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# ON THE COMPACTNESS OF A CLASS OF RIEMANNIAN MANIFOLDS

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**A class of Riemannian manifolds is studied in this paper. The main conditions are 1) the injectivity is bounded away from 0; 2) a norm of the Riemannian curvature is bounded; 3) volume is bounded above; 4) the Ricci curvature is bounded above by a constant divided by square of the distance from a point. Note the last condition is scaling invariant. It is shown that there exists a sequence of such manifolds whose metric converges to a continuous metric on a manifold.**

**Introduction.** Let $\mathcal{L} = \mathcal{L}(H, K, V, n, i_0)$ be the set of $n$-dimensional Riemannian manifolds $(M, g)$, *s.t.*,

(0.1) $M$ is diffeomorphic to $(B_2, g_0)$, the standard Euclidean ball of radius 2, center $= 0$;
(0.2) $(M, g)$ has $C^\infty$ curvature tensor in $M$;
(0.3) for any $x \in M$, the Ricci curvature at $x$ $|Ric(g)(x)| \leq Hr^{-2}$, where $r = dist(x, 0)$;
(0.4) the injectivity of $(M, g) \geq i_0 > 0$;
(0.5) $\int_M |Rm(g)|^{\frac{n}{2}} dg < K$;
(0.6) volume of $(M, g) \leq V$.

In the case when the condition (0.3) is replaced by $|Ric(g)| \leq H$, and (0.6) is replaced by a diameter bound, a compactness property is proved by the first author in a more general setting. The purpose of this paper is to extend some of his results to the present situation where the bound om Ricci curvature of $(M, g)$ blows up like $r^{-2}$ at a point. As an application, we will discuss the compactness of orbifolds with a finite number of singularities.

The main result is:

THEOREM 0.7. *Let $(M_k, g_k) \in \mathcal{L}$, $k = 1, 2, 3, \ldots$. Then there exists a subsequence (again denoted by $(M_k, g_k)$), a $C^\infty$ manifold $M'$ diffeomorphic to $B_2(0)$, and a $C^0$ metric $g'$ on $M'$* s.t. *$g_k \to g'$ in $C^0$-norm on $M'$ and the convergence is in $C^{1,\alpha}$-norm away from 0.*

In Section 1 we study the geodesic balls centered at 0. A compactness estimate of the metric $g$ will be derived. In Section 2, a small geodesic sphere is shown to have a small diameter. In Section 3, some $L^{n/2}$-curvature pinching results are derived, which will be used in Section 4 to show the existence of harmonic coordinates. We will prove in Section 4 the above main result and a slightly different version.

In the definition of $\mathcal{L}$, if (0.3) is replaced by a 1-sided condition

$$Ric(g) \geq -Hr^{-2}g, \tag{0.3$'$}$$

then the above compactness result should be modified as follows. Denote the set of such Riemannian manifolds by $\mathcal{L}'$.

THEOREM 0.8. *Let $(M_k, g_k) \in \mathcal{L}'$, $k = 1, 2, 3, \ldots$. Then there exists a subsequence of $(M_k, g_k)$, which converges in $C^\circ$-norm to a $C^\infty$ manifold $M'$ with a $C^\circ$ metric $g'$.*

**1.** In this section, we assume that for some $H > 0$, $i_0 > 0$, $(M, g)$ is a Riemannian manifold diffeomorphic to $B_2$ satisfying

$$Ric(g) \geq -Hr^{-2}g; \tag{1.1}$$

$$inj(g) \geq i_0 > 0. \tag{1.2}$$

Let $B_\rho(0) = \{x \in M | d(0, x) \leq\}$ be the geodesic ball of $M$ centered at 0. Consider a geodesic polar coordinate system $\{r, x^1, \cdots, x^{n-1}\}$ on $B_\rho(0)$, we have

$$ds(g)^2 = dr^2 + \sum_{i=1}^{n-1} g_{ij}(r, x)dx^i dx^j; \tag{1.3}$$

$$R_{irrj} = -\frac{1}{2}\frac{\partial^2}{\partial r^2}g_{ij}(r,x) + \frac{1}{4}\sum g^{kl}\frac{\partial}{\partial r}g_{ik}\frac{\partial}{\partial r}g_{jl}. \tag{1.4}$$

