

*Pacific  
Journal of  
Mathematics*

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# THE BOUNDARY DISTORTION OF A QUASICONFORMAL MAPPING

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**We investigate how quasiconformal mappings  $f: \mathbf{B}^n \rightarrow \mathbf{R}^n$  distort Hausdorff measures of sets on the boundary  $\partial\mathbf{B}^n$ .**

**1. Introduction.** Recent exciting developments in the study of the boundary behavior of conformal mappings, orchestrated by Makarov ([M1], [M2], [M3]), lead to the natural question: what is the boundary distortion of a quasiconformal mapping  $f$  of the unit ball  $\mathbf{B}^n$  into  $\mathbf{R}^n$ ? Makarov's results assert that for any conformal mapping of the unit disk in the plane there is a set of full measure on the boundary whose image has Hausdorff dimension precisely one. There is no hope for such results for general quasiconformal mappings as shown by well-known examples. In fact, given  $0 < \alpha < 1$  there is a quasiconformal self-homeomorphism of the unit disk carrying a set of full measure on the circle into a set of Hausdorff dimension  $\alpha$  (see [Ro], [T2]). In the other direction, given  $1 < \alpha < 2$  it is not difficult to construct a quasiconformal mapping of the disk onto a Jordan domain such that the image of any boundary set of positive measure has Hausdorff dimension  $\alpha$ .

In this article we discuss various questions related to the boundary distortion of quasiconformal mappings; in particular, we demonstrate that in spite of the discouraging counterexamples at least some features of the restricted expansion/contraction phenomenon are retained. Naturally, our results lag behind the deep information available in the case of conformal mappings, but we feel that some of the techniques used in this paper are of interest in the higher dimensional quasiconformal theory. It is also our hope that the modest beginning here will inspire future research in this area. Several open questions are listed at the end of the paper.

**2. Main results.** In this section we describe our main results. The proofs will follow in subsequent sections.

**2.1. Notation.** We let  $B(x, r)$  stand for the open  $n$ -ball centered at  $x$  with radius  $r$ , and we assume  $n \geq 2$ . For short,  $B_r = B(0, r)$

and  $\mathbf{B}^n = B(0, 1)$ . The boundary, the complement, and the closure of a set  $E \subset \mathbf{R}^n$  are denoted by  $\partial E$ ,  $\mathbf{C}E$ , and  $\overline{E}$ , respectively. We write  $\mathcal{H}_s$  for the  $s$ -dimensional Hausdorff measure and usually, for simplicity, we write  $\mathcal{H}_{n-1}(E) = |E|$ . Also,  $\dim_H E$  denotes the Hausdorff dimension of a set  $E$ . We often let  $C$  denote a generic positive constant which is not necessarily the same at each occurrence. Moreover, sometimes we write  $A \approx B$  if there is  $C$  such that  $C^{-1}A \leq B \leq CA$ .

The conformal ( $n$ )-modulus of a path family  $\Gamma$  will be denoted by  $\text{mod } \Gamma$ . Often  $\Gamma = \Delta(E, F; A)$ , the family of all paths joining two sets  $E$  and  $F$  in  $A$ . For the basic properties of the modulus, see [V1].

The *outer dilatation*  $K_O(f)$  of a homeomorphism  $f: D \rightarrow D'$  is defined by

$$K_O(f) = \sup \frac{\text{mod } \Gamma}{\text{mod } f\Gamma},$$

where the supremum is over all path families  $\Gamma$  in  $D$  such that  $\text{mod } f\Gamma \neq 0$ . Then the *inner dilatation* is  $K_I(f) = K_O(f^{-1})$  and  $f$  is said to be  $K$ -quasiconformal if

$$K(f) = \max\{K_O(f), K_I(f)\} \leq K < \infty.$$

We recall the basic relations

$$K_O(f) \leq K_I(f)^{n-1}, \quad K_I(f) \leq K_O(f)^{n-1}.$$

If  $f$  is a quasiconformal mapping of  $\mathbf{B}^n$  into  $\mathbf{R}^n$ , then an analog of Beurling's theorem ensures that  $f$  has angular limits at each point on the boundary  $\partial\mathbf{B}^n$  except possibly in a set  $E_f$  of zero  $n$ -capacity. In particular,  $E_f$  has Hausdorff dimension zero (see, for instance, [N, Corollary 7.14], [Vu, Chapter IV]). In the following we understand that  $f$  is defined on  $\partial\mathbf{B}^n$  via its radial extension; it is immaterial what the value of  $f$  is in the exceptional set  $E_f$ .

The standard reference to the basic theory of quasiconformal mappings is [V1].

2.2. *Contraction.* Our first three results concern the boundary compression of quasiconformal mappings.

**THEOREM A.** *Suppose that  $f$  is a quasiconformal mapping of  $\mathbf{B}^n$  into  $\mathbf{R}^n$ . Then for any set  $E \subset \partial\mathbf{B}^n$  with positive  $\mathcal{H}_{n-1}$ -measure, the Hausdorff dimension of  $f(E)$  is at least*

$$(n-1)(2K_O(f))^{1/(1-n)}.$$

If  $n = 2$  and  $f$  is conformal, we recover the classical Beurling estimate for the support of harmonic measure. In fact, the idea of the proof for Theorem A goes back to Beurling. The lower bound in Theorem A is undoubtedly not sharp. It is perhaps too optimistic to hope for the bound  $(n - 1)(K_O(f))^{1/(1-n)}$ , which would give Makarov's result as a special case, but a reasonable conjecture is  $(n - 1)(2K_O(f))^{1/(1-n)} + \varepsilon$  for some  $\varepsilon = \varepsilon(n, K) > 0$ . This would correspond to Carleson's result [C] for conformal mappings. Although Carleson's ingenious method morally works for quasiconformal mappings, it is not clear to us how to transfer the final steps in his argument into  $n$ -space. If we place an additional constraint on the target domain, an improvement to Theorem A is immediate.

Recall that a domain  $D \subset \mathbf{R}^n$  is *b-uniform* if each pair of points  $x, y \in D$  can be joined by an arc  $L$  in  $D$  such that

$$\text{diam } L \leq b|x - y|$$

and

$$\min\{\text{diam } L[x, z], \text{diam } L[y, z]\} \leq b \text{dist}(z, \partial D)$$

whenever  $z \in L$  and  $L[w, z]$  denotes the subarc of  $L$  between  $w$  and  $z$ .

**THEOREM B.** *Suppose that  $f$  is a quasiconformal mapping of  $\mathbf{B}^n$  onto a  $b$ -uniform domain. Then for any set  $E \subset \partial \mathbf{B}^n$  with positive  $\mathcal{H}_{n-1}$ -measure, the Hausdorff dimension of  $f(E)$  is at least*

$$(n - 1)(2K_O(f))^{1/(1-n)} + \varepsilon,$$

where  $\varepsilon > 0$  depends only on  $n, K_O(f)$ , and  $b$ .

Theorem B is a consequence of the following result which extends [NP, Theorem 9].

