(\overline{U} \cap X \cap \Delta) \geq n$ for all $U \in \mathscr{U}$. Let $V \in \mathscr{U}$. Then we can find $U \in \mathscr{U}$ with $\overline{U} \subset V$. Hence $\overline{U} \cap X \cap \Delta \subset V \cap X \cap \Delta$ and so $D(V \cap X \cap \Delta) \geq n$. This shows that $X \in H$. An application of Lemma 2.5 shows finally that $H = \overline{D}^{\geq n}(\Delta)$. □

4. Apparatus of absorbing sequences. Let us recall [Tor] that a closed subset A of $\mathbf{Q}$ is a Z-set if every map of the n-dimensional cube I^n, $n = 1, 2, \ldots$, can be approximated by a map whose image misses A. An arbitrary set A with the above property is called a locally homotopy negligible set. This is equivalent to the statement that there exists a homotopy $h_t: \mathbf{Q} \to \mathbf{Q}$ which instantaneously maps $\mathbf{Q}$ off A, i.e., $h_t(\mathbf{Q}) \cap A = \varnothing$ for $t > 0$ and $h_0 = \mathrm{id}$. A set that is a countable union of Z-sets is called a σZ-set. An embedding whose image is a Z-set is called a Z-embedding.

Let $\mathscr{C}_n$ be a topological class that is closed hereditary, $n \geq 1$, and write $\mathscr{C} = \{\mathscr{C}_n\}_{n=1}^{\infty}$. By a graded sequence $X = \{X_n\}_{n=1}^{\infty}$ in a space E we mean a sequence of subsets of E such that $X_1 \supseteq X_2 \supseteq \cdots$. A graded sequence $\{A_n\}_{n=1}^{\infty}$ in a compact space Y is called a $\mathscr{C}$-sequence if $A_n \in \mathscr{C}_n$, $n \geq 1$. We say that the graded sequence X in a copy E of the Hilbert cube is *strongly* $(Y, \{A_n\}_{n=1}^{\infty})$*-universal*, if for every map $f: Y \to E$ which restricts to a Z-embedding on some compact subset K of Y, there exists a Z-embedding $v: Y \to E$ which is arbitrarily close to f and satisfies:

(a) $v|K = f|K$,
(b) $v^{-1}(X_n)\backslash K = A_n \backslash K$.

The sequence X will be called *strongly* $\mathscr{C}$*-universal* if it is strongly $(\mathbf{Q}, \{A_n\}_{n=1}^{\infty})$-universal for every $\mathscr{C}$-sequence $\{A_n\}_{n=1}^{\infty}$ in the Hilbert cube $\mathbf{Q}$. (This is equivalent to the definition given in **[DvMM]**.) The graded sequence X in E is called $\mathscr{C}$*-absorbing*, if

(1) it is a $\mathscr{C}$-sequence,
(2) X_1 is contained in a σZ-set in E, and
(3) X is strongly $\mathscr{C}$-universal.

For the graded sequences studied in this paper, verification of the $\mathscr{C}$-absorbing property will be done in the following manner. Let X be such a graded sequence in a copy E of the Hilbert cube. By its definition and the results of §3, each such X will be a $\mathscr{C}$-sequence, and hence (1) will be true. The tool for verifying (2) will be Lemma 5.3.

Our proof of (3), the strong $\mathscr{C}$-universality will go as follows. We will select a certain closed subset E_0 of E. Then we will show that

(i) the sequence $\{X_n \cap E_0\}_{n=1}^{\infty}$ is $\mathscr{C}$-universal in E_0, i.e., given a $\mathscr{C}$-sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathbf{Q}$, there exists an embedding $w: \mathbf{Q} \to E_0$ with $w^{-1}(X_n \cap E_0) = A_n$, $n \geq 1$, and

(ii) the sequence X is strongly $(E_0, \{X_n \cap E_0\}_{n=1}^{\infty})$-universal in E.

As shown in **[DvMM**, Theorem 1**]** (the uniqueness theorem) two $\mathscr{C}$-absorbing sequences $X = \{X_n\}_{n=1}^{\infty}$ and $Y = \{Y_n\}_{n=1}^{\infty}$ in respective copies E and E' of $\mathbf{Q}$ are homeomorphic, i.e., there exists a homeomorphism h of E onto E' so that $h(X_n) = Y_n$; in particular

$h(X_n \setminus X_{n+1}) = h(Y_n \setminus Y_{n+1})$ and $h(\bigcap_{n=1}^{\infty} X_n) = \bigcap_{n=1}^{\infty} Y_n$.

In this paper our choices of $\mathscr{C} = \{\mathscr{C}_n\}_{n=1}^{\infty}$ will always be such that $\mathscr{C}_1 = \mathscr{C}_2 = \cdots$. Hence we shall specialize the above definitions for that situation.

Suppose $\mathscr{C}$ is a topological, closed hereditary class and let $\mathscr{C}_n = \mathscr{C}$ for $n \geq 1$. We say that a graded sequence $\{X_n\}_{n=1}^{\infty}$ in a copy E of the Hilbert cube is *$\mathscr{C}$-absorbing* if the sequence $\{X_n\}_{n=1}^{\infty}$ is $\{\mathscr{C}_n\}_{n=1}^{\infty}$-absorbing. If, additionally, $X_n = X$ then the statement that X is a *relative $\mathscr{C}$-absorbing* set in E will mean that $\{X_n\}_{n=1}^{\infty}$ satisfies the conditions (1), (2) and is strongly $(\mathbf{Q}, \{A_n\}_{n=1}^{\infty})$-universal for every constant $\mathscr{C}$-sequence $\{A_n\}_{n=1}^{\infty}$ (i.e., $A_n = A$ for $n \geq 1$ and $A \in \mathscr{C}$) in the Hilbert cube $\mathbf{Q}$. It directly follows from the definitions that such X is a $\mathscr{C}$-absorbing set in E in the sense of **[BM]**. In particular, the uniqueness theorem of **[DvMM]** yields that two relative $\mathscr{C}$-absorbing sets are relatively homeomorphic.

Herein we shall apply only the cases where either $\mathscr{C} = \mathscr{A}$ is the class of all metric σ-compacta or $\mathscr{C} = \mathscr{F}_{\sigma\delta}$ is the class of absolute $F_{\sigma\delta}$-sets. Let $B = [0, 1]^{\infty} \setminus (0, 1)^{\infty} \subset \mathbf{Q}$ be the pseudoboundary of $\mathbf{Q}$ and $s = \mathbf{Q} \setminus B = (0, 1)^{\infty}$ be its pseudointerior. The next two theorems show how $\mathscr{A}$- and $\mathscr{F}_{\sigma\delta}$-absorbing sequences compare with standard models based on these spaces.

4.1. THEOREM. *Let $X = \{X_n\}_{n=1}^{\infty}$ be an $\mathscr{A}$-absorbing sequence in a copy E of the Hilbert cube. Then for $n \geq 1$ we have*

(a) *X_n is homeomorphic to B,*

(b) *$E \setminus X_n$ is homeomorphic to s,*

(c) *$X_n \setminus X_{n+1}$ is homeomorphic to $B \times s$,*

(d) *$(E \setminus X_{n+1}, X_n \setminus X_{n+1})$ is homeomorphic to $(\mathbf{Q} \times s, B \times s)$, i.e., $X_n \setminus X_{n+1}$ is a Z-absorbing set* **[vM]** *in a copy of $s \cong R^{\infty}$,*

(e) *$\bigcap_{n=1}^{\infty} X_n$ is a relative $\mathscr{F}_{\sigma\delta}$-absorbing set in E; in particular $\bigcap_{n=1}^{\infty} X_n$ is homeomorphic to B^{∞}.*

Proof (*cf.* **[DvMM]**). Let

$$S_n = \underbrace{B \times B \times \cdots \times B}_{n\text{-times}} \times \mathbf{Q} \times \mathbf{Q} \times \cdots \subset \mathbf{Q}^{\infty}$$

and

$$S_n'' = \{(x_i) \in \mathbf{Q}^{\infty} | x_i \in B \text{ for some } i \geq n\}.$$

As shown in **[DvMM]**, both $\{S_n\}_{n=1}^{\infty}$ and $\{S_n''\}_{n=1}^{\infty}$ are $\mathscr{A}$-absorbing sequences.