For the Ricci curvature in the radial direction, we have

$$R_{rr} = -\frac{\partial^2}{\partial r^2}\ln\sqrt{g(r)} - \frac{1}{4}\left|\frac{\partial}{\partial r}g(r)\right|^2_{g(r)}, \tag{1.5}$$

where $g(r) = g(r,x)$,

$$\sqrt{g}\,dV_0 = \sqrt{\det(g_(ij)}\,dx^1\wedge\ldots\wedge\,dx^{n-1}, \tag{1.6}$$

($dV_0 =$ the volume element of the standard Euclidean sphere)
and

$$\left|\frac{\partial g}{\partial r}\right|^2_g = \sum g^{ij}g^{kl}\frac{\partial}{\partial r}g_{ij}\frac{\partial}{\partial r}g_{kl}.$$

We start out with the following estimate:

PROPOSITION 1.7. *For $\rho \le \frac{i_0}{2}$, there exists $C_1 = C_1(H,n) > 0$ s.t.* $\int_0^\rho r^2\left|\frac{\partial}{\partial r}g\right|^2 dr \le C_1\rho$.

*Proof.* The function is essentially the same as that given in [**12**], p.5-6. For any piecewise $C^\infty$ function $\phi$ of $r$ with $\phi(\rho) = 0$, we have

$$\begin{aligned}\left(\frac{1}{4}-\epsilon\right)&\int_0^\phi r^2\phi^2\left|\frac{\partial}{\partial r}g\right|^2 dr \\ &\le \frac{n-1}{2\epsilon}\int_0^\rho(r^2\phi'^2+\phi^2)dr - \int_0^\rho r^2\phi^2R_n dr.\end{aligned} \tag{1.8}$$

Take $\epsilon = \frac{1}{8}$, $\phi = \rho - r$, and use $-R_{rr} \le Hr^{-2}$, we get

$$\begin{aligned}&\int_0^\phi r^2(\phi-r)^2\left|\frac{\partial}{\partial r}g\right|^2 dr \\ &\le 32(n-1)\int_0^\phi(r^2+(\phi-r)^2)\,dr + H\int_0^\phi(r^2(\phi-r)^2)r^{-2}\,dr \\ &\le C(H,n)\rho^3.\end{aligned}$$

Thus,

$$\int_0^{\frac{\phi}{2}} r^2 \left|\frac{\partial}{\partial r} g\right|^2 dr \le \frac{1}{\left(\frac{\rho}{2}\right)^2} \int_0^{\phi} r^2 (\phi - r)^2 \left|\frac{\partial}{\partial r} g\right|^2 dr \le \frac{1}{2} C_1(H, n)\rho.$$

□

PROPOSITION 1.9. *There exists* $C_2 = C_2(H, i_0, n) > 0$ s.t. *for any* $r \in \left(0, \frac{i_0}{2}\right)$, *we have*

$$r\left|\frac{\partial}{\partial r} \ln \sqrt{g}\right| \le C_2.$$

*Proof.* From (1.5) and integration by parts,

$$\int_0^{\phi} r^2 R_n \, dr = -\frac{1}{2} r^2 \frac{\partial}{\partial r} \ln g + \frac{1}{2} \int_0^{\phi} 2r \frac{\partial}{\partial r} \ln g - \frac{1}{2} \int_0^{\phi} r^2 \left|\frac{\partial}{\partial r} g\right|^2 dr.$$