**THEOREM B'.** *Suppose that  $f$  is a quasiconformal mapping of  $\mathbf{B}^n$  onto a  $b$ -uniform domain  $D$ . Then  $f^{-1}$  is Hölder continuous in  $D$  with exponent  $\alpha = (2K_O(f))^{1/(1-n)} + \varepsilon$  for some  $\varepsilon = \varepsilon(n, K_O(f), b) > 0$ .*

The distortion of general Hausdorff measures is described in the next theorem. We let  $Df$  denote the differential of a quasiconformal mapping  $f$ , which exists almost everywhere in the domain of  $f$ .

**THEOREM C.** *Suppose that  $f$  is a quasiconformal mapping of  $\mathbf{B}^n$  into  $\mathbf{R}^n$  such that  $|Df|^{-r} \in L^1(\mathbf{B}^n)$  for  $r > 0$ . Then for any set  $E \subset \partial\mathbf{B}^n$  with positive  $\mathcal{H}_s$ -measure,  $0 < s \leq n - 1$ , the Hausdorff dimension of  $f(E)$  is at least*

$$\frac{sr}{n - s + r}.$$

Astala and Koskela [AK] have shown that if  $f$  is a  $K$ -quasiconformal mapping of  $\mathbf{B}^n$ , then *always*  $|Df|^{-r} \in L^1(\mathbf{B}^n)$  for some  $r = r(n, K) > 0$ . Thus Theorem C is never vacuous. The exact value of  $r(n, K)$  is not known but for  $n \geq 3$  it satisfies  $r(n, K) \rightarrow \infty$  as  $K \rightarrow 1$  (see [Re, Chapter 4]). For conformal mappings Pommerenke [P1] has obtained the lower bound  $r(2, 1) \geq 1.39$ . Moreover, Carleson and Makarov [CM] have recently established that if  $f$  is univalent in the unit disk  $\mathbf{B}^2$  and  $E \subset \partial\mathbf{B}^2$  with positive  $\mathcal{H}_s$ -measure,  $0 < s < 1$ , then the Hausdorff dimension of  $f(E)$  is at least  $s/(2 - as)$ , where  $0 < a < 2$  is an absolute constant (see also [M2]). By using this result and factorization, we see that

$$\dim_H f(E) \geq \frac{s}{2K(f) - as}$$

whenever  $E \subset \partial\mathbf{B}^2$  is as above and  $f$  is quasiconformal in  $\mathbf{B}^2$ ; note that  $K(f) = K_O(f) = K_I(f)$  in the plane.

We shall show in Lemma 5.1 below that if  $n \neq 4$  and if  $f$  is as in Theorem C, then there is a quasiconformal mapping  $g$  of  $\mathbf{B}^n$  such that  $g = f$  on  $\partial\mathbf{B}^n$  and that  $|Dg|^{-r} \in L^1(\mathbf{B}^n)$  for all

$$0 < r < \frac{1}{(2K_O(f))^{1/(n-1)} - 1};$$

thus for  $n \neq 4$  Theorem A follows from Theorem C.

If  $f$  is a  $K$ -quasiconformal self mapping of  $\mathbf{R}^n$ , then it follows from [G1, Theorem 3] that

$$\frac{\dim_H f(E)}{\dim_H E} \geq \frac{p(n, K) - n}{p(n, K) - \dim_H E}$$

for any set  $E \subset \mathbf{R}^n$ , where  $p(n, K) > n$  is defined to be the supremum of all  $p > n$  such that  $|Df| \in L^p_{\text{loc}}(\mathbf{R}^n)$ . Also, if  $n = 2$  and if  $f$  has a  $K$ -quasiconformal extension to the whole plane, Astala's recent affirmative solution to the area distortion problem [A] guarantees the lower bound  $2(1 + K)^{-1}$  in Theorem A. We emphasize that in Theorems A and C  $f$  need not have to have a continuous extension to  $\partial\mathbf{B}^n$ .

2.3. *Expansion.* Now we turn our attention to the other aspect of the Makarov theory, namely expansion. Makarov’s insight was to show that if the derivative of a conformal mapping  $f$  satisfies

$$\liminf_{r \rightarrow 1} |f'(rw)| = 0$$

for all  $w$  in a set  $E \subset \partial \mathbf{B}^2$ , then  $f$  cannot expand at almost all points in  $E$ . Consequently, since  $f'$  is analytic, Plessner’s theorem implies that the expansion occurs in a subset of zero measure on  $\partial \mathbf{B}^2$ . For a quasiconformal mapping  $f: D \rightarrow D'$  there is, in general, no pointwise defined derivative but the integral average

$$(2.4) \quad a_f(x) = \exp \left\{ \frac{1}{n} \int_{B_x} \log J_f(y) dy \right\}$$

has proved to be a good substitute; here  $B_x = B(x, \text{dist}(x, \partial D)/2)$  (see [AG1], [AG2], [AK], [Ha], [K2]). Take notice that if  $f$  is conformal, then  $\log J_f(x) = 2 \log |f'(x)|$  is harmonic, whence  $a_f(x) = |f'(x)|$ .

Pommerenke [P2] simplified and strengthened Makarov’s argument in [M1] as regards the expansion. We simplify it a bit further and show that more or less the same proof goes through for quasiconformal mappings. Of course Plessner’s theorem is not applicable in this case and we have to be content with the following formulation.

**THEOREM D.** *Suppose that  $f$  is a quasiconformal mapping of  $\mathbf{B}^n$  into  $\mathbf{R}^n$ . Let*

$$E_\infty = \left\{ w \in \partial \mathbf{B}^n : 0 < \liminf_{r \rightarrow 1} a_f(rw) \leq \limsup_{r \rightarrow 1} a_f(rw) = \infty \right\}.$$

Then

$$\partial \mathbf{B}^n = E_0 \cup E_1 \cup E_\infty \quad (\text{disjoint union})$$

where  $|E_0| = 0$  and  $f(E_1)$  has  $\sigma$ -finite  $\mathcal{H}_{n-1}$ -measure. Moreover, if  $E \subset E_1$ , then  $|E| = 0$  implies  $|f(E)| = 0$ .

Pommerenke’s version of Makarov’s theorem follows from Theorem D. Indeed, if  $f$  is conformal, then  $a_f(x) = |f'(x)|$  and Plessner’s theorem implies  $|E_\infty| = 0$ .

**DEFINITION.** We say that a quasiconformal mapping  $f$  of  $\mathbf{B}^n$  into  $\mathbf{R}^n$  has the *Makarov property* if there is a set  $E$  of full measure on the boundary of  $\mathbf{B}^n$  such that  $f(E)$  has  $\sigma$ -finite  $(n - 1)$ -measure.

By Theorem D,  $f$  has the Makarov property if for almost every  $w \in \partial \mathbf{B}^n$  either

$$\liminf_{r \rightarrow 1} a_f(rw) = 0 \quad \text{or} \quad \limsup_{r \rightarrow 1} a_f(rw) < \infty.$$

As mentioned in the introduction, this need not always be the case. For instance, one can construct a quasiconformal mapping  $f$  of  $\mathbf{R}^2$  that carries the unit circle onto the van Koch snowflake such that

$$|f(z) - f(w)| \approx |z - w|^\alpha, \quad \alpha = \frac{\log 3}{\log 4},$$

for all  $z, w \in \partial \mathbf{B}^2$  (see [T1]). It is easy to see that then the image of any set  $E \subset \partial \mathbf{B}^2$  of positive length has Hausdorff dimension  $1/\alpha > 1$ . In fact, in this case  $\lim_{r \rightarrow 1} a_f(rw) = \infty$  for all  $w \in \partial \mathbf{B}^2$ .