To obtain (a) and (b) apply the uniqueness theorem for the $\mathscr{A}$-absorbing sequences $\{X_n\}$ and $\{S_n\}$ and the fact (see [vM]) that $(\mathbf{Q}^\infty, S_n)$ is homeomorphic to $(\mathbf{Q}, B)$. To get (c) and (d) apply the uniqueness theorem for $\{X_n\}$ and $\{S''_n\}$. For (c) use the fact that

$$S''_n \backslash S''_{n+1} = \underbrace{\mathbf{Q} \times \mathbf{Q} \times \cdots \times \mathbf{Q}}_{(n-1)\text{-times}} \times B \times s \times s \times \cdots ,$$

which is homeomorphic to $B \times s$. For (d) use the fact that

$$(\mathbf{Q}^\infty \backslash S''_{n+1}, S''_n \backslash S''_{n+1}) = (\underbrace{\mathbf{Q} \times \mathbf{Q} \times \cdots \times \mathbf{Q}}_{(n-1)\text{-times}} \times \mathbf{Q} \times s \times s \times \cdots ,$$
$$\underbrace{\mathbf{Q} \times \mathbf{Q} \times \cdots \times \mathbf{Q}}_{(n-1)\text{-times}} \times B \times s \times s \times \cdots),$$

which is homeomorphic to $(\mathbf{Q} \times s, B \times s)$. Item (e) follows from the fact that $\bigcap_n S_n = B^\infty$. □

4.2. THEOREM. *Let $X = \{X_n\}_{n=1}^\infty$ be an $\mathscr{F}_{\sigma\delta}$-absorbing sequence in a copy E of the Hilbert cube. Then for $n \geq 1$ we have X_n is homeomorphic to B^∞. Furthermore, $\bigcap_{n=1}^\infty X_n$ is also homeomorphic to B^∞.*

Proof. Since, by 4.1(e), B^∞ is a relative $\mathscr{F}_{\sigma\delta}$-absorbing set in $\mathbf{Q}^\infty$, the sequence $\{T_n\}_{n=1}^\infty$, where

$$T_n = B^\infty \times B^\infty \times \cdots \times B^\infty \times \mathbf{Q}^\infty \times \mathbf{Q}^\infty \times \cdots ,$$

is an $\mathscr{F}_{\sigma\delta}$-absorbing sequence in $(\mathbf{Q}^\infty)^\infty$, a copy of $\mathbf{Q}$ (see Theorem 3.1 of [DvMM]). By the uniqueness theorem on absorbing sequences there is a homeomorphism h of E onto $(\mathbf{Q}^\infty)^\infty$ with $h(X_n) = T_n$. Now the assertion follows from the fact (see [vM]) that T_n is homeomorphic to B^∞. □

5. Absorbing $\mathscr{A}$-sequences in $C(\mathbf{Q})$ and $2^{\mathbf{Q}}$ related to dimension. Throughout this section, D will denote a true dimension function having an enhanced Bockstein basis (see §§2 and 3). We will be interested in the following graded sequences in $2^{\mathbf{Q}}$ related to D:

$$D^* = \{D^{\geq n}\}_{n=1}^\infty \quad \text{and} \quad D^*|C(\mathbf{Q}) = \{D^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^\infty ,$$

where $D^{\geq n} = D^{\geq n}(\mathbf{Q})$ (see 3.3(b) for the definition of $D^{\geq n}(\mathbf{Q})$). The sequences D^* and $D^*|C(\mathbf{Q})$ will be considered in $2^{\mathbf{Q}}$ and in $C(\mathbf{Q})$, the hyperspace of all continua in $\mathbf{Q}$, respectively. By [C-S2], both $2^{\mathbf{Q}}$

and $C(\mathbf{Q})$ are copies of $\mathbf{Q}$. Since D is a true dimension function, by 2.8, we have $\dim^{\geq n} \supseteq D^{\geq n}$ for every n. We may treat D^* and $D^*|C(\mathbf{Q})$ as "subsequences" of $\dim^*$ and $\dim^*|C(\mathbf{Q})$, respectively. By Corollary 3.4(b), the sequences

$$(*) \qquad D^* \text{ and } D^*|C(\mathbf{Q}) \text{ are } \mathscr{A}\text{-sequences.}$$

The sequence $\dim^*$ was analyzed in detail in **[DvMM]**, where it was established that $\dim^*$ is an $\mathscr{A}$-absorbing sequence.

We now come to the main theorem of this section.

5.1. THEOREM. *For any true dimension function D having an enhanced Bockstein basis, the sequences D^* and $D^*|C(\mathbf{Q})$ are $\mathscr{A}$-absorbing in $2^{\mathbf{Q}}$ and $C(\mathbf{Q})$, respectively.*

Before proceeding with our lengthy proof of this theorem, let us provide several significant consequences. Recall that $D^{\geq n} = \{A \in 2^{\mathbf{Q}} | D(A) \geq n\}$. We define $D^{\leq n}$ and D^n similarly. Applying 4.1, 5.1 and 3.4, we have

5.2. COROLLARY. *For $n = 1, 2, \dots,$ we have*

(a) *$D^{\geq n}$ and $D^{\geq n+1} \cap C(\mathbf{Q})$ are homeomorphic to B,*
(b) *$D^{\leq n-1}$ and $D^{\leq n} \cap C(\mathbf{Q})$ are homeomorphic to s,*
(c) *D^n and $D^{n+1} \cap C(\mathbf{Q})$ are homeomorphic to $B \times s$,*
(d) *$(D^{\leq n}, D^n)$ and $(D^{\leq n+1} \cap C(\mathbf{Q}), D^{n+1} \cap C(\mathbf{Q}))$ are homeomorphic to $(\mathbf{Q} \times s, B \times s)$,*
(e) *D^∞ and $D^\infty \cap C(\mathbf{Q})$ are homeomorphic to B^∞.*

This is true in particular if $D = \dim$ or $D = \dim_G$ for any abelian group G. □

To prove Theorem 5.1, we must verify (1)–(3) of §4. By $(*)$, (1) is true, so the next step is to show (2), which is done in the next lemma.

5.3. LEMMA. *The spaces $D^{\geq 1}$ and $D^{\geq 2} \cap C(\mathbf{Q})$ are σZ-sets in $2^{\mathbf{Q}}$ and in $C(\mathbf{Q}))$, respectively.*

Proof. It is known **[Cu]** that $\dim^{\geq 1}$, is a σZ-set in $2^{\mathbf{Q}}$ (see the first paragraph of the proof of Theorem 4.6 in **[DvMM]**). Since $D^{\geq 1} \subseteq \dim^{\geq 1}$, $D^{\geq 1}$ is also a σZ-set. Lemma 5.4 below with $p = \infty$ and the fact that $D^{\geq 2} \cap C(\mathbf{Q}) \subseteq \dim^{\geq 2} \cap C(\mathbf{Q})$ show similarly that $D^{\geq 2} \cap C(\mathbf{Q})$ is a σZ-set. □

Denote by $C(I^p)$ the subspace consisting of all continua of the hyperspace of the cube I^p. By [C-S2], $C(I^p)$ is a copy of $\mathbf{Q}$ for $2 \leq p \leq \infty$. Write

$$C^{\geq n}(I^p) = \{A \in C(I^p) | \dim(A) \geq n\}.$$

For any compactum X, $\mathscr{F}(X)$, will designate the hyperspace of finite subsets of X.

5.4. LEMMA. *The space* $C^{\geq 2}(I^p)$, $2 \leq p \leq \infty$, *is a* σZ*-set in* $C(I^p)$.

Proof. The metric d indicated in the introduction has the property that if $B(x, \epsilon) \cap B(y, \delta) \neq \varnothing$, then the line segment $[x, y] \subset B(x, \epsilon) \cup B(y, \delta)$. We shall use this fact in a moment.