Thus

$$\frac{1}{2} r^2 \frac{\partial}{\partial r} \ln \sqrt{g} \le H \int_0^{\phi} r^{-2} r^2 \, dr + \frac{1}{4} C_1 r + \left(\int_0^{\phi} r^2 \left|\frac{\partial}{\partial r} \ln g\right|^2\right)^{\frac{1}{2}} r^{\frac{1}{2}}$$

$$\le \tfrac{1}{3} H r + \tfrac{1}{4} C_1 r + (n-1)^{\frac{1}{2}} \left(\int_0^{\phi} r^2 \left|\frac{\partial}{\partial r} g\right|^2 dr\right)^{\frac{1}{2}} r^{\frac{1}{2}}$$

$$\le C_2(H, i_0, n) r.$$

□

Next we study the induced metric $g(r) = \sum g_{ij}(r, x) \, dx^i \, dx^j$ on the geodesic sphere

$$S_r(0) = \{x \in M : d(x, 0) = r\}, \quad r \le \frac{i_0}{2}.$$

PROPOSITION 1.10. *There exists* $C_3 = C_3(H, n) > 0$ s.t. *for* $0 < r_1 < r_2 \le \frac{i_0}{2}$, *we have*

$$e^{C_3 r_2 r_1^{-1}} g(r_1) \le g(r_2) \le e^{C_3 r_2 r_1^{-1}} g(r_1).$$

*Proof.* From Proposition 1.7, we have, for any vector $\nu = (\nu^1, \dots, \nu^n) \in TS_1$,

$$\left|\ln \frac{h(r_2)}{h(r_1)}\right| \le \int_{r_1}^{r_2} \left|\frac{\partial}{\partial r} \ln h(r)\right| dr \le \left(\int_{r_1}^{r_2} \left|\frac{\partial}{\partial r} g\right| r dr\right) r_1^{-1}$$
$$\le \sqrt{r_2}(C_1 r_2)^{\frac{1}{2}} r_1^{-1} = \sqrt{C_1 \frac{r_2}{r_1}},$$

where $h(r) = g_{ij}(r)\, d\nu^i d\nu^j$. Hence $e^{C_3 r_2 r_1^{-1}} \le \frac{h(r_2)}{h(r_1)} \le e^{C_3 r_2 r_1^{-1}}$, where $c_3 = \sqrt{c_1}$. □

Before we go any further, let us make some remarks regarding conditions (0.3) and (0.5). Let $\tau > 0$ be small. Define a new metric $g^\tau$ on $M$ by $g^\tau(x) = \tau^{-2} g(\tau x)$.

REMARK.

(1.11) If $g$ satisfyes $(0.3)'$, so does $g^\tau$.

(1.12) $$\int_{B_1} |R(g^\tau)|^{\frac{n}{2}}\, dg^\tau = \int_{B_\tau} |R(g)|^{\frac{n}{2}}\, dg.$$

Therefore, by a scaling of this type if necessary, we can assume that $g$ satisfies (0.3) and (0.5) with $K \ll 1$.

Once we have Proposition 1.10 we can control the $L^{n/2}$ norm of the Riemannian curvature tensor $Rm(r)$ of $g(r)$, the induced metric on $S(0, r)$.

THEOREM 1.13. *If $(M, g) \in \mathcal{L}'$ then for any $\rho \le \frac{i_0}{4}$, there exist* $r_\rho \in (\frac{\rho}{2}, \rho)$, $C_4 = C_4(H, K, i_0, n) > 0$, s.t.

(1.15) $$\int_{S(0,r_\rho)} |Rm(r_\rho)|_{g(r_\rho)}^{\frac{n}{2}}\, dg(r_\rho) \le C_4 r_\rho^{-1}.$$

*Proof.* By Lemma 1.17 in [**12**],$\exists C_5 = C_5(H, i_0, n)$ *s.t.* for $\rho < \frac{i_0}{4}$,

$$\int_{\frac{\rho}{2}}^{\rho} \left|\frac{\partial}{\partial r} g\right|^n dr \le C_5 \left(\frac{1}{\rho^n} + \int_{\frac{\rho}{2}}^{\rho} |Rm(g)|^{\frac{n}{2}}\, dr\right).$$