Incidentally, we do not know of any examples in dimensions  $n \geq 3$  where  $E_\infty$  has positive  $\mathcal{H}_{n-1}$ -measure. Väisälä [V6] has constructed a quasiconformal mapping of the unit ball  $\mathbf{B}^n$  onto a Jordan domain whose boundary has positive volume; it would be interesting to know the behavior of  $a_f$  for such a mapping. Another interesting open problem is to find conditions (other than  $|E_\infty| = 0$ ) that would guarantee the Makarov property. In dimension  $n = 2$  we have the following sufficient condition which is an immediate consequence of a recent work of Fefferman, Kenig, and Pipher [FKP].

**THEOREM E.** *Suppose that  $f$  is a quasiconformal mapping of  $\mathbf{B}^2$  into  $\mathbf{R}^2$ . Let  $\mu(z)$  denote the complex dilatation of  $f$  for  $z \in \mathbf{B}^2$ . If the function*

$$\frac{|\mu(z)|^2}{1 - |z|}$$

*is a Carleson measure in  $\mathbf{B}^2$ , then there is a set  $E \subset \partial \mathbf{B}^2$  such that  $|E| = 2\pi$  and  $f(E)$  has Hausdorff dimension 1. Moreover,  $f$  has the Makarov property.*

Recall that a measure  $\nu$  in  $\mathbf{B}^2$  is a Carleson measure if there is a constant  $C > 0$  such that  $\nu(B \cap \mathbf{B}^2) \leq C \operatorname{diam} B$  for each disk  $B$  centered at a point on  $\partial \mathbf{B}^2$ .

Theorem E tells us that if the dilatation of a quasiconformal mapping becomes small on the boundary in a certain uniform sense, the boundary distortion is not different from that of a conformal mapping. We do not know to what extent Theorem E is sharp for the class of mappings that become “conformal at the boundary”.

It is clear that a quasiconformal mapping can have the Makarov property without its dilatation tending to zero as we approach the boundary, but it seems difficult to give nontrivial conditions guaranteeing this.

As our final result we state an analog of Theorem E for  $n \geq 3$ . Because 1-quasiconformal mappings in space are Möbius transformations, the following theorem comes as no surprise.

**THEOREM F.** *Suppose that  $n \geq 3$  and that  $f$  is a quasiconformal mapping of  $\mathbf{B}^n$  into  $\mathbf{R}^n$  such that*

$$K(f|(\mathbf{B}^n \setminus \overline{B}_r)) \rightarrow 1,$$

where  $r_j \uparrow 1$ . Then for any set  $E \subset \partial \mathbf{B}^n$  with Hausdorff dimension  $s$ ,  $0 < s \leq n - 1$ , the Hausdorff dimension of  $f(E)$  is  $s$ .

We do not know whether the mapping  $f$  in Theorem F has the Makarov property.

**3. Proof of Theorem A.** Let  $f: \mathbf{B}^n \rightarrow D$  be a  $K$ -quasiconformal mapping; we normalize  $f(0) = 0$  and  $\text{dist}(0, \partial D) = 1$ . Fix a point  $y \in \partial D$  and let  $0 < r < d/2$ , where

$$d = d(n, K) = \text{dist}(f\overline{B}_{1/2}, \partial D).$$

Then for the path family

$$\Gamma' = \Lambda(f\overline{B}_{1/2}, \overline{B}(y, r) \cap \partial D; D)$$

we have the estimate

$$\text{mod } \Gamma' \leq \frac{\omega_{n-1}}{(\log d/r)^{n-1}},$$

and so

$$(3.1) \quad \text{mod } \Gamma \leq \frac{K_O \omega_{n-1}}{(\log d/r)^{n-1}},$$

where  $\Gamma = f^{-1}\Gamma'$  and  $K_O = K_O(f)$ . Let  $E$  denote an arbitrary compact subset of the set of all boundary points  $w \in \partial \mathbf{B}^n$  such that the radial limit  $\lim_{t \rightarrow 1} f(tw) = f(w)$  exists and satisfies  $|f(w) - y| \leq r$ . Let

$$\Gamma_1 = (\overline{B}_{1/2}, E; \mathbf{B}^n).$$

If  $w \in E$  and  $\gamma \in \Gamma_1$ , then by Lindelöf's theorem for quasiconformal mappings either  $f(\gamma)$  is nonrectifiable or  $f$  has the limit  $f(w) \in$



$\overline{B}(y, r)$  along  $\gamma$ . Because the modulus of nonrectifiable paths is zero, we conclude that

$$(3.2) \quad \text{mod } \Gamma_1 \leq \text{mod } \Gamma.$$

By reflecting one can show that

$$(3.3) \quad \text{mod } \Gamma_0 \leq 2 \text{mod } \Gamma_1,$$

where

$$\Gamma_0 = (E, \partial B(0, 2); B(0, 2)),$$

(see e.g. [HM, Lemma 3.8]). Next, if we replace  $E$  by a (possibly degenerate) spherical cap  $E^* \subset \partial \mathbf{B}^n$  such that  $|E| = |E^*|$ , then

$$(3.4) \quad \text{mod } \Gamma_0^* \leq \text{mod } \Gamma_0,$$

where  $\Gamma_0^*$  is the symmetrized family

$$\Gamma_0^* = (E^*, \partial B(0, 2); B(0, 2));$$

see [S]. Let

$$r_1 = \text{diam } E^* \approx |E^*|^{1/(n-1)}.$$

A symmetrization argument (see [HM, (3.14)]) combined with the above inequalities (3.1)–(3.4) yield

$$\begin{aligned} r_1 &\leq c_n \exp\{-\omega_{n-1}^{1/(n-1)} (\text{mod } \Gamma_0^*)^{1/(1-n)}\} \\ &\leq c_n \exp\left\{-\omega_{n-1}^{1/(n-1)} \left(\frac{2K_O \omega_{n-1}}{(\log d/r)^{n-1}}\right)^{1/(1-n)}\right\} \\ &\leq c_n \exp\{-(2K_O)^{1/(1-n)} \log d/r\} \leq C(n, K)r^\alpha, \end{aligned}$$

where  $\alpha = (2K_O)^{1/(1-n)}$ . Finally since  $r_1 \approx |E^*|^{1/(n-1)} = |E|^{1/(n-1)}$  and since  $E$  was arbitrary, we infer that

$$|f^{-1}(\overline{B}(y, r) \cap \partial D)| \leq C(n, K)r^\beta,$$

where  $\beta = (n-1)(2K_O)^{1/(1-n)}$ .