Given $\epsilon > 0$, we will find a map

$$\varphi \colon C(I^p) \to C(I^p)$$

such that $d(A, \varphi(A)) \leq 2\epsilon$ and $\varphi(A) \notin C^{\geq 2}(I^p)$ for every $A \in C(I^p)$. By Proposition 2.2 of [Cu], there exists a map $\psi \colon C(I^p) \to \mathscr{F}(I^p)$ with $d(\psi, \mathrm{id}) < \epsilon$. Now, to every $A \in C(I^p)$ we will assign a continuum $\varphi(A)$ with the properties:

(1) $\varphi(A)$ is 1-dimensional,
(2) $\psi(A) \subset \varphi(A)$,
(3) $d(\varphi(A), A) \leq 2\epsilon$.

Consider the finite collection of balls $\{B(x, \epsilon) | x \in \psi(A)\}$. For every $x_1, x_2 \in \psi(A)$, we let

$$A_{x_1 x_2} = [x_1, x_2] \cap \left(\overline{B}(x_1, \epsilon) \cup \overline{B}(x_2, \epsilon)\right),$$

where $[x_1, x_2]$ denotes the segment with x_1 and x_2 as its endpoints (clearly, if $B(x_1, \epsilon) \cap B(x_2, \epsilon) \neq \varnothing$ then $A_{x_1 x_2} = [x_1, x_2]$). We set

$$\varphi(A) = \bigcup_{x_1, x_2 \in \psi(A)} A_{x_1 x_2}.$$

Since $d(\psi(A), A) < \epsilon$ and A is a continuum, $\varphi(A)$ is also a (1-dimensional) continuum. Moreover, we have $d(\varphi(A), A) \leq 2\epsilon$. One checks that $A \to \varphi(A)$ is continuous. □

Let A_0 be a nonempty closed subset of $\mathbf{Q}$. Consider the relative hyperspaces

$$C(\mathbf{Q}, A_0) = \{A \in C(\mathbf{Q}) | A \cap A_0 \neq \varnothing\},$$
$$2^{\mathbf{Q}}(A_0) = \{A \in 2^{\mathbf{Q}} | A \cap A_0 \neq \varnothing\}$$

which are, by **[C-S1]**, copies of the Hilbert cube. We write $C_0(\mathbf{Q}) = C(\mathbf{Q}, \{0\})$ and $2^{\mathbf{Q}}_0 = 2^{\mathbf{Q}}(\{0\})$. Note that, by homogeneity of $\mathbf{Q}$, 0 can be replaced by an arbitrary $q \in \mathbf{Q}$. We denote $C_0(I^p) = \{A \in C(I^p) | 0 \in A\}$ and $C_0^{\geq n}(I^p) = C^{\geq n}(I^p) \cap C_0(I^p)$.

Now we set out to prove (3) of §4 for D^* and $D^*|C(\mathbf{Q})$. This will be done in the two stages (i) and (ii) of that section. As E_0 we select $2^{\mathbf{Q}}_0$ when dealing with the sequence D^*, and $C_0(\mathbf{Q})$ when dealing with the sequence $D^*|C(\mathbf{Q})$. Our approach will be to give the proof for $D^*|C(\mathbf{Q})$ first and then indicate the modifications needed for D^*. We start with (ii).

5.5. PROPOSITION. *The sequence* $D^*|C(\mathbf{Q}) = \{D^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty}$ *is strongly* $(C_0(\mathbf{Q}), \{D^{\geq n} \cap C_0(\mathbf{Q})\}_{n=2}^{\infty})$*-universal in* $C(\mathbf{Q})$.

We need two auxiliary facts.

5.6. LEMMA. *There are homotopies* $\psi\colon 2^{\mathbf{Q}} \times [0, 1] \to 2^{\mathbf{Q}}$ *and* $\varphi\colon C(\mathbf{Q}) \times [0, 1] \to C(\mathbf{Q})$ *satisfying the following properties*:

(a) $\psi_0 = \mathrm{id}_{2^{\mathbf{Q}}}$ *and* $\varphi_0 = \mathrm{id}_{C(\mathbf{Q})}$;

(b) $\psi(2^{\mathbf{Q}} \times (0, 1]) \subset \mathscr{F}(\mathbf{Q})$;

(c) $\varphi(A, t) \notin C^{\geq 2}(\mathbf{Q})$ *for* $(A, t) \in C(\mathbf{Q}) \times (0, 1]$;

(d) *writing* π_n *for the projection of* $\mathbf{Q} = \prod_{i=1}^{\infty} I_i$ *onto* $\prod_{i=n+1}^{\infty} I_i$, $I_i = I$, *we have* $\pi_n(\psi(A, t)) = \{0\}$ *for* $t > \frac{1}{n}$ *and* $A \in 2^{\mathbf{Q}}$ (*resp.*, $\pi_n(\varphi(A, t)) = \{0\}$ *for* $t > \frac{1}{n}$ *and* $A \in C(\mathbf{Q})$);

(e) $d(\psi_t(A), A) \leq 2t$ *for* $(A, t) \in 2^{\mathbf{Q}} \times [0, 1]$ (*resp.*, $d(\varphi_t(A), A) \leq 7t$ *for* $(A, t) \in C(\mathbf{Q}) \times [0, 1]$);

(f) $\psi(A, t) \subset \varphi(A, t)$ *for* $(A, t) \in C(\mathbf{Q}) \times [0, 1]$.

Proof. Since $\mathscr{F}(\mathbf{Q})$ is locally homotopy negligible in $2^{\mathbf{Q}}$ **[Cu]**, there exists a homotopy $H\colon 2^{\mathbf{Q}} \times (0, 1] \to \mathscr{F}(\mathbf{Q})$ such that $d(H_t, \mathrm{id}) \leq t$ for all $t > 0$. Set $H_0 = \mathrm{id}$ to extend H over $2^{\mathbf{Q}} \times [0, 1]$. Pick a homotopy $p\colon \mathbf{Q} \times [0, 1] \to \mathbf{Q}$ so that

(1) $p_0 = \mathrm{id}$,

(2) $\pi_n p_t(q) = 0$, $q \in \mathbf{Q}$, provided $t > \frac{1}{n}$,

(3) $d(p_t, \mathrm{id}) \leq t$.

We may view p as a map from $2^{\mathbf{Q}} \times [0, 1]$ to $2^{\mathbf{Q}}$ via $(A, t) \mapsto p(A \times \{t\})$. With this, write $\psi_t = p_t \circ H_t$ and note that the conditions (a), (b), (d), (e) are satisfied for ψ_t (e.g., we have $d(\psi_t(A), A) \leq d(p_t H_t(A), H_t(A)) + d(H_t(A), A)) \leq t + t = 2t$).