From Proposition 1.10, there exists $C = C(H, i_0, n)$ *s.t.*

$$C^{-1}\sqrt{g}(\rho) \leq \sqrt{g}(r) \leq C_3\sqrt{g}(\rho)$$

for $r \in (\frac{\rho}{2}, \rho)$, i.e., $\sqrt{g}(r)$ is equivalent to $\sqrt{g}(\rho)$. Thus for some constant $C_6 = C_6(H, i_0, n) > 0$, we have

$$\int_{\frac{\rho}{2}}^{\rho} \left|\frac{\partial}{\partial r}g\right|^n \sqrt{g}(r)\,dr \leq C_6\left(\rho^{-n}\sqrt{g}(\rho) + \int_{\frac{\rho}{2}}^{\rho} |Rm(g)|^{\frac{n}{2}} \sqrt{g}(r)\,dr\right).$$

Integrating over $S_\rho(0)$, we get

$$\int_{B_\rho \backslash B_{\frac{\rho}{2}}} \left|\frac{\partial}{\partial r}g\right|^n dg \leq C_6\rho^{-n}\int_{S_\rho} dg(\rho) + C_6\int_{B_\rho} |Rm(g)|^{\frac{n}{2}}\,dg.$$

Taking $\rho = \frac{i_0}{4}$, we get

$$\int_{B_{\frac{i_0}{4}} \backslash B_{\frac{i_0}{8}}} \left|\frac{\partial}{\partial r}g\right|^n dg \leq C_6\left(\frac{i_0}{4}\right)^{-n} vol\left(S_{\frac{i_0}{4}}\right) + C_6\int_{B_{\frac{i_0}{4}}} |Rm(g)|^{\frac{n}{2}}\,dg.$$

By Bishop's volume estimate [**1**], $\exists C_7 = C_7(H, i_0, n)$ *s.t.* $vol\left(S_{\frac{i_0}{4}})\right) \leq C_7$. Thus we get a constant $C_8 = C_8(H, i_0, n) > 0$ *s.t.*

$$\int_{B_{\frac{i_0}{4}} \backslash B_{\frac{i_0}{8}}} \left|\frac{\partial}{\partial r}g\right|^n dg \leq C_8 + C_8\int_{B_{\frac{i_0}{4}}} |Rm(g)|^{\frac{n}{2}}\,dg. \tag{1.16}$$

Define $g^\tau = r^{-2}g$ with $r = \frac{4\rho}{i_0}$. Noticing that $Ric(g^\tau) \geq -Hr^{-2}$, $inj(g^\tau) \geq i_0$, we can apply (1.16) to $g^\tau$. By the scaling invariance of (1.16), we get

$$\begin{aligned}
\int_{B_\rho \backslash B_{\frac{\rho}{2}}} \left|\frac{\partial}{\partial r}g\right|^n dg &= \int_{B_{\frac{i_0}{4}} \backslash B_{\frac{i_0}{8}}} \left|\frac{\partial}{\partial r}g^\tau\right|^n dg^\tau \\
&\leq C_8 + C_8\int_{B_{\frac{i_0}{4}}} |Rm(g^\tau)|^{\frac{n}{2}}\,dg^\tau \\
&= C_8 + C_8\int_{B_\rho} |Rm(g^\tau)|^{\frac{n}{2}}\,dg \\
&\leq C_8 + C_8K = C_9.
\end{aligned}$$

Hence

$$\int_{\frac{\rho}{2}}^{\rho} \left( \int_{S_r} \left| \frac{\partial}{\partial r} g \right|^n dg(r) \right) dr \le C_9. \tag{1.17}$$