To complete the proof, suppose that  $F \subset \partial D$  is such that  $\mathcal{H}_\beta(F) = 0$ , where  $\beta$  is as above. Fix  $\varepsilon > 0$  and cover  $F$  by balls  $B(y_i, r_i)$ ,  $r_i < d/2$ , such that  $\sum_i r_i^\beta < \varepsilon$ . The preceding argument shows that

$$|f^{-1}(\overline{B}(y_i, r_i) \cap \partial D)| \leq C(n, K)r_i^\beta$$

for each  $i$ ; in particular,

$$|f^{-1}F| \leq \left| \bigcup_i f^{-1}(\overline{B}(y_i, r_i) \cap \partial D) \right| \leq C \sum_i r_i^\beta < C\varepsilon,$$

where  $C = C(n, K)$ . Letting  $\varepsilon \rightarrow 0$  yields  $|f^{-1}F| = 0$ , as desired. The proof of Theorem A is complete.

**4. Proofs of Theorems B and B'.** We begin with a lemma.

4.1. LEMMA. Suppose that  $D \subset \mathbf{R}^n$  is a domain such that  $\partial D = \partial D^*$ , where  $D^* = \mathbf{R}^n \setminus \overline{D}$  is an unbounded  $b$ -uniform domain, and let  $y_0 \in D$ . Then there exists  $\varepsilon = \varepsilon(n, b) > 0$  such that

$$\text{mod } \Gamma_{r, x_0} \leq \frac{(1 - \varepsilon)\omega_{n-1}}{(\log d/r)^{n-1}}$$

for  $0 < r < d = \text{dist}(y_0, \partial D)/2$  and  $x_0 \in \partial D$ , where

$$\Gamma_{r, x_0} = \Delta(\partial B(x_0, r) \cap D, \partial B(x_0, d) \cap D; D).$$

*Proof.* Fix  $x_0 \in \partial D$ ,  $0 < r < d$ , and define

$$\rho(x) = |x - x_0|^{-1}(\log d/r)^{-1}$$

if  $x \in (B(x_0, d) \cap D) \setminus B(x_0, r)$  and  $\rho(x) = 0$  otherwise. Then

$$\text{mod } \Gamma_{r, x_0} \leq \int_{\mathbf{R}^n} \rho^n dx.$$

Because  $D^*$  is  $b$ -uniform, for each  $r < t < d$  there is a point  $z \in \partial B(x_0, t)$  such that  $B(z, t/2b) \subset D^*$ . Hence a simple calculation verifies that

$$\int_{\mathbf{R}^n} \rho^n dx \leq \frac{(1 - \varepsilon)\omega_{n-1}}{(\log d/r)^{n-1}}$$

with  $\varepsilon = \varepsilon(b, n) > 0$  as desired.

To prove Theorem B', we first observe that because  $D$  is uniform,  $f$  extends to a *quasimöbius embedding* of  $\partial \mathbf{B}^n$  into  $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$  by [V4, 5.6], and hence  $D^* = \mathbf{R}^n \setminus \overline{D}$  is an unbounded  $b'$ -uniform domain for some  $b' = b'(n, K, b)$  by [V5, 5.10]. Then we apply the argument of [AK, Theorem 2.3] together with the previous lemma (see also [NP, Theorem 7]) to conclude that

$$(4.2) \quad 1 - |f^{-1}(x)| \leq C \text{dist}(x, \partial D)^\alpha$$

holds for all  $x \in D$ , where  $\alpha$  is as described in the theorem. The reasoning here is so similar to that in [AK], that the details are best left to the reader. Now condition (4.2) guarantees that  $f^{-1}$  is  $\alpha$ -Hölder continuous in  $D$  (see [GM, 2.24, 3.4]), and the theorem follows.

Finally, Theorem B is an immediate consequence of Theorem B', as  $f^{-1}$  extends to an  $\alpha$ -Hölder continuous mapping of  $\partial D$ .

**5. Proof of Theorem C.** We first establish the following lemma and indicate how Theorem A follows from Theorem C for  $n \neq 4$ .

5.1. LEMMA. *Let  $f$  be a quasiconformal mapping from  $\mathbf{B}^n$  into  $\mathbf{R}^n$  and suppose that  $n \neq 4$ . Then there is a quasiconformal mapping  $g$  from  $\mathbf{B}^n$  into  $\mathbf{R}^n$  such that  $g = f$  on  $\partial\mathbf{B}^n$  and  $|Dg|^{-r} \in L^1(\mathbf{B}^n)$  for any*

$$0 < r < \frac{1}{(2K_o(f))^{1/(n-1)} - 1}.$$

*Proof.* By [TV, Corollary 7.12] there is, for any  $\varepsilon > 0$ , a quasiconformal mapping  $g$  from  $\mathbf{B}^n$  onto  $D = f(\mathbf{B}^n)$  such that  $k_D(f(x), g(x)) < \varepsilon$  and

$$(5.2) \quad L^{-1}k_{\mathbf{B}^n}(x, y) \leq k_D(g(x), g(y)) \leq Lk_{\mathbf{B}^n}(x, y)$$

for all  $x, y \in \mathbf{B}^n$ ; here  $k_*$  denotes the quasihyperbolic metric in a domain. Standard estimates for the quasihyperbolic metric give

$$\log \frac{|f(x) - g(x)| + d}{d} \leq k_D(f(x), g(x)) < \varepsilon,$$

where  $d = \min\{\text{dist}(f(x), \partial D), \text{dist}(g(x), \partial D)\}$ . Thus

$$|f(x) - g(x)| \leq (e^\varepsilon - 1)d < d/2$$

for  $\varepsilon > 0$  small enough. This implies

$$\text{dist}(g(x), \partial D) \geq \text{dist}(f(x), \partial D)/2.$$

On the other hand, the reader can verify that the proof of [AK, Theorem 2.3] gives

$$\text{dist}(f(x), \partial D) \geq C^{-1}(1 - |x|)^{(2K_o(f))^{1/(n-1)}}$$

for some  $C \geq 1$  independent of  $x$ , so that

$$(5.3) \quad \text{dist}(g(x), \partial D) \geq C^{-1}(1 - |x|)^{(2K_o(f))^{1/(n-1)}}.$$

Because (5.2) guarantees that  $g$  is locally bilipschitz satisfying

$$|Dg(x)| \geq C^{-1} \frac{\text{dist}(g(x), \partial D)}{1 - |x|}$$

for almost every  $x \in \mathbf{B}^n$ , we arrive at the desired conclusion by combining this with (5.3) and integrating.

Suppose now that  $f$  is as in Theorem A. In Theorem C take  $s = n - 1$ ; then

$$\frac{sr}{n - s + r} = \frac{(n - 1)r}{r + 1}$$

which equals  $(n - 1)(2K_O(f))^{1/(1-n)}$  if

$$r = \frac{1}{(2K_O(f))^{1/(n-1)} - 1}.$$

We thus can conclude Theorem A for  $n \neq 4$  from Theorem C and Lemma 5.1.

Before the proof of Theorem C, we recall some facts about  $p$ -modulus and Hausdorff measure.