To obtain φ fix $A \in C(\mathbf{Q})$. The set $\varphi_t(A)$ $(t > 0)$ will be an "enlargement" of the finite set $\psi_t(A)$ to a 1-dimensional continuum that "passes through" $\psi_t(A)$. We will apply the reasoning of 5.4. Note that the finite family $\{B(x, \frac{5}{2}t) | x \in \psi_t(A)\}$ of balls covers $\psi_t(A)$. For every $x_1, x_2 \in \psi_t(A)$, we let

$$A^t_{x_1 x_2} = [x_1, x_2] \cap (\overline{B}(x_1, \tfrac{5}{2}t) \cup \overline{B}(x_2, \tfrac{5}{2}t)).$$

We set

$$\varphi_t(A) = \bigcup_{x_1, x_2 \in \psi_t(A)} A^t_{x_1 x_2}.$$

Since A is connected, $\varphi_t(A)$ is also connected. The continuity of φ follows from the continuity of ψ restricted to $C(\mathbf{Q})$. Moreover we have

$$\begin{aligned} d(\varphi_t(A), A) &\leq d(\varphi_t(A), \psi_t(A)) + d(\psi_t(A), A) \\ &\leq 5t + 2t = 7t. \end{aligned}$$

Thus (e) follows. The property in (d) is a consequence of the same property for ψ. □

5.7. Remark. Observe that the proof of 5.4 works for the space $C_0(I^p)$. We simply can assume $0 \in \psi(A)$. No other changes are necessary to obtain that $C_0^{\geq 2}(I^p)$ is a σZ-set in $C_0(I^p)$. Similarly, the proof of 5.6 works for $2_0^{\mathbf{Q}}$ and $C_0(\mathbf{Q})$. Here we can assume $0 \in H(A, t)$ and get homotopies $\psi: 2_0^{\mathbf{Q}} \times [0, 1] \to 2_0^{\mathbf{Q}}$ and $\varphi: C_0(\mathbf{Q}) \times [0, 1] \to C_0(\mathbf{Q})$ satisfying suitable properties (a)–(f) of 5.6. □

5.8. Lemma. *Fix pairwise disjoint infinite subsets* $N_1, N_2, \ldots$ *of the set of positive integers* $\mathbb{N}$ *such that* $\{1, 2, \ldots, n+1\} \cap N_n = \varnothing$ *and fix* $k_n \in N_n$. *There is an injective map*

$$\chi: C_0(\mathbf{Q}) \times (0, 1] \to C_0(\mathbf{Q})$$

such that

(a) $\chi_t^{-1}(D^{\geq n} \cap C_0(\mathbf{Q})) = D^{\geq n} \cap C_0(\mathbf{Q})$ *for* $t > 0$ *and* $n \geq 2$,

(b) *if* $\frac{1}{n+1} < t \leq \frac{1}{n}$ *then* $p_{k_n}\chi_t(A) \neq \{0\}$, *while* $p_k\chi_t(A) = \{0\}$ *for all* $k \in \mathbb{N}\backslash(N_n \cup N_{n+1})$ *and* $A \in C_0(\mathbf{Q})$ *(*p_j *denotes projection on the* j*-th coordinate of* $\mathbf{Q}$*)*,

(c) $d(\chi_t(A), \{0\}) \leq t$ *for* $(A, t) \in C_0(\mathbf{Q}) \times (0, 1]$.

Proof. Select a homeomorphism λ_n of $\mathbf{Q}$ onto $\prod_{k \in N_n \backslash \{k_n\}} I_k$ having the property that 0 is carried to the element of $\prod_{k \in N_n \backslash \{k_n\}} I_k$ all of whose coordinates are zero (which we also denote 0). Let h_n be the homeomorphism of $C_0(\mathbf{Q})$ onto $C_0(\prod_{k\in N_n\setminus\{k_n\}} I_k)$ induced by λ_n. Set

$$\begin{aligned}\overline{h}_n(A) &= \{0\}\times h_n(A)\times\{0\}\cup\ \{0\}\times\{0\}\times[0,1]\\ &\subset\left(\prod_{k\in\mathbb{N}\setminus N_n} I_k\right)\times\left(\prod_{k\in N_n\setminus\{k_n\}} I_k\right)\times I_{k_n}=\prod_{k\in\mathbb{N}} I_k=\mathbf{Q}.\end{aligned}$$

Define

$$\chi\left(A,\,s\frac{1}{n}+(1-s)\frac{1}{n+1}\right)=s\,\overline{h}_n(A)\cup(1-s)\,\overline{h}_{n+1}(A).$$

It is clear that $\chi(A,\,s\frac{1}{n}+(1-s)\frac{1}{n+1})$ is the pointed union of $sh_n(A)$, $(1-s)h_{n+1}(A)$, $[0,s]$ and $[0,1-s]$ (the basepoint is $0\in\mathbf{Q}$). Consequently, using 2.1 and the fact that D is a true dimension function, (a) easily follows. The construction assures (b). For $j\in\{1,2,\dots,n+1\}$, $j\notin N_n\cup N_{n+1}$, by (b) it follows that $p_j\chi_t(A)=\{0\}$, $\frac{1}{n+1}<t\le\frac{1}{n}$; thus $d(\chi_t(A),\{0\})\le\frac{1}{n+1}<t$.

To show that χ is injective, suppose $\chi(A,t)=\chi(A',t')$. Assuming $t'<t$ and $\frac{1}{n+1}<t\le\frac{1}{n}$, we get $\frac{1}{n+1}<t'\le\frac{1}{n}$ (otherwise $p_{k_\ell}\chi(A',t')$ would be $\{0\}$ for $\ell\ge n+2$ while $p_{k_\ell}\chi(A,t)\ne\{0\}$). If $\frac{1}{n+1}<t,t'\le\frac{1}{n}$, then clearly $t=t'$ and thus $A=A'$ because h_n is a homeomorphism. □

Proof of 5.5. Let $F\colon C_0(\mathbf{Q})\to C(\mathbf{Q})$ be a map and let K be a closed subset of $C_0(\mathbf{Q})$ such that $F|K$ is a Z-embedding. We may assume that F itself is a Z-embedding and that we have

(1) $F(C_0(\mathbf{Q})\setminus K)\cap F(K)=\varnothing$.

Let $\epsilon_0>0$ be given, $\epsilon_0\le 1$.

Pick a map $\epsilon\colon C(\mathbf{Q})\to[0,\epsilon_0]$ such that

(2) $\epsilon^{-1}(\{0\})=F(K)$, and

(3) $d(F(A),F(K))\ge 12\epsilon(F(A))$ for $A\in C_0(\mathbf{Q})$.

Note from 5.8(c) that χ extends to $C_0(\mathbf{Q})\times[0,1]$ via $\chi(A,0)=\{0\}$. Let $\epsilon^*=\epsilon\circ F\colon C_0(\mathbf{Q})\to[0,\epsilon_0]$ and take φ and ψ from 5.6. Our ϵ_0-approximation G of F is given by

$$G(A)=\varphi(F(A),\epsilon^*(A))\cup[\psi(F(A),\epsilon^*(A))+\chi(A,\epsilon^*(A))].$$

We use $V+W$ to denote $\{v+w\mid(v,w)\in V\times W\}$. If $A\in K$ then one checks that $G(A)=F(A)$.

Suppose $A \in C_0(\mathbf{Q})\backslash K$. Since

$$\begin{aligned}\psi(F(A), \epsilon^*(A)) + \chi(A, \epsilon^*(A)) \\ = \bigcup\{x + \chi(A, \epsilon^*(A)) \mid x \in \psi(F(A), \epsilon^*(A))\},\end{aligned}$$

by 5.6(f), $G(A)$ is a continuum. Putting it informally, $G(A)$ is the union of the 1-dimensional continuum $\varphi(F(A), \epsilon^*(A))$ and a pairwise disjoint, finite collection of continua T with $D(T) = D(A)$, each of which intersects it. An application of 2.1 yields $D(G(A)) = D(A)$; therefore $G^{-1}(D^{\geq n} \cap C(\mathbf{Q})\backslash K) = D^{\geq n} \cap C_0(\mathbf{Q})\backslash K$. (In those instances where we do not need $G(A)$ to be a continuum, as in the proof of reflexive universality of D^* in 5.9 and $\overline{D}^*$ in 6.1, we just omit $\varphi(F(A), \epsilon^*(A))$ in defining G. For such G we will have $D(G(A)) = D(A)$ in 5.9 and $D(G(A)) = D(A)$ uniformly in 6.1. In the proof of 6.1 for continua, where we desire $D(G(A)) = D(A)$ uniformly, then we shall produce a different φ denoted $\overline{\varphi}$ so that $D(\overline{\varphi}(F(A), \epsilon^*(A))) = \infty$ uniformly.)