(1.17) and the Gauss formula on $S$,

$$Rm(g)_{ijkl} = Rm(g(r))_{ijkl} + \frac{1}{4}\left( \frac{\partial}{\partial r} g_{ik} \frac{\partial}{\partial r} g_{jl} - \frac{\partial}{\partial r} g_{jk} \frac{\partial}{\partial r} g_{il} \right)$$

imply that there exists a constant $C = C(H, K, i_0, n) > 0$ *s.t.*

$$\begin{aligned} &\int_{\frac{\rho}{2}}^{\rho} \left( \int_{S_r} |Rm(g(r))|^{\frac{n}{2}} \, dg(r) \right) dr \\ &\qquad \le C + C \int_{\frac{\rho}{2}}^{\rho} \left( \int_{S_r} |Rm(g)|^{\frac{n}{2}} \, dg(r) \right) dr \\ &\qquad \le C + CK. \end{aligned}$$

This implies the existence of $r_\rho \in \left[\frac{\rho}{2}, \rho\right]$ and $C_4 = C_4(H, K, i_0, n) > 0$ *s.t.*

$$\int_{S_{r_\rho}} |Rm(r_\rho)|^{\frac{n}{2}} \, dg(r_\rho) \le C_4 r_\rho^{-1}.$$

□

We now state and prove the compactness estimate of the induced metric on small geodesic spheres.

Let $(M, g) \in \mathcal{L}'$, $\rho \le \frac{i_0}{4}$, let $r_\rho \in \left[\frac{\rho}{2}, \rho\right]$ as in Theorem 1.13. We have the following

THEOREM 1.18. *There exists* $C_{10} = c_{10}(H, K, i_0, n) > 0$ *and a* $C^\infty$ *Riemannian metric* $h(r_\rho)$ *on the geodesic sphere* $S_{r_\rho}$ s.t.

$$C_{10}^{-1} g(r_\rho) \le {r_\rho}^2 h(r_\rho) \le C_{10} g(r_\rho); \tag{1.19}$$

$$|Rm(h(r_\rho))| \le C_{10}. \tag{1.20}$$

*Proof.* Proposition 1.10 and Theorem 1.13 are sufficient for carrying through the argument in [**12**]. □

**2.** In this section, we show that the diameter of a small geodesic sphere is small. More precisely,

THEOREM 2.1. *There exists* $C_{11} = C_{11}(H, K, i_0, V, n)$ s.t. *for any* $(M, g) \in \mathcal{L}'$, *any* $r \in \left(0, \frac{i_0}{2}\right)$, $diam(g(r)) \leq C_{11} r$.

*Proof.* First observe that there exists a constant $C = C(H, K, i_0, V, n) > 0$ *s.t.*

$$diam\left(S_{\frac{i_0}{4}}\right) \leq C. \tag{2.2}$$

To prove (2.2), we normalize by scaling so that $i_0 = 4$. Let $\gamma$ be a minimal geodesic on the geodesic sphere $S_1(0)$. We show that there exists $\tilde{C} = \tilde{C}(H, i_0, V)$ *s.t.*

$$length\, \gamma \leq \tilde{C}.$$

Let $\alpha$ be any curve in the annulus $B_{\frac{3}{2}}(0) \backslash B_{\frac{1}{2}}(0)$ *s.t.* for $0 \leq t_1 < t_2 < \cdots \leq 1$, $\alpha|[t_i, t_{i+1}]$ is a minimal geodesic in the annulus. The geodesic balls centered at $\gamma(t_i)$ with radius $\delta$ can be made mutually disjoint by choosing $\delta > 0$ sufficiently small. Let $N$ be the number of these balls. By Gromov's relative volume estimate [**6**], the volume of each small bal is bounded from below by a constant $C' = C'(H, i_0, V, n)$. But the total volume of the mannifold $M$ is bounded from above by $V$ (cf. (0.6)). Hence $N \leq V/C'$. Since the induced metric $g(r_1)$ and $g(r_2)$ are equivalent (by Proposition 1.10), we can project $\alpha|[t_i, t_{i+1}]$ into $S_1(0)$, to get (2.2).

