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**CORRECTION TO: "TRACE RINGS FOR VERBALLY PRIME
ALGEBRAS"**

ALLAN BERELE

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In [1] and [2] we incorrectly state a theorem of Razmyslov from [3]. We quoted Razmyslov as saying:

For all k and l , $M_{k,l}$ satisfies a trace identity of the form

$$(*) \quad p(x_1, \dots, x_n, a) = c(x_1, \dots, x_n)\text{tr}(a)$$

where $p(x_1, \dots, x_n, a)$ and $c(x_1, \dots, x_n)$ are central polynomials.

This statement is true if $k \neq l$ and false if $k = l$. We will indicate why this is true and what effect it has on the results of [1] and [2]. It turns out that [1] needs only a very minor comment, but that [2] requires a modification to the main theorem and a longer proof in the case of $k = l$.

First, here is a correct version of Razmyslov's theorem:

For all k and l , $M_{k,l}$ satisfies a trace identity of the form

$$(**) \quad p(x_1, \dots, x_n, a) = \text{tr}(c'(x_1, \dots, x_n))\text{tr}(a)$$

where $p(x_1, \dots, x_n, a)$ is a central polynomial and $c'(x_1, \dots, x_n)$ does not involve any traces.

If $k \neq l$, then the trace of the identity matrix equals $k - l$ which is not zero. So, if we set $a = I$ in (**) we get

$$\text{tr}(c'(x_1, \dots, x_n)) = (k - l)^{-1}p(x_1, \dots, x_n, I).$$

Hence, in this case $\text{tr}(c'(x_1, \dots, x_n))$ equals a central polynomial modulo the identities for $M_{k,l}$, and so (*) is true in this case. To see that (*) is false if $k = l$ it is useful to have the following lemma.

LEMMA 1. *Let $f(x_1, \dots, x_n)$ be a pure trace identity for $M_{k,k}$ and write $f(x_1, \dots, x_n) = f_0(x_1, \dots, x_n) + f_1(x_1, \dots, x_n)$, where each monomial in f_0 involves an even number of traces and each monomial in f_1 involves an odd number of traces. Then $f_0(x_1, \dots, x_n)$ and $f_1(x_1, \dots, x_n)$ are each trace identities for $M_{k,k}$.*

Proof. We define an automorphism on $M_{k,k}$. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of $M_{k,k}$, where A, B, C and D are $k \times k$ blocks, and

define $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^*$ to be the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$. Then $-^*$ is an automorphism and $\text{tr}(x^*) = -\text{tr}(x)$ for any matrix x . Hence $M_{k,k}$ satisfies the trace identity $f(x_1^*, \dots, x_n^*) = f_0(x_1^*, \dots, x_n^*) + f_1(x_1^*, \dots, x_n^*) = f_0(x_1, \dots, x_n) - f_1(x_1, \dots, x_n)$. The lemma follows.

COROLLARY. $M_{k,k}$ does not satisfy $(*)$.

Proof. Multiply $(*)$ by a new variable x_{n+1} and take trace. The left-hand side becomes a product of two traces which is not an identity, and the right-hand side becomes a product of three traces, contradicting Lemma 1.

To fix up the proof in [1] in the case $k = l$ all that is required is this simple remark: Let x_1, \dots, x_n and y_1, \dots, y_n be $2n$ variables. Then $M_{k,k}$ satisfies the identity

$$(1) \quad \begin{aligned} \text{tr}(c'(x_1, \dots, x_n))\text{tr}(c'(y_1, \dots, y_n)) \\ = p(x_1, \dots, x_n, c'(y_1, \dots, y_n)). \end{aligned}$$

Hence,

$$c(x_1, \dots, x_n)c(y_1, \dots, y_n) = \text{tr}(c'(x_1, \dots, x_n))\text{tr}(c'(y_1, \dots, y_n))$$

is a central polynomial for $M_{k,l}$ even if $k \neq l$. This is all that [1] requires. (We will now resume using the shorthand notation from [2] and we will write $p(x, a)$, $c(x)$, $c'(x)$, $p(y, a)$, etc.)

DEFINITION. Let R be any ring and let J^2 be the ideal of R generated by all evaluations of $p(x_1, \dots, x_n, c'(y_1, \dots, y_n))$ on R .

We remark for future reference this easy consequence of (1): $M_{k,k}$ satisfies the identity

$$(2) \quad p(x, c'(y)) = p(y, c'(x)).$$

Hence we may denote it as $c(x)c(y)$ to emphasize its symmetric nature.

Here is the main result:

THEOREM 3. Assume that R is p.i. equivalent to some $M_{k,k}$ and that the annihilator of J^2 is (0) . Then there is an embedding of R into a $\mathbf{Z}/2\mathbf{Z}$ -graded ring with trace $\bar{R} = R_0 + R_1$, such that $R \subset R_0$, $\text{tr}(R_0) \subset R_1$ and $\text{tr}(R_1) = (0)$; such that \bar{R} is generated by R and $\text{tr}(R)$; and such that

(a) the trace on \bar{R} is a non-degenerate,

- (b) there is a faithful R -submodule of R_1 , J such that for all homogeneous r in \overline{R} there exists an integer n such that $J^n r \subset R$, and
 (c) \overline{R} satisfies the same trace identities as $M_{k,k}$.

Proof. The construction of \overline{R} will be in two parts, first R_0 and then R_1 . Much of the construction will be very similar to [2] and so we will omit a number of details.