Using 5.6(c), we get

$$\begin{aligned}d(\psi(F(A), \epsilon^*(A)) + \chi(A, \epsilon^*(A)), \psi(F(A), \epsilon^*(A))) \\ \leq d(\chi(A, \epsilon^*(A)), \{0\}) \leq \epsilon^*(A).\end{aligned}$$

Consequently, by 5.6(e) we have

(4) $d(G(A), F(A)) \leq d(\varphi(F(A), \epsilon^*(A))) \cup [\psi(F(A), \epsilon^*(A)) + \chi(A, \epsilon^*(A))], \psi(F(A), \epsilon^*(A)) + d(\psi(F(A), \epsilon^*(A)), F(A)) \leq 9\epsilon^*(A) + 2\epsilon^*(A) = 11\epsilon^*(A) = 11\epsilon(F(A))$.

If $A \in C_0(\mathbf{Q})\backslash K$, then (1) and (2) imply that $\epsilon(F(A)) > 0$; this fact and items (3), (4) imply that $G(A) \notin F(K)$. The reader may check that for $A \in K$, $G(A) = F(A)$. Thus to show that G is an embedding it suffices to check that $G|C_0(\mathbf{Q})\backslash K$ is injective.

Assume $A, A' \in C_0(\mathbf{Q})\backslash K$, $G(A) = G(A')$ and $\epsilon^*(A') \leq \epsilon^*(A)$. Suppose there are s and n with $\frac{1}{s+1} < \epsilon^*(A') \leq \frac{1}{s} \leq \frac{1}{n+1} < \epsilon^*(A) \leq \frac{1}{n}$. In case $\frac{1}{s} < \frac{1}{n+1}$, then $s \neq n, n+1$; hence $k_s \in N_s \subseteq \mathbb{N}\backslash(N_n \cup N_{n+1})$. Using 5.6(d), 5.8(b) and the definition of G, we get $p_{k_s}(G(A)) = \{0\}$. Let $t = \epsilon^*(A')$; then using 5.8(b) we have $p_{k_s}(\chi_t(A')) \neq \{0\}$. From this and the definition of G, one sees that $p_{k_s}(G(A')) \neq \{0\}$, a contradiction. Consider the case $\frac{1}{s} = \frac{1}{n+1}$. Arguing in a manner similar to the above, one concludes that $p_{k_n}(G(A)) \neq 0$, whereas $p_{k_n}(G(A')) = \{0\}$, another contradiction. We infer that $\frac{1}{n+1} < \epsilon^*(A') \leq \epsilon^*(A) \leq \frac{1}{n}$, $p_k(G(A')) = \chi(A', \epsilon^*(A'))$

and $p_k(G(A)) = \chi(A, \epsilon^*(A))$ for all $k \in N_n \cup N_{n+1}$. Since χ is injective, we conclude that $A = A'$.

It remains to check that $G(C_0(\mathbf{Q}))$ is a Z-set in $C(\mathbf{Q})$. To get this, note that $G(A) \subset \mathbf{Q}$ is nowhere dense. Consider the map $(A, t) \to \lambda_t(A) = \{x \in \mathbf{Q} | d(x, A) \leq t\}$. Clearly (λ_t) is an instantaneous deformation of $C(\mathbf{Q})$ off $G(C_0(\mathbf{Q})\backslash K)$. It follows that $G(C_0(\mathbf{Q})) = F(K) \cup G(C_0(\mathbf{Q})\backslash K)$ is the union of a Z-set and a locally homotopy negligible set, hence a Z-set in $C(\mathbf{Q})$ **[Tor]**. □

5.9. Proposition. *The sequence* $D^* = \{D^{\geq n}\}_{n=1}^{\infty}$ *is strongly* $(2_0^{\mathbf{Q}}, \{D^{\geq n} \cap 2_0^{\mathbf{Q}}\}_{n=1}^{\infty})$*-universal in* $2^{\mathbf{Q}}$.

Proof. Note that Lemma 5.8 holds in the $2_0^{\mathbf{Q}}$-setting, i.e., there is an injective map $\chi: 2_0^{\mathbf{Q}} \times (0, 1] \to 2_0^{\mathbf{Q}}$ such that suitable versions of (a)–(c) hold (e.g., the condition (a) reads $\chi^{-1}(D^{\geq n} \cap 2_0^{\mathbf{Q}}) = D^{\geq n} \cap 2_0^{\mathbf{Q}}$ for $t > 0$ and $n \geq 1$). The role of $[0, 1]$ in defining $\overline{h}_n$ can be replaced by $\{0, 1\} \subset [0, 1]$. Now, correct the definition of G from the proof of 5.5 to let

$$G(A) = \psi(F(A), \epsilon^*(A)) + \chi(A, \epsilon^*(A)),$$

for $A \in 2_0^{\mathbf{Q}}$. □

Let us pass to proving $\mathscr{A}$-universality, step (i) of verifying strong $\mathscr{A}$-universality.

5.10. Proposition. *The sequence* $\{D^{\geq n} \cap C_0(\mathbf{Q})\}_{n=2}^{\infty}$ *is* $\mathscr{A}$*-universal in* $C_0(\mathbf{Q})$.

Proof. Fix an $\mathscr{A}$-sequence $\{A_n\}_{n=2}^{\infty}$ in $\mathbf{Q}$. We need to show that there is an embedding $w: \mathbf{Q} \to C_0(\mathbf{Q})$ with

(1) $w^{-1}(D^{\geq n} \cap C_0(\mathbf{Q})) = A_n$ for $n \geq 2$.

Represent $\mathbf{Q}$ as the product of two Hilbert cubes $\mathbf{Q}' \times \mathbf{Q}''$. Assume we are able to construct a map $\Phi: \mathbf{Q} \to C_0(\mathbf{Q}')$ satisfying (1). Then we could correct Φ to a required embedding in the following way. Write $\mathbf{Q}'' = \prod_{k=1}^{\infty} I_k$, $I_k = I$. For $x = (x_i) \in \mathbf{Q}$, $(0 \leq x_i \leq 1)$, set

$$v(x) = \bigcup_{n=1}^{\infty} \Big(\underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{(n-1)\text{-times}} \times [0, x_n] \times \{0\} \times \{0\} \times \cdots \Big) \subset \mathbf{Q}.$$

It is clear that $v: \mathbf{Q} \to C_0(\mathbf{Q}'')$ is an embedding and $\dim(v(x)) = 1$ for every $x \in \mathbf{Q}$. Now, assuming $\Phi: \mathbf{Q} \to C_0(\mathbf{Q}')$ is a map with (1), we can combine Φ and v to get w:

$$w(x) = (\Phi(x) \times \{0\}) \cup (\{0\} \times v(x)) \subset \mathbf{Q}' \times \mathbf{Q}''.$$

It is clear that $w(x) \in C_0(\mathbf{Q})$. Since $v(x)$ is 1-dimensional and Φ satisfies (1), by 2.8 and 2.1, w satisfies (1).

To produce Φ we need the following modification of [**DMM**, Lemma 5.4].

5.11. Lemma. *Suppose Z and Y are subsets of a copy E of the Hilbert cube so that $Z \subseteq Y$, Y is a σZ-set and $E \backslash Z$ is locally homotopy negligible in E. Then, given a σ-compactum $A \subseteq \mathbf{Q}$ there is $f: \mathbf{Q} \to E$ with $f(A) \subseteq Z$ and $f(\mathbf{Q} \backslash A) \subseteq E \backslash Y$.* □

We continue the proof of 5.10. We will apply 5.11 with $E = C_0(I^n)$, $Y_n = C_0^{\geq 2}(I^n)$, $Z_n = C_0^{\geq n}(I^n)$ and $A_n \subset \mathbf{Q}$, $n \geq 2$. First note that according to 5.7, $C_0^{\geq 2}(I^n)$ is a σZ-set in $C_0(I^n)$, a copy of the Hilbert cube. Moreover, $C_0(I^n) \backslash C_0^{\geq n}(I^n)$ is locally homotopy negligible in $C_0(I^n)$, because the map $(A, t) \to \{x \in I^n : d(A, x) \leq t\}$ is an instantaneous homotopy of $C_0(I^n)$ off $C_0(I^n) \backslash C_0^{\geq n}(I^n)$. Represent $\mathbf{Q}'$ as the product

$$\mathbf{Q}' = I^2 \times I^3 \times I^4 \times \cdots .$$

Apply 5.11, to find $\Phi_2: \mathbf{Q} \to C_0(I^2)$ so that

$$\Phi_2^{-1}(C_0^{\geq 2}(I^n)) = A_2.$$

Let $n > 2$. Use 5.11, to get $\Phi_n: \mathbf{Q} \to C_0(I^n)$ such that

$$\Phi_n(A_n) \subset C_0^{\geq n}(I^n) \quad \text{and} \quad \Phi_n(\mathbf{Q} \backslash A_n) \subset C_0(I^n) \backslash C_0^{\geq 2}(I^n).$$