For the definition of  $M_p(\Gamma)$ , the  $p$ -modulus of a path family  $\Gamma$ ,  $1 < p < \infty$ , we refer to [V1, Chapter 1]. Notice in particular that  $M_n(\Gamma) = \text{mod } \Gamma$ . If  $A$  is a set in  $\mathbf{R}^n$ , we say that  $A$  has zero  $p$ -modulus if

$$M_p(\Delta(A', \partial B; B)) = 0$$

for each compact  $A' \subset A$  and for some (each) ball  $B$  containing  $A'$ . Next we suppose that  $1 < p \leq n$ , for if  $p > n$ , no nonempty subset of  $\mathbf{R}^n$  has zero  $p$ -modulus. Then we have

$$M_p(A) = 0 \Rightarrow \dim_H(A) \leq n - p$$

and

$$\mathcal{H}_{n-p}(A) < \infty \Rightarrow M_p(A) = 0,$$

whenever  $A \subset \mathbf{R}^n$ . See e.g. [V2].

To begin the proof, fix  $0 < s \leq n - 1$  and let  $E \subset \partial \mathbf{B}^n$  be a set such that  $0 < \mathcal{H}_s(E) < \infty$ . We may assume that  $E$  is compact, that  $f$  has radial limits everywhere in  $E$ , and that those limits lie in a fixed ball.

Let  $1 < q < p \leq n$ . We are going to estimate the  $q$ -modulus of the path family  $\Gamma = \Delta(\overline{B}_{1/2}, E; \mathbf{B}^n)$ . Because  $f$  is  $\text{ACL}_q$  in the sense of [V1, 26.5], we may assume by Fuglede's theorem [V1, 28.2] that  $f$  is absolutely continuous on each path in  $\Gamma$ . Thus if  $\rho$  is admissible for the image family  $\Gamma' = \Delta(f(\overline{B}_{1/2}), f(E); D)$ , the function  $\tilde{\rho}(x) = \rho(f(x))|f'(x)|$  is admissible for  $\Gamma$ . Therefore, we have the following  $q$ -modulus estimate

$$\begin{aligned} M_q(\Gamma) &\leq \int_{\mathbf{B}^n} \rho^q(f(x))|f'(x)|^q dx \\ &\leq \left( \int_{\mathbf{B}^n} \rho^p(f(x))|f'(x)|^n dx \right)^{q/p} \left( \int_{\mathbf{B}^n} |f'(x)|^{q(p-n)/(p-q)} dx \right)^{(p-q)/p} \end{aligned}$$

by Hölder's inequality. The last integral converges if

$$(5.4) \quad p \geq q \frac{n+r}{q+r}.$$

Consequently, for these values of  $p$  and  $q$ ,

$$M_q(\Gamma)^{p/q} \leq C \int_{\mathbf{B}^n} \rho^p(f(x)) |f'(x)|^n dx \leq C \int_D \rho(y)^p dy,$$

where the quasiconformality of  $f$  was used in the last inequality. Since  $\rho$  was arbitrary,

$$(5.5) \quad M_q(\Gamma)^{p/q} \leq CM_p(\Delta(f(\overline{B}_{1/2}), f(E); \mathbf{R}^n)).$$

On the other hand, by using the theory of  $QED_p$ -domains [K1] one can show that

$$M_q(\Delta(E, \partial B(0, 2); B(0, 2))) \leq C(q, n)M_q(\Gamma),$$

and hence

$$M_q(E) > 0 \Rightarrow M_p(\Delta(f(\overline{B}_{1/2}), f(E); \mathbf{R}^n)) > 0,$$

by (5.5). Finally, by again invoking the  $QED_p$ -theory [K1], it is not difficult to see that  $M_p(f(E)) > 0$  if  $M_p(\Delta(f(\overline{B}_{1/2}), f(E); \mathbf{R}^n)) > 0$ . We conclude that

$$M_q(E) > 0 \Rightarrow M_p(f(E)) > 0$$

for all  $1 < q < p < n$  such that (5.4) holds. Now the theorem follows from the indicated relationship between modulus and Hausdorff measure. More precisely, choose  $q = n - s$  and observe that  $\dim_H(fE) \geq n - p$  for all  $p$  satisfying (5.4).

**6. Proof of Theorem D.** We assume throughout this section that  $f$  is a  $K$ -quasiconformal mapping of  $\mathbf{B}^n$  onto a domain  $D$ . We recall two basic distortion properties.

If  $x, y$  and  $z$  belong to a hyperbolic ball of radius  $R > 0$  in  $\mathbf{B}^n$ , then

$$|x - y| \leq t|x - z| \Rightarrow |f(x) - f(y)| \leq \eta(t)|f(x) - f(z)|$$

for an increasing homeomorphism  $\eta: [0, \infty) \rightarrow [0, \infty)$  that depends only on  $n, K$ , and  $R$ ; see [V3, Theorem 2.4]. This property will be called the *local quasismmetry of  $f$* .

Similarly, if  $x$  and  $y$  lie in a hyperbolic ball with radius  $R$ , we have

$$(6.1) \quad C^{-1}a_f(x) \leq a_f(y) \leq Ca_f(x),$$

where  $C$  depends only on  $n, K$ , and  $R$ . This follows e.g. from the local quasismmetry combined with the Koebe type result

$$(6.2) \quad C^{-1}a_f(x) \leq \frac{\text{dist}(f(x), \partial D)}{1 - |x|} \leq Ca_f(x),$$

where  $C$  depends only on  $n$  and  $K$  (see [AG1, Theorem 1.8]).

We start with two lemmata, similar to those in [P2]. The first was proved in [K2, Lemma 2.6].

6.3. LEMMA. *If  $\gamma \subset \mathbf{B}^n$  is a rectifiable curve with  $\text{diam } \gamma \geq \text{dist}(\gamma, \partial \mathbf{B}^n)$ , then*

$$\text{diam } f(\gamma) \leq C \int_{\gamma} a_f(x) |dx|,$$

where  $C = C(n, K)$ .

6.4. LEMMA. *Let  $A \subset \partial \mathbf{B}^n$ . If  $a_f(rw) \leq M$  for all  $0 < r < 1$  and  $w \in A$ , then*

$$|f(w_1) - f(w_2)| \leq CM|w_1 - w_2|$$

for all  $w_1, w_2 \in A$ , where  $C = C(n, K)$ .

*Proof.* We observe first that the assumption and Lemma 6.3 guarantee that  $f(w)$  exists as a radial limit for each  $w \in A$ , so the assertion makes sense. Next, we may suppose that  $|w_1 - w_2| < 1/10$ , for otherwise Lemma 6.3 implies

$$|f(w_1) - f(w_2)| \leq \text{diam } fA \leq CM \leq 10CM|w_1 - w_2|.$$

Fix  $0 < r < 1$  such that

$$|rw_1 - rw_2| \approx (1 - r) \approx |w_1 - w_2|.$$

Then

$$\begin{aligned} |f(w_1) - f(w_2)| &\leq |f(w_1) - f(rw_1)| \\ &\quad + |f(rw_1) - f(rw_2)| + |f(rw_2) - f(w_2)|. \end{aligned}$$

Because, by the local quasisymmetry and (6.2),

$$\begin{aligned} |f(rw_1) - f(rw_2)| &\approx \text{dist}(f(rw_1), \partial D) \approx a_f(rw_1)(1 - r) \\ &\leq M(1 - r) \approx M|w_1 - w_2|, \end{aligned}$$

it suffices to show, by symmetry, that

$$|f(rw_1) - f(w_1)| \leq CM(1 - r).$$

But this follows from Lemma 6.3, and the proof is thereby complete.