Next, apply (2.2) to the metric $g^\tau$ defined by $g^\tau(x) = \tau^{-2} g(\tau x)$. By scaling properties, we get

$$diam\,(g(r)) \leq C \frac{4r}{i_0}.$$

□

**3.** Let $(M,g)$ be in $\mathcal{L}'$. As before we use the geodesic polar coordinates at 0, i.e.,

$$g = dr^2 + \sum_{i,j=1}^{n-1} g_{ij}(x,r)\, dx^i\, dx^j = dr^2 + g(r),$$

where $g(r) = g(x,r)$ is the induced metric on the geodesic sphere $S_r(0)$.

We will begin with the following estimate:

PROPOSITION 3.1. *For* $\rho \le \frac{i_0}{4}$, $\eta \in (0,\rho)$, *we have*

$$\int_{T\left(\frac{\eta}{4},\frac{\eta}{2}\right)} \left( \max_{\eta \le \rho} \int_{S(x,r)} \left| B(x,r) + \frac{1}{r} g(x,r) \right|^{\frac{n}{2}} dg(r) \right) dg(x)$$

$$\le C(H,n,\eta,\rho) \int_{B(\rho+\eta)} |R_m(g)|^{\frac{n}{2}}\, dg,$$

*where* $B(x,r)$ *is the second fundamental form of* $S(x,r)$,

$$T\left(\frac{\eta}{4},\frac{\eta}{2}\right) = \left\{ x \in M | dist(x,0) \in \left(\frac{\eta}{4},\frac{\eta}{2}\right) \right\}.$$

*Proof.* Let $x \in T\left(\frac{\eta}{4},\frac{\eta}{2}\right)$, $y \in M$ *s.t.* $d(x,y) = \rho \le \frac{i_0}{2}$. Let $\gamma$ be the minimal geodesic from $x$ to $y$ with $\gamma(0) = x$, $\gamma(\rho) = y$, $d(x,y) = \rho$. Observe that, as a consequence of Proposition 1.10, there exists a constant $C_{12} = C_{12}(H, i_0, n) > 0$ *s.t.* for any Jacobi field $X$ on $\gamma$ with $X(\gamma(0)) = 0$, $< X(\gamma(l)), \gamma'(l) >= 0$, we have

$$|X(\gamma(t))| \le C_{12} |X(\gamma(l))|$$

$\forall\, t \in [0,l]$, where $l =$ the length of $\gamma$.

Let $E$ be the parallel vector field along $\gamma$ with

$$E(\gamma(l)) = X(\gamma(l)),$$

then the vector field $A$, defined by $A = X - \frac{t}{l}E$, is again a Jacobi field. Assume $|X(\gamma(l))| = 1$. We have

$$\int_0^l |A'|^2 = \int_0^l < A'', A > \; dt \le \int_0^l |Rm||X||A|\, dt$$

$$\le C_{12}(C_{12}+1) \int_\gamma |Rm| = C_{13} \int_\gamma |Rm|,$$

where $C_{13} = C_{13}(H, i_0, n)$.

Next, by a cut-off function argument, one can show that (c.f. [**12**], p.31)

$$|A'|^2(\gamma(l)) \leq C_{14} \int_\gamma |Rm|^2. \tag{3.2}$$

We claim that there exists $C_{15} = C_{15}(H, K, i_0, n)$ *s.t.*

$$\left|B(x,r) + \frac{1}{l} g(\gamma(l))\right|^2 (\gamma(l)) \leq C_{15} \int_\gamma |Rm|^2.$$