Now, define $\Psi_n: \mathbf{Q} \to (I^2 \times I^3 \times \cdots \times I^n \times \{0\} \times \{0\} \times \cdots) \subset \mathbf{Q}'$ by letting

$$\Psi_n(x) = \bigcup_{i=2}^{n} (\underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{(i-2)\text{-times}} \times \Phi_i(x) \times \{0\} \times \{0\} \times \cdots).$$

The sequence $\{\Psi_n\}_{n=2}^{\infty}$ is a Cauchy sequence of maps from $\mathbf{Q}$ into $C_0(\mathbf{Q}')$. Hence its limit $\Phi = \lim \Psi_n$ is a continuous map. Notice that

$$\Phi(x) = \bigcup_{i=2}^{\infty} (\underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{(i-2)\text{-times}} \times \Phi_i(x) \times \{0\} \times \{0\} \times \cdots).$$

From the above formula, it follows that whenever $x \in A_k \backslash A_{k+1}$ then $\Phi(x)$ is the pointed union of A, B, C, where $\dim(A) < k$, B is a

k-dimensional subset of I^k and C is 1-dimensional. Since D is a true dimension function, then by 2.8 and 2.1 condition (1) follows. □

5.12. PROPOSITION. *The sequence* $\{D^{\geq n} \cap 2^{\mathbf{Q}}_0\}_{n=1}^{\infty}$ *is* $\mathscr{A}$*-universal in* $2^{\mathbf{Q}}_0$.

Proof. Represent $\mathbf{Q}' = I \times I^2 \times \cdots$ and repeat the proof of 5.10 replacing the segment $[0, x_n]$ by $\{0, x_n\}$. □

Proof of 5.1. Apply 5.3, 5.5, 5.9, 5.10 and 5.12. □

Let A_0 be a nonempty closed subset of $\mathbf{Q}$. We can define in a natural way the following relative graded $\mathscr{A}$-sequences $\{D(A_0)^{\geq n}\}_{n=1}^{\infty}$ in $2^{\mathbf{Q}}(A_0)$ and $\{D(A_0)^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty}$ in $C(\mathbf{Q}, A_0)$, where $D(A_0)^{\geq n} = D^{\geq n} \cap 2^{\mathbf{Q}}(A_0)$. Another variation might be to consider the following sequences $\{D|A_0^{\geq n}\}_{n=1}^{\infty}$ and $\{D|A_0^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty}$, where $D|A_0^{\geq n} = \{A \in 2^{\mathbf{Q}} | D(A \cap A_0) \geq n\}$. By 3.4(b), the above four sequences are also $\mathscr{A}$-sequences. It is reasonable to ask what are restrictions on A_0 in order that these sequences are absorbing (an obvious sufficient condition for the last pair of sequences is $D(A_0) = \infty$). In general, it is not clear how to adjust the arguments of the proof of 5.1 in this case (even for $D = \dim$). However, it can easily be done if $A_0 = \{\mathrm{pt}\}$.

5.13. REMARK. The sequences

$$\{D^{\geq n} \cap 2^{\mathbf{Q}}_0\}_{n=1}^{\infty} \quad \text{and} \quad \{D^{\geq n} \cap C_0(\mathbf{Q})\}_{n=2}^{\infty}$$

are $\mathscr{A}$-absorbing in $2^{\mathbf{Q}}_0$ and $C_0(\mathbf{Q})$, respectively. □

6. Absorbing $\mathscr{F}_{\sigma\delta}$-sequences related to uniform dimension. Let D be a true dimension function having an enhanced Bockstein basis. Consider the following graded sequences related to D:

$$\overline{D}^* = \{\overline{D}^{\geq n}\}_{n=1}^{\infty} \quad \text{and} \quad \overline{D}^*|C(\mathbf{Q}) = \{\overline{D}^{\geq n} \cap C(\mathbf{Q})\}_{n=2}^{\infty},$$

where $\overline{D}^{\geq n} = \overline{D}^{\geq n}(\mathbf{Q})$ (see 3.5 for the definition of $\overline{D}^{\geq n}(\mathbf{Q})$). The sequences $\overline{D}^*$ and $\overline{D}^*|C(\mathbf{Q})$ will be considered in $2^{\mathbf{Q}}$ and in $C(\mathbf{Q})$, respectively. By 3.5, the sequences $\overline{D}^*$ and $\overline{D}^*|C(\mathbf{Q})$ are $\mathscr{F}_{\sigma\delta}$-sequences. In [DvMM] it was shown that $\overline{\dim}^*$ is an $\mathscr{F}_{\sigma\delta}$-absorbing sequence. We extend this result by proving

6.1. THEOREM. *For any true dimension function D having an enhanced Bockstein basis, the sequences $\overline{D}^*$ and $\overline{D}^*|C(\mathbf{Q})$ are $\mathscr{F}_{\sigma\delta}$-absorbing sequences in $2^{\mathbf{Q}}$ and $C(\mathbf{Q})$, respectively.*

Before we give a proof of this theorem, let us provide a consequence of it, 4.2 and 3.5.

6.2. COROLLARY. *For $n = 1, 2, \dots, \infty$, the spaces $\overline{D}^{\geq n}$ and $\overline{D}^{\geq n+1} \cap C(\mathbf{Q})$ are homeomorphic to B^∞ ($\overline{D}^{\geq\infty} = \bigcap_n \overline{D}^{\geq n}$). This is true if $D = \dim$ or $D = \dim_G$ for any abelian group G.* □

The proof of the following auxiliary fact is implicitly contained in [**DvMM**, Lemma 5.1].

6.3. LEMMA. *Suppose $\{Y_k\}_{k=1}^\infty$ is $\mathscr{A}$-universal in $\mathbf{Q}$. Form the sequence $X = \{X_k\}_{k=1}^\infty$, where*

$$X_k = \{(x_n) \in \mathbf{Q}^\infty | \, x_n \in Y_k \textit{ for infinitely many } n\}.$$

Then X is $\mathscr{F}_{\sigma\delta}$-universal in $\mathbf{Q}^\infty$.

Proof (*cf.* [**DvMM**]). Pick an $F_{\sigma\delta}$ graded sequence $\{A_k\}_{k=1}^\infty$ in $\mathbf{Q}$. Since each A_k is an $F_{\sigma\delta}$-subset of $\mathbf{Q}$, it can be expressed $A_k = \bigcap_{n=1}^\infty A_k^n$, where A_k^n are σ-compacta. We can assume that $\{A_k^n\}$ decreases with n and k. Since $\{Y_k\}_{k=1}^\infty$ is $\mathscr{A}$-universal, there is $v_n \colon \mathbf{Q} \to \mathbf{Q}$ with $v_n^{-1}(Y_k) = A_k^n$ for $k \geq 1$. Set $v = (v_n) \colon \mathbf{Q} \to \mathbf{Q}^\infty$. If $x \in A_k$, then $x \in A_k^n$ for every n and $v_n(x) \in Y_k$ for every n; hence $v_n(x) \in X_k$. If $x \notin A_k$, then $x \notin A_k^{n_0}$ for some n_0; consequently $x \notin A_k^n$ for $n \geq n_0$. This implies that $v_n(x) \notin Y_k$ for $n \geq n_0$; hence $v(x) \notin X_k$. □

6.4. PROPOSITION. *Consider the $\mathscr{A}$-universal sequence*

$$\{D^{\geq k} \cap C_0(\mathbf{Q})\}_{k=2}^\infty$$

in $C_0(\mathbf{Q})$ and form

$$X_k = \{(A_n) \in (C_0(\mathbf{Q}))^\infty \, | A_n \in D^{\geq k} \textit{ for infinitely many } n\}.$$

Then the sequence $X = \{X_k\}_{k=2}^\infty$ embeds into $\{\overline{D}^{\geq k} \cap C_0(\mathbf{Q})\}_{k=2}^\infty$, meaning that there is an embedding $\alpha \colon (C_0(\mathbf{Q}))^\infty \to C_0(\mathbf{Q})$ such that

(1) *$\alpha^{-1}(\overline{D}^{\geq k} \cap C_0(\mathbf{Q})) = X_k$ for $k \geq 2$.*

We will modify the proof of [**DvMM**, Proposition 5.2].