6.5. *Dyadic decomposition of  $\partial \mathbf{B}^n$ .* We require a decomposition of  $\partial \mathbf{B}^n$  into sets that correspond to the dyadic intervals on the boundary of the unit disk (or the dyadic subcubes of  $\mathbf{R}^n$ ). To be precise

with this construction, let  $\mathbf{Q}^{n-1} = [-1, 1]^{n-1}$  be the closed unit cube in  $\mathbf{R}^{n-1}$  and consider the radial stretching  $F: \mathbf{Q}^{n-1} \rightarrow \overline{\mathbf{B}}^{n-1}$  defined by

$$F(x) = \frac{x}{|v_x|}, \quad x \in \mathbf{Q}^{n-1} \setminus \{0\},$$

where  $v_x$  is the unique point on  $\partial\mathbf{Q}^{n-1}$  where the ray  $tx, t \geq 0$ , meets  $\partial\mathbf{Q}^{n-1}$ . Also, we set  $F(0) = 0$ . Then  $F$  is an  $L$ -bilipschitz mapping with  $L$  depending only on  $n$ . Next, let  $\pi: \partial\mathbf{B}^n \setminus \{e_n\} \rightarrow \mathbf{R}^{n-1}$  be the stereographic projection

$$\pi(x) = e_n + \frac{2(x - e_n)}{|x - e_n|^2}.$$

Then

$$H = \pi^{-1} \circ F: \mathbf{Q}^{n-1} \rightarrow \partial\mathbf{B}_-^n = \partial\mathbf{B}^n \cap \{x \in \mathbf{R}^n : x_n \leq 0\}$$

defines an  $L$ -bilipschitz mapping with  $L = L(n)$ .

By *dyadic cubes* on  $\partial\mathbf{B}^n$  we mean the members of the family

$$\mathcal{D} = \{H(Q), H^*(Q) : Q \text{ is a closed dyadic cube in } \mathbf{Q}^{n-1}\},$$

where  $H^*$  denotes  $H$  followed by the reflection  $(x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, -x_n)$ . The members of  $\mathcal{D}$  are also denoted by  $Q$ . By construction, it is readily seen that they share the following properties of the ordinary dyadic cubes:

(i) for a.e. point  $w$  on  $\partial\mathbf{B}^n$  there is a unique descending chain of dyadic cubes converging to  $w$ ;

(ii)  $|Q| \approx |Q'|$  if  $Q$  and  $Q'$  are within one generation apart.

Next, if  $Q \in \mathcal{D}$ , we let  $w_Q \in Q$  denote the *center* of  $Q$  and it is, by definition, the image of the center of the corresponding cube under  $H$  (or  $H^*$ ). Moreover,  $z_Q \in \mathbf{B}^n$  is a *tent-point* of  $Q$  if it satisfies

$$1 - |z_Q| \approx |z_Q - w_Q| \approx \text{diam } Q \approx |Q|^{1/(n-1)}$$

for some dimensional constants. Also observe that for each point  $x \in \mathbf{B}^n$  there is  $Q_x \in \mathcal{D}$  such that

$$1 - |x| \approx \text{dist}(x, Q_x) \approx \text{diam } Q_x$$

for some dimensional constants.

From now on, we assume that a fixed choice of the point  $z_Q$  has been made for each  $Q \in \mathcal{D}$ .

6.6. LEMMA. *Let  $Q \in \mathcal{D}$ . Then there is  $A \subset Q$  and  $\delta = \delta(n) > 0$  such that  $|A| \geq \delta|Q|$  and that*

$$(6.7) \quad |f(z_Q) - f(w)| \leq M(n, K) \text{dist}(f(z_Q), \partial D)$$

for each  $w \in A$ .

*Proof.* We normalize  $f(z_Q) = 0$  and  $\text{dist}(0, \partial D) = 1$ . Fix  $t > 1$  and let  $A_t \subset Q$  be the set of all points  $w$  in  $Q$  such that  $|f(w)| > t$ . Consider the mapping  $g = f \circ U_Q$ , where  $U_Q$  is a Möbius self mapping of the unit ball that carries 0 to  $z_Q$ . Then  $g$  is a quasiconformal mapping with  $K(g) = K(f) = K$ , and  $g(0) = 0$ . Let  $\tilde{A}_t = U_Q^{-1}(A_t)$ ,  $\tilde{Q} = U_Q^{-1}(Q)$ , and let  $\Gamma_t$  denote the family of all radial segments joining  $\bar{B}_{1/2}$  to  $\tilde{A}_t$ . We have

$$\text{mod } \Gamma_t \geq (\log 2)^{1-n} |\tilde{A}_t|$$

and hence

$$\text{mod } g\Gamma_t \geq (\log 2)^{1-n} |\tilde{A}_t|/K.$$

The local quasisymmetry together with our normalization implies that  $g(\bar{B}_{1/2}) \subset B(0, H)$  for some  $H = H(n, K)$ . Thus

$$\text{mod } g\Gamma_t \leq \frac{\omega_{n-1}}{(\log t/H)^{n-1}}$$

for  $t > H$  so that

$$|\tilde{A}_t| \leq C(n, K)(\log t/H)^{1-n} \equiv \varphi(t),$$

and since  $|\tilde{Q}| \approx 1$ , we can choose  $t = t(n, K)$  so large that

$$|\tilde{Q} \setminus \tilde{A}_t| \geq |\tilde{Q}| - \varphi(t) \geq C(n) > 0.$$

Moreover, for any  $E \subset \partial \mathbf{B}^n$

$$|U_Q(E)| = \int_E J_{U_Q}(w) d\mathcal{H}_{n-1}(w) = \int_E |DU_Q(w)|^{n-1} d\mathcal{H}_{n-1}(w)$$

and because  $|DU_Q| \approx \text{diam } Q$  on  $\tilde{Q}$ , we obtain

$$|\tilde{Q} \setminus \tilde{A}_t| \approx \frac{|U_Q(\tilde{Q} \setminus \tilde{A}_t)|}{|Q|}.$$

Finally, since for  $w \in \tilde{Q} \setminus \tilde{A}_t$  we have  $|g(w)| \leq t$ , the assertion follows.



*Proof of Theorem D.* Our proof closely resembles that of Pommerenke's in [P2]. First observe  $\partial \mathbf{B}^n \setminus E_\infty = \tilde{E}_0 \cup \tilde{E}_1$ , where

$$\tilde{E}_0 = \left\{ w \in \partial \mathbf{B}^n : \liminf_{r \rightarrow 1} a_f(rw) = 0 \right\} \quad \text{and} \quad \tilde{E}_1 = \partial \mathbf{B}^n \setminus E_\infty \setminus \tilde{E}_0.$$

Now

$$\tilde{E}_1 = \bigcup_{k=1}^{\infty} A_k, \quad \text{where } A_k = \left\{ w \in \tilde{E}_1 : \limsup_{r \rightarrow 1} a_f(rw) < k \right\}.$$

By Lemma 6.4  $f$  is Lipschitz on  $A_k$  so that  $|f(A_k)| < \infty$ ; in particular,  $f(\tilde{E}_1)$  has  $\sigma$ -finite  $\mathcal{H}_{n-1}$ -measure and  $f$  is absolutely continuous in  $\tilde{E}_1$ .