For any $a, b \in R$ we construct an R -map $t(a, b): J^2 \rightarrow R$ via $t(a, b)(c(x)c(y)r) = p(x, a)p(y, b)r$. The reader should think of $t(a, b)$ as $\text{tr}(a)\text{tr}(b)$. The proof that $t(a, b)$ is well-defined is similar to the corresponding proof in [2]. We note that $t(a, b)$ is symmetric, bilinear and vanishes if either argument is a commutator. Here are a few of its other properties:

$$(3) \quad \text{if } \sum_i r_i t(a_i, b_i) = 0, \quad \text{then for all } s, \quad \sum_i r_i s t(a_i, b_i) = 0,$$

$$(4) \quad t(a, b)t(c, d) = t(c, b)t(a, d),$$

$$(5) \quad t(t(a, b), c) = 0.$$

Finally, as in [2], R_0 can be constructed as the subring of $\varprojlim \text{hom}_R((J^2)^n, R)$ generated by R and all $t(a, b)$. Note that t extends to a map from $R_0 \times R_0$ to its center.

To define R_1 we start with the free R_0 -module on the symbols $\text{tr}(a)$, $a \in R$ and then mod out by the relation (&)

$$\text{if } \sum_i \alpha_i t(a_i, b) = 0 \quad \text{for all } b \in R \quad \text{then } \sum_i \alpha_i \text{tr}(a_i) = 0,$$

where the α_i are in R_0 and the a_i are in R .

This relation has a number of implications for tr . Regarded as a map from R_0 to R_1 it is linear over the center of R_0 and it vanishes on commutators. Equations (3)–(5) all have counterparts for tr :

$$(3') \quad \text{if } \sum_i \alpha_i \text{tr}(a_i) = 0 \quad \text{then for all } s, \quad \sum_i \alpha_i s \text{tr}(a_i) = 0,$$

$$(4') \quad t(a, b)\text{tr}(c) = t(c, b)\text{tr}(a),$$

$$(5') \quad \text{tr}(t(a, b)) = 0.$$

It follows from (3') that we may define a bimodule structure on R_1 via $(\sum_i \alpha_i \text{tr}(a_i))s = \sum_i \alpha_i s \text{tr}(a_i)$. Then we define a bilinear pairing $R_1 \times R_1 \rightarrow R_0$ via $(\alpha \text{tr}(a))(\beta \text{tr}(b)) = \alpha\beta t(a, b)$. Using (&) it is straightforward to show that this pairing is well-defined. Finally, we

construct a multiplicative structure on $\bar{R} = R_0 + R_1$ via

$$\begin{aligned} & \left(a + \sum_i b_i \operatorname{tr}(c_i) \right) \left(d + \sum_j e_j \operatorname{tr}(f_j) \right) \\ &= \left(ad + \sum_{i,j} b_i e_j t(c_i, f_j) \right) + \left(\sum_j a e_j \operatorname{tr}(f_j) + \sum_i b_i d \operatorname{tr}(c_i) \right). \end{aligned}$$

That it is associative follows from (4'). We now prove that \bar{R} has the properties (a), (b) and (c) that we claimed in the statement of the theorem.

It is useful at this point to prove that \bar{R} satisfies the identity (**), namely

$$(**) \quad p(x, a) = \operatorname{tr}(c'(x))\operatorname{tr}(a).$$

In order to prove this it suffices to take x and a in R . Consider $\operatorname{tr}(c'(x))\operatorname{tr}(a) = t(c'(x), a)$ as a map from J^2 to R . This map takes $c(y)c(z)$ to

$$\begin{aligned} p(y, c'(x))p(z, a) &= \quad (\text{by (2)}) \\ p(x, c'(y))p(z, a) &= \quad (\text{by (2) of [2]}) \\ p(x, a)p(z, c'(y)) &= \end{aligned}$$

$p(x, a)$ times $c(y)c(z)$. This proves (**).

Let $J = R \operatorname{tr}(c'(R^n)) \subset R_1$. Note that the square of J equals the ideal of R we denoted J^2 by (**), and so $\operatorname{ann}(J) = (0)$. Continuing the proof of (b), let $r \in R_0$. It follows from the construction of R_0 that $(J^2)^n r = J^{2n} r$ is contained in R , for some n . And, if $r \in R_1$ then we may assume without loss of generality that $r = \alpha \operatorname{tr}(a)$ for some $\alpha \in R_0, a \in R$. But then, $J^{2n} \alpha \subset R$ for some n as above, and $J \operatorname{tr}(\alpha) \subset R$ by (**). Hence $J^{2n+1} r \subset R$.

The proof of (c) follows from (b) as in [2]. Let $f(x) = f(x_1, \dots, x_m)$ be a trace polynomial in which either term has an even number of traces or each term has an odd number of traces. Then it follows from (b) that $M_{k,k}$ and R satisfy an identity of the form $j(y)f(x) = g(x, y)$, where x and y are disjoint sets of variables and $g(x, y)$ doesn't involve any traces. Since $M_{k,k}$ is verbally prime, $f(x)$ is a trace identity for $M_{k,k}$ if and only if $g(x, y)$ is a p.i. for $M_{k,k}$. Moreover, since \bar{R} is a central extension of R , they satisfy the same p.i.'s. Hence, if $f(x)$ is a trace identity for \bar{R} then $p(x, y)$ will be an identity for R and so for $M_{k,k}$, and so $f(x)$ will also be an identity

for $M_{k,k}$. Conversely, if $f(x)$ is a trace identity for $M_{k,k}$, then it follows that $j(y)f(x)$ is a trace identity for \bar{R} . But this implies that the evaluations of $f(x)$ would annihilate some power of J and so $f(x)$ is forced to be an identity.

The proof of (a) is also similar to the corresponding proof in [2] and we omit it.

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