Proof. Represent $\mathbf{Q} = (I \times \mathbf{Q}')^\infty$, where $\mathbf{Q}'$ is a Hilbert cube subfactor of $\mathbf{Q}$. We find α in the form $\alpha(P) = \lim \widehat{A}_n(P)$, where $P = (P_i) \in (C_0(\mathbf{Q}))^\infty$ and $P \to \widehat{A}_n(P)$ is a sequence of maps that converges uniformly to α.

For convenience we shall treat $I \times \mathbf{Q}' \subset (I \times \mathbf{Q}') \times (I \times \mathbf{Q}') \subset \cdots \subset (I \times \mathbf{Q}')^n \subset \cdots$ as subsets of $(I \times \mathbf{Q}')^\infty$ with all final coordinates equal zero. The idea behind our construction is as follows. For each n and P, $\widehat{A}_n(P)$ will be a continuum in $(I \times \mathbf{Q}')^n$ containing 0. We shall identify in $\widehat{A}_n(P)$ a countable subset $A_n(P)$. The continuum $\widehat{A}_{n+1}(P) \subset (I \times \mathbf{Q}')^{n+1}$ will be obtained as the union of $\widehat{A}_n(P)$ and two sets R and S. The set R will be a countable, pairwise disjoint union of segments, each of which intersects $\widehat{A}_n(P)$ in exactly one point of $A_n(P)$. The set S will be a countable, pairwise disjoint union of continua, each of which intersects $\widehat{A}_n(P) \cup R$ in exactly one point of R. Certainly such a set must be connected, and we shall design the construction so that $\widehat{A}_{n+1}(P)$ is compact.

Define $G = \{0\} \cup \{2^{-m} | m \geq 1\}$. Let $\widehat{A}_1(P) = [0, \frac{1}{2}] \times \{0\}$, $A_1(P) = (G \backslash \{0\}) \times \{0\} \subset I \times \mathbf{Q}'$. Put

$$F_g(P) = \begin{cases} gP_m, & \text{if } g = 2^{-m}, \\ \{0\}, & \text{if } g = 0. \end{cases}$$

Apply 5.7, to obtain homotopies $(\psi_t)\colon C_0(\mathbf{Q}) \to 2_0^{\mathbf{Q}'}$ and $(\varphi_t)\colon C_0(\mathbf{Q}) \to C_0(\mathbf{Q}')$, $0 \leq t \leq 1$, so that $\psi_0 = \varphi_0 = \mathrm{id}$, $\psi_t(A) \in \mathscr{F}(\mathbf{Q}')$, $t > 0$, and $\psi(A) \subset \varphi(A)$, $A \in C_0(\mathbf{Q})$.

Assume inductively that $\widehat{A}_n(P)$ and $A_n(P)$ subsets of $(I \times \mathbf{Q}')^{n-1} \times (I \times \{0\})$ have been defined for $n \geq 1$ so that $P \to \widehat{A}_n(P)$ is continuous, $\widehat{A}_n(P)$ is a continuum containing 0 and $A_n(P) \subset \widehat{A}_n(P) \backslash (I \times \mathbf{Q}')^{n-1} \times \{(0, 0)\}$ is countable. Assume further that

(2) $A_n(P) \cap (I \times \mathbf{Q}')^{n-1} \times ([\epsilon, 1] \times \{0\})$ is finite for all $\epsilon > 0$.

We let $\widehat{A}_{n+1}(P) = \widehat{A}_n(P) \cup R \cup S$, where

$$R = \bigcup \left\{ \{x\} \times \{(a, 0)\} \times \left(\left[0, \frac{a}{2}\right] \times \{0\} \right) | (x, a, 0) \in A_n(P) \right\},$$

and

$$S = \bigcup \{ \{x\} \times (\{a\} \times \varphi_{ab}(F_a(P))) \times \{(ab, 0)\} | (x, a, 0) \in A_n(P), b \in G \}.$$

We need to show that $\widehat{A}_{n+1}(P)$ is compact. First, $\widehat{A}_n(P) \cup R$ is compact by an application of (2) and the fact that $\widehat{A}_n(P)$ is compact.

Second, suppose $(z_i) = ((x_i, a_i, v_i, a_i b_i, 0))$ is a sequence in S converging to $z \in (I \times \mathbf{Q}')^{n+1}$. The proof that $z \in \widehat{A}_{n+1}(P)$ comes down to checking three cases, $b_i = 0$ for all i; $b_i \geq \epsilon > 0$ for all i and some ϵ; (b_i) is a positive sequence converging to 0. We leave the details to the reader.

Let

$$A_{n+1}(P) = \bigcup \{\{x\} \times (\{a\} \times \psi_{ab}(F_a(P))) \\ \times \{(ab, 0)\} \mid (x, a, 0) \in A_n(P),\, b \in G \backslash \{0\}\}.$$

To prove (2) for $A_{n+1}(P)$, note that if $ab \geq \epsilon > 0$, then a, b are both bounded away from 0. Hence a, b occur only finitely many times in the expression for $A_{n+1}(P)$. Also, because of (2), the elements $(x, a, 0) \in A_n(P)$ occur only finitely many times in this expression. Since each $\psi_{ab}(F_a(P))$ is finite, then (2) is true for $A_{n+1}(P)$.

Using a suitable metric on $\mathbf{Q}$ we could achieve that

$$\sum_{n=1}^{\infty} d(\widehat{A}_n, \widehat{A}_{n+1}) < \infty.$$

Then $P \to \widehat{A}_n(P)$ is a Cauchy sequence of maps. Moreover, $\alpha(P) \in C_0(\mathbf{Q})$ has the following form:

$$\alpha(P) = \operatorname{cl}\left(\bigcup_{n=1}^{\infty} \widehat{A}_n(P)\right).$$

To show that α is an embedding we shall demonstrate that α is injective. If $P \neq P'$, then there exists $g \in G$ such that $F_g(P) \neq F_g(P')$. In the part of $\widehat{A}_2(P)$ designated S, set $b = 0$ and $a = g$. Then $\alpha(P) \cap (\{g\} \times \mathbf{Q}') \times \{(0, 0)\} = \{g\} \times F_g(P) \times \{(0, 0)\}$. Similarly, $\alpha(P') \cap (\{g\} \times \mathbf{Q}') \times \{(0, 0)\} = \{g\} \times F_g(P') \times \{(0, 0)\}$.

To show (1), let $P = (P_i) \in X_k$, i.e., $D(P_i) \geq k$ for infinitely many i. As in the proof of 5.2 of **[DvMM]**, one checks that $D(\widehat{A}_{n+1}(P)) \geq k$ at every point $p \in \widehat{A}_n(P)$ of the form $(x, 0, 0) \in (I \times \mathbf{Q}')^{n-1} \times I \times \{0\}$, i.e., every neighborhood of p in $\widehat{A}_{n+1}(P)$ has D-dimension $\geq k$. Since each $p \in \widehat{A}_n(P)$ has the form $(x, 0, 0) \in (I \times \mathbf{Q}')^n \times I \times \{0\} \subset \widehat{A}_{n+1}(P)$, then this implies that each point of $\widehat{A}_n(P)$ has in $\widehat{A}_{n+2}(P)$ arbitrarily small neighborhoods with D-dimension $\geq k$. Since $\bigcup_{n=1}^{\infty} \widehat{A}_n(P)$ is dense in $\alpha(P)$, we conclude that $\alpha(P) \in \overline{D^{\geq k}} \cap C_0(\mathbf{Q})$.