We next show that  $\tilde{E}_0 = E_0 \cup A_0$ , where  $|E_0| = 0$  and  $|f(A_0)| = 0$ . Then we can set  $E_1 = \tilde{E}_1 \cup A_0$ .

To this end, for each  $w \in \tilde{E}_0$  and  $j = 1, 2, \dots$  there is  $r_{j,w}$ ,  $1 - 1/j < r_{j,w} < 1$ , such that

$$(6.8) \quad a_f(r_{j,w}w) < 2^{-j}.$$

Let  $Q_{j,w}$  be the largest dyadic cube in  $\mathcal{D}$  containing  $w$  with  $\text{diam } Q_{j,w} \leq 1 - r_{j,w}$ ; such a cube exists for almost every  $w \in \partial \mathbf{B}^n$ . Then the collection

$$\mathcal{D}_k = \{Q_{j,w} : w \in \tilde{E}_0, j \geq k\}$$

covers almost all of  $\tilde{E}_0$ , and by the Vitali covering theorem we can select a countable collection of essentially disjoint cubes  $\{Q_{k,l}\} \subset \mathcal{D}_k$  such that

$$\left| \tilde{E}_0 \cap \left( \bigcup Q_{k,l} \right) \right| = |\tilde{E}_0|.$$

Next, Lemma 6.6 implies the existence of sets  $A_{k,l} \subset Q_{k,l}$  and  $\delta > 0$  such that

$$(6.9) \quad |A_{k,l}| \geq \delta |Q_{k,l}|$$

and that

$$(6.10) \quad f(A_{k,l}) \subset B(f(z_{k,l}), M \text{dist}(f(z_{k,l}), \partial D)),$$

where  $z_{k,l} = z_{Q_{k,l}}$  and  $M = M(n, K)$ . We define

$$E_k = \tilde{E}_0 \cap \left( \bigcup Q_{k,l} \right), \quad F_k = E_k \cap \left( \bigcup A_{k,l} \right),$$

and

$$A_0 = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} F_k.$$

Next, we show that  $|f(A_0)| = 0$ . It follows from (6.10) that each  $f(F_k)$  can be covered by balls

$$B(f(z_{k,l}), M \operatorname{dist}(f(z_{k,l}), \partial D)) \equiv B_{k,l}.$$

Since the hyperbolic distance between  $z_{k,l}$  and  $r_{j,w}$  (where  $z_{k,l}$  corresponds to  $Q_{k,l} = Q_{j,w}$  for some  $j \geq k$  and  $w \in \tilde{E}_0$ ) is bounded by a constant  $C = C(n)$ , we have by (6.1), (6.2), and (6.8) that

$$\begin{aligned} \operatorname{diam} B_{k,l} &= 2M \operatorname{dist}(f(z_{k,l}), \partial D) \\ &\leq C(1 - |z_{k,l}|)a_f(z_{k,l}) \leq C \operatorname{diam} Q_{k,l} 2^{-k}. \end{aligned}$$

Thus

$$\sum_l (\operatorname{diam} B_{k,l})^{n-1} \leq C 2^{-k(n-1)} \sum_l (\operatorname{diam} Q_{k,l})^{n-1} \leq C 2^{-k(n-1)},$$

where  $C = C(n, K)$ ; here we used the fact that the cubes  $Q_{k,l}$  are essentially disjoint. In consequence, since  $f(A_0)$  belongs to  $f(F_i) \cup f(F_{i+1}) \cup \dots$  for any  $i = 1, 2, \dots$ ,  $f(A_0)$  can be covered by balls  $B_{k,l}$ ,  $k \geq i$ , such that

$$\sum_{k \geq i} \sum_l (\operatorname{diam} B_{k,l})^{n-1} \leq C \sum_{k \geq i} 2^{-k(n-1)} \leq C 2^{-i(n-1)}.$$

This implies  $|f(A_0)| = 0$ .

It remains to show that  $|E_0| = 0$ , where  $E_0 = \tilde{E}_0 \setminus A_0$ . Now

$$E_0 \subset \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} \bigcup_{l=1}^{\infty} (Q_{k,l} \setminus A_{k,l}) \cup E_{0,0},$$

where  $|E_{0,0}| = 0$ , and we want to show that  $|G_i| = 0$  for each  $i = 1, 2, \dots$ , where

$$G_i = \bigcap_{k=i}^{\infty} \bigcup_{l=1}^{\infty} (Q_{k,l} \setminus A_{k,l}).$$

Fix  $i$ . By construction, for almost every point  $w$  in  $G_i$  there is a descending chain of cubes of the form  $Q_{k,l}$ ,  $k \geq i$ , that converge to  $w$ . It is then easy to see that  $w$  cannot be a point of density of  $G_i$  because

$$\frac{|Q_{k,l} \cap G_i|}{|Q_{k,l}|} \leq \frac{|Q_{k,l} \setminus A_{k,l}|}{|Q_{k,l}|} \leq 1 - \delta < 1$$

for all  $k$  by (6.9). This completes the proof of Theorem D.

**7. Proofs of Theorems E and F.** Suppose that  $f$  is as in Theorem E. We can factor  $f$  as  $f = \varphi \circ h$ , where  $h: \mathbf{B}^2 \rightarrow \mathbf{B}^2$  is a quasiconformal

mapping with  $h(0) = 0$  and  $\varphi: \mathbf{B}^2 \rightarrow f(\mathbf{B}^2)$  is conformal. An easy argument shows that the complex dilatation  $\mu_h$  of  $h$  satisfies the Carleson measure condition of Theorem E, i.e.  $|\mu_h(z)|^2/(1 - |z|)$  is a Carleson measure in  $\mathbf{B}^2$ . Then [FKP, Theorem 3.3] shows that the boundary correspondence of  $h$  induces an  $A_\infty$  measure on the boundary of  $\mathbf{B}^2$ . To be more precise, the quasiconformal mapping  $h$  gives rise to an elliptic operator

$$L = -\operatorname{div} A(x)\nabla$$

in the disk, where  $A(x) = J_h(x)[Dh(x)^{-1}][Dh(x)^{-1}]^t$  for a.e.  $x \in \mathbf{B}^2$ . A straightforward calculation shows that the ratio of

$$\sup_{x \in B_z} \|A(x) - Id\|, \quad B_z = B(z, (1 - |z|)/2),$$

and  $|\mu_h(z)|$  is bounded from above and below by a constant independent of  $z \in \mathbf{B}^2$ ; because the associated elliptic measure  $\omega_L$  satisfies

$$\omega_L^0(E) = \frac{|h(E)|}{2\pi}$$

for  $E \subset \partial\mathbf{B}^2$  (see e.g. [GLM] or [HM]), the assertion follows from [FKP, 3.3].