To see this, let $X, Y$ be vector fields on $S(x,l)$ *s.t.*

$$|X(\gamma(l))| = |Y(\gamma(l))| = 1,$$

and let $E, \overline{E}$ be parallel vector fields on $\gamma$ with

$$E(\gamma(l)) = X(\gamma(l)),$$

$$\overline{E}(\gamma(l)) = Y(\gamma(l)).$$

Extended $X, Y$ to the geodesic ball $B(x,l)$ *s.t.* they are Jacobi fields on each radial geodesic. Then, clearly $B(X,Y) = - < \nabla_{\gamma'}, X, Y >= - < X', Y >$. We have, from (3.2), that

$$\begin{aligned}
&|B(X,Y) + \frac{1}{l} < X, Y > |^2(\gamma(l)) \\
&= | < X', Y > - \frac{1}{l} < E, Y > |^2(\gamma(l)) \\
&= | < X' - \frac{1}{l} E, Y > |^2(\gamma(l)) \\
&\leq C_{14} |Y(\gamma(l))|^2 \int_\gamma |Rm|^2 = C_{14} \int_\gamma |Rm|^2.
\end{aligned}$$

To finish the proof, we define $f(x,y)$, for $x, y$ with $d(x,y) = \rho + \frac{\eta}{2} \leq \frac{i_0}{2}$, by

$$f(x,y) = \max_{\eta \leq r \leq \rho} \left|B(x,r) + \frac{1}{r} g(x,r)\right|^{\frac{n}{2}} (\gamma(r)),$$

where $\gamma$ is the minimal geodesic from $x$ to $y$, $r =$ distance from $x$.

Let

$$\Omega = \bigcup_{x\in T\left(\frac{\eta}{4},\frac{\eta}{2}\right)} S\left(x,\rho+\frac{\eta}{2}\right) \subset M,$$

and

$$\Sigma = \bigcup_{x\in T\left(\frac{\eta}{4},\frac{\eta}{2}\right)} \left(x, S\left(x,\rho+\frac{\eta}{2}\right)\right) \subset M\times M.$$

Then

$$\begin{aligned}\int_\Sigma\int f(x,y) &= \int_{x\in T\left(\frac{\eta}{4},\frac{\eta}{2}\right)}\left(\int_{S\left(x,\rho+\frac{\eta}{2}\right)} f(x,y)\,dg_x(y)\right)dg(x)\\ &= \int_\Omega\left(\int_{\Omega_y} f(x,y)\,dg_y(x)\right)\,dg(y),\end{aligned}$$

where $g_x$ is the induced metric of $S\left(x,\rho+\frac{\eta}{2}\right)$, and $\Omega_y = T\left(\frac{\eta}{4},\frac{\eta}{2}\right)\cap S\left(y,\rho+\frac{\eta}{2}\right)\subset S\left(y,\rho+\frac{\eta}{2}\right)$. We have

$$\int_\Sigma\int f(x,y) \le \int_\Omega\left(\int_{\Omega_y} f(x,y)\,dg_y(x)\right)\,dg(y).$$

Define $\overline{\gamma}(t)=\gamma(t)$ for $t\in[0,\rho]$. From (3.3) we get

$$\begin{aligned}&\int_{\Omega_y} f(x,y)\,dg_y(x)\\ &\le C(H,\eta,\rho)\int_{\Omega_y}\left(\int_{\overline{\gamma}}|Rm(g)|^{\frac{n}{2}}\right)dg_y\\ &\le C(H,\eta,\rho)\int_\delta^{\rho+\delta}\left(\int_{\Omega_y}|Rm(g)|^{\frac{n}{2}}\left(\gamma\left(\rho+\frac{\eta}{2}-t\right)\right)dg_y\right)\,dt.\end{aligned}$$

By Proposition 1.10,

$$dg_y\left(\gamma\left(\rho+\frac{\eta}{2}-t\right)\right)\ge C\left(H,n,\frac{\rho}{\eta}\right)dg_y(x).$$

Therefore

$$\int_{\Omega_y} f(x,y)\,dg_y(x)\le C\left(H,n,\eta,\frac{\rho}{\eta}\right)\int_{B(\rho+\eta)}|Rm(g)|^{\frac{n}{2}}dg.$$

Finally we have

$$\int_{\Omega_y} f(x,y) \leq C\left(H, n, \eta, \frac{\rho}{\eta}\right) vol\left(T\left(\frac{\eta}{4}, \rho+\eta\right)\right) \int_{B(\rho+\eta)} |Rm(g)|^{\frac{n}{2}} dg$$
$$\leq C\left(H, n, \eta, \frac{1}{\eta}, \rho, V, i_0\right) \int_{B(\rho+\eta)} |Rm(g)|^{\frac{n}{2}} dg.$$

□

Let $\dot{R}m(r)$ be the scalar curvature free curvature tensor of $g(r)$. We have the following proposition.