Now, assume $P \notin X_k$, i.e., there exists m such that $D(P_i) < k$ for all $i \geq m$. Then we have $D(F_g(P)) < k$ for $g \leq 2^{-m}$. We have that $D(\widehat{A}_n(P) \cap C) < k$ for every n, where $C = \{(q_i, u_i) \in (I \times \mathbf{Q}')^\infty | q_1 \leq 2^{-m}\}$. To show this inductively, observe that $D(R) = 1$ and $D(S \cap C) < k$ by an application of 2.1.

We claim that $D(\alpha(P) \cap C) < k$. To verify this, note that

$$\alpha(P) \cap C = \bigcap_{n=1}^{\infty} \pi_n^{-1}(\widehat{A}_n(P) \cap C),$$

and $\pi_n(\widehat{A}_{n+1}(P) \cap C) = \widehat{A}_n(P) \cap C$, where π_n is the projection of $\mathbf{Q}$ onto $(I \times \mathbf{Q}')^{n-1} \times (I \times \{0\})$. Moreover, we have

$$\widehat{A}_1(P) \cap C \xleftarrow{\pi_1} \widehat{A}_2(P) \cap C \xleftarrow{\pi_2} \widehat{A}_3(P) \cap C \longleftarrow \cdots.$$

Assign to every $x \in \alpha(P) \cap C$ an element $t = (\pi_n(x)) \in \prod_{n=1}^{\infty} \widehat{A}_n(P)$. Clearly, we have $t \in \varprojlim(\widehat{A}_n(P), \pi_n)$ and the map $x \to t$ is a homeomorphism of $\alpha(P) \cap C$ onto $A = \varprojlim(\widehat{A}_n(P), \pi_n)$. Since $D(\widehat{A}_n(P) \cap C) < k$ for every n, by 2.2, $D(A) = D(\alpha(P) \cap C) < k$.

This argument shows that if $P \notin X_k$ then $\alpha(P) \notin \overline{D}^{\geq k} \cap C_0(\mathbf{Q})$. The proof of (1) is complete. □

Proof of 6.1. We will verify the conditions (1)–(3) of §4. To get (1) apply Theorem 3.5. Use the inclusions, $\overline{D}^{\geq 1} \subset D^{\geq 1}$ and $\overline{D}^{\geq 2} \cap C(\mathbf{Q}) \subset D^{\geq 2} \cap C(\mathbf{Q})$, and 5.3 to obtain (2). The verification of (3) will be done in the two stages (i) and (ii) of §4. As previously we take E_0 to be $2_0^{\mathbf{Q}}$ when dealing with the sequence $\overline{D}^*$ and $C_0(\mathbf{Q})$ when dealing with $\overline{D}^*|C(\mathbf{Q})$. Proposition 6.4 together with Lemma 6.3 show that $\{\overline{D}^{\geq n} \cap C_0(\mathbf{Q})\}_{n=2}^{\infty}$ is $\mathscr{F}_{\sigma\delta}$-universal in $C_0(\mathbf{Q})$, yielding (i). The argument of 6.4 can be adjusted to get the $\mathscr{F}_{\sigma\delta}$-universality of $\{\overline{D}^{\geq n} \cap 2_0^{\mathbf{Q}}\}_{n=1}^{\infty}$ in $2_0^{\mathbf{Q}}$. The proof then will be a repetition of the proof of Proposition 5.2 in **[DvMM]**, except one should use our inverse limit reasoning from the proof of 6.4 to conclude that $D(\alpha(P) \cap C) < k$. The fact that $\overline{D}^*$ is strongly $(2_0^{\mathbf{Q}}, \{\overline{D}^{\geq n} \cap 2_0^{\mathbf{Q}}\}_{n=1}^{\infty})$-universal in $2^{\mathbf{Q}}$, i.e., condition (ii) of §4, follows from the proof of 5.9 and 5.5 (see the comments in parentheses therein).

In what follows, we describe an adjustment of the proof of 5.5 to secure the strong $(C_0(\mathbf{Q}), \{\overline{D}^{\geq n} \cap C_0(\mathbf{Q})\}_{n=2}^{\infty})$-universality of $\overline{D}^*|C(\mathbf{Q})$ in $C(\mathbf{Q})$. Let $N_0, N_1, \ldots$ be pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\{1, 2, \ldots, n+1\} \cap N_n = \varnothing$, $n \geq 1$, and let χ be as in Lemma 5.8. Then, by (b) of 5.8, $p_k \chi_t(A) = \{0\}$ for every $k \in N_0$. Find a homotopy $\pi: \mathbf{Q} \times (0, 1] \to \mathbf{Q}$ such that if $q \in \mathbf{Q}$ and

$q_k = 0$, then $p_k\pi_t(q) = 0$ for all t and such that if $\frac{1}{n+1} < t \leq \frac{1}{n}$, then $p_k(\pi_t(x)) = 0$ for $k \in \mathbb{N}\backslash(N_0 \cup \{1, \ldots, n+1\})$ and $p_k(\pi_t(x)) = p_k(x)$ for $k \in P_n$, where $P_n = \{1, \ldots, n\} \cup (\{n+4, n+5, \ldots\} \cap N_0)$ (consider suitable projections and connect them using segment homotopies). Since $\hat{d}$ has the property that whenever q and q' agree on the first n-coordinates, then $d(q, q') \leq \frac{1}{n+1}$, we have $d(\pi_t(x), x)) < t$. Let φ be as in 5.6. Define for $t > 0$, $\overline{\varphi}(A, t) = \pi_t(\overline{B}(\varphi(A, t), t))$.

By the construction it is clear that $\overline{\varphi}(A, t)$ is a continuum. One can see that $\varphi(A, t) \subset \overline{\varphi}(A, t)$. For if $\frac{1}{n+1} < t \leq \frac{1}{n}$ and $z \in \varphi(A, t)$, then by 5.6(d), $z = (z_1, \ldots, z_n, 0, 0, \ldots)$. Hence $p_k\pi_t(z) = z_k$, $1 \leq k \leq n$, and $p_k\pi_t(z) = 0$, $k > n$. Thus $z = \pi_t(z) \in \pi_t(\varphi(A, t)) \subset \pi_t(\overline{B}(\varphi(A, t), t)) = \overline{\varphi}(A, t)$. We claim that for such z, $D(\pi_t(B(z, t)))$ $= \infty$ uniformly. For if $w = (w_k) \in B(z, t)$, then π_t is injective on $\{q = (q_i) \in \mathbf{Q} \mid q_k = w_k \text{ for } k \in \mathbb{N}\backslash P_n\}$. Consequently every neighborhood of w in $B(z, t)$ contains a copy of $\mathbf{Q}$ on which π_t is injective; this yields $D(\pi_t(B(z, t))) = \infty$ uniformly. Since $\pi_t(B(\varphi(A, t), t))$ is dense in $\overline{\varphi}(A, t)$ we conclude that also $D(\overline{\varphi}(A, t)) = \infty$ uniformly. Now replace φ by $\overline{\varphi}$ in the formula describing G and $12\epsilon(F(A))$ by $14\epsilon(F(A)))$ in the proof of 5.5. Since $d(\overline{\varphi}(A, t), \varphi(A, t)) \leq 2t$ the estimates of the proof of 5.5 work. The fact that $D(\overline{\varphi}(A, t)) = \infty$ uniformly assures that whenever A is of $D(A) \geq n$ uniformly, then so is $G(A)$ (see the comment in parentheses in the proof of 5.5). Our proof is now complete. □

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