In particular, we have  $|A| = 0$  if and only if  $|h(A)| = 0$  for  $A \subset \partial\mathbf{B}^2$ . Makarov's theorem guarantees that there is a set  $E' \subset \partial\mathbf{B}^2$  such that  $|E'| = 2\pi$  and the Hausdorff dimension of  $\varphi(E')$  equals 1. Then  $h^{-1}(E') = E$  is the desired set.

Next we prove Theorem F. Suppose that  $f$  is as in Theorem F and let  $E \subset \partial\mathbf{B}^n$  be such that  $\dim_H E = s$ ,  $0 < s \leq n - 1$ . The proof will show that there is no loss of generality in assuming that  $f$  is bounded. Fix  $\varepsilon > 0$ . For  $w \in \partial\mathbf{B}^n$  and  $r > 0$  the domain

$$D_{w,r} = B(w, r) \cap \mathbf{B}^n$$

is  $b$ -uniform with  $b$  independent of  $w$  and  $r$ . By [AH, Theorem 1.3] there is  $K_1 = K_1(n, b) > 1$  such that every bounded  $K$ -quasiconformal mapping  $g$  of  $D_{w,r}$  with  $K \leq K_1$  can be extended to a  $K_2$ -quasiconformal mapping of  $\mathbf{R}^n$ ; moreover,  $K_2 \rightarrow 1$  as  $K \rightarrow 1$ . Thus for fixed  $\varepsilon > 0$  we can choose  $r > 0$  so small that  $f_{w,r} \equiv f|_{D_{w,r}}$  extends to a  $K_2$ -quasiconformal mapping  $\tilde{f}_{w,r}: \mathbf{R}^n \rightarrow \mathbf{R}^n$  with  $s - \varepsilon \leq sK_2^{1/(1-n)} \leq sK_2^{1/(n-1)} \leq s + \varepsilon$ . Because  $K$ -quasiconformal mappings of  $\mathbf{R}^n$  are Hölder continuous with exponent  $K^{1/(1-n)}$  on compact subsets,  $f$  satisfies

$$M_w^{-1}|x - y|^{1/\alpha} \leq |f(x) - f(y)| \leq M_w|x - y|^\alpha, \quad \alpha = K_2^{1/(1-n)},$$

for all  $x, y \in \overline{D}_{w,r}$  and for any  $w \in \partial\mathbf{B}^n$ . In particular, the Hausdorff dimension of  $f(E \cap \overline{D}_{w,r})$  is at most  $s + \varepsilon$  and at least  $s - \varepsilon$  by the choice of  $r$ . Finally, since  $\partial\mathbf{B}^n$  can be covered by finitely many sets  $\overline{D}_{w_1,r}, \dots, \overline{D}_{w_k,r}$  we conclude that

$$s - \varepsilon \leq \dim_H f(E) \leq s + \varepsilon,$$

and the theorem follows by letting  $\varepsilon \rightarrow 0$ .

**8. Open problems.** In the following,  $f$  is a  $K$ -quasiconformal mapping of  $\mathbf{B}^n$  into  $\mathbf{R}^n$ .

*Problem 1.* Denote by  $\mu(n, K)$  the supremum of all numbers  $\alpha$  such that  $\dim_H f(E) \geq \alpha$  whenever  $E \subset \partial\mathbf{B}^n$  has positive  $\mathcal{H}_{n-1}$ -measure. By Theorem A,  $\mu(n, K) \geq (n-1)(2K)^{1/(1-n)}$ . Find a better lower bound for  $\mu(n, K)$ .

*Problem 2.* Similarly, define  $\mu(n, K, s)$ ,  $0 < s \leq n - 1$ , to be the supremum of all numbers  $\alpha$  such that  $\dim_H f(E) \geq \alpha$  whenever  $E \subset \partial\mathbf{B}^n$  has positive  $\mathcal{H}_s$ -measure. By Theorem C,  $\mu(n, K, s) \geq sr/(n - s + r)$ , where  $r = r(n, K) > 0$ . Find a better lower bound for  $\mu(n, K, s)$ . This is related to the unknown value of  $r(n, K)$ .

*Problem 3.* Let  $r(n, K) > 0$  be the supremum of all  $r > 0$  such that  $|Df|^{-r} \in L^1(\mathbf{B}^n)$ . Show that

$$r(n, K) \geq \frac{1}{(2K_O(f))^{1/(n-1)} - 1}.$$

Note that the inequality  $r(2, 1) < 1$  follows from elementary distortion theorems for univalent functions. Brennan [B] has shown that  $r(2, 1) > 1$  (see also [P1]) and hence it would be natural to expect that

$$r(n, K) > \frac{1}{(2K_O(f))^{1/(n-1)} - 1}.$$

This latter inequality would then give a better bound for  $\mu(n, K, s)$  in Problems 1 and 2.

*Problem 4.* Give conditions that guarantee that  $f$  has the Makarov property. Improve the Carleson measure condition in Theorem E.

*Problem 5.* Let  $n \geq 3$  and suppose that

$$K(r) = K(f|\mathbf{B}^n \setminus \overline{B}_r) \rightarrow 1, \quad r \rightarrow 1.$$

Under what conditions on  $K(r)$  does  $f$  have the Makarov property? Is there a higher dimensional analog of the Carleson measure condition in Theorem E? Mattila and Vuorinen [MV] have studied a related question.

*Problem 6.* Let  $n \geq 3$  and define  $E_\infty$  as in Theorem D. Construct a quasiconformal mapping such that  $|E_\infty| > 0$ . Is it possible that  $E_\infty = \partial\mathbf{B}^n$ ?

*Problem 7.* Let  $n \geq 3$  and suppose that  $|\partial D| < \infty$ , where  $D = f(\mathbf{B}^n)$ . Is  $f|_{\partial\mathbf{B}^n}$  absolutely continuous? What about  $f^{-1}|_{\partial D}$ ? This question was asked by Baernstein and Manfredi [BM, p. 846]. Gehring [G2] showed that  $f|_{\partial\mathbf{B}^n}$  is absolutely continuous if  $f$  has an extension to a global quasiconformal mapping; Väisälä [V3] proved the same result under the weaker assumption that  $f(D)$  be uniform. Recently the first author showed that  $f|_{\partial\mathbf{B}^n}$  is absolutely continuous if  $\partial D$  has an  $(n-1)$ -tangent plane  $\mathcal{H}_{n-1}$  almost everywhere [He]. The absolute continuity of  $f^{-1}|_{\partial D}$  is open in all these cases.

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Received September 15, 1992 and in revised form January 26, 1993. The first author was supported by NSF Grant DMS 91-48183 and The Academy of Finland.

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Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

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The Pacific Journal of Mathematics at University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,  
Berkeley, CA 94720, A NON-PROFIT CORPORATION

This publication was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$ ,  
the American Mathematical Society's  $\mathcal{T}\mathcal{E}\mathcal{X}$  macro system.  
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Volume 165    No. 1    September 1994

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