PROPOSITION 3.4. *For any $x \in T\left(\frac{\eta}{4}, \frac{\eta}{2}\right)$, where $\eta \in (0, \rho)$ with $\rho \leq \frac{i_0}{4}$, we have*

$$\int_{\eta}^{\rho} \left( \int_{S(x,r)} |\dot{R}m(r)|^{\frac{n}{4}} dg_x(r) \right) dr$$
$$\leq C(H, n, \eta, \rho, i_0) \left( \left( \int_{B_x(\rho)} |Rm(g)|^{\frac{n}{2}} dg \right)^{\frac{1}{2}} \right.$$
$$+ \left( \max_{\eta \leq \rho} \int_{S(x,r)} \left| A(r) + \frac{1}{r} g_x(r) \right|^{\frac{n}{2}} dg_x(r) \right)^{\frac{1}{2}}$$
$$\left. + \max_{\eta \leq \rho} \int_{S(x,r)} \left| A(r) + \frac{1}{r} g_x(r) \right|^{\frac{n}{2}} dg_x(r) \right).$$

*Proof.* $\dot{R}m(r)$ can be expressed as

$$(\dot{R}m(r))_{ijkl}$$
$$= (Rm(r))_{ijkl} - \frac{R(r)}{(n-1)(n-2)}(g_{ik}(r)g_{jl}(r) - g_{il}(r)g_{jk}(r)),$$

where $R(r)$ is the scalar curvature of $g(r)$. We have

$$\int_{S(x,r)} \left| B_{ik}(r)B_{jl}(r) - \frac{1}{r^2} g_{ik}(r) g_{jl}(r) \right|^{\frac{n}{4}} dg(r)$$

$$
\begin{aligned}
&= \int_{S(x,r)} B_{ik}(r)\left(B_{jl}(r) + \frac{1}{r} g_{jl}(r)\right) \\
&\qquad -\frac{1}{r} g_{jl}(r)\left(B_{ik}(r) + \frac{1}{r} g_{ik}(r)\right)^{\frac{n}{4}} \, dg(r) \\
&\leq C \int_{S(x,r)} |B|^{\frac{n}{4}} \left|B(r) + \frac{1}{r} g(r)\right|^{\frac{n}{4}} dg(r) \\
&\qquad + C \int_{S(x,r)} \left|B(r) + \frac{1}{r} g(r)\right|^{\frac{n}{4}} dg(r) \\
&\leq C \left(\int_{S(x,r)} \left|B(r) + \frac{1}{r} g(r)\right|^{\frac{n}{2}} dg(r)\right)^{\frac{1}{2}} \\
&\qquad + C \int_{S(x,r)} \left|B(r) + \frac{1}{r} g(r)\right|^{\frac{n}{2}} dg(r).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\int_{S(x,r)} \left|(B_{ik}B_{jl} - B_{il}B_{jk}) - \frac{1}{r^2}(g_{ik}g_{jl} - g_{il}g_{jk})\right|^{\frac{n}{4}} dg(r) \\
&\leq C(H, K, i_0, n) \left(\int_{S(x,r)} \left|B(r) + \tfrac{1}{r} g(r)\right|^{\frac{n}{2}} dg(r)\right)^{\frac{1}{2}} \\
&\qquad + C(H, K, i_0, n) \int_{S(x,r)} \left|B(r) + \tfrac{1}{r} g(r)\right|^{\frac{n}{2}} dg(r).
\end{aligned}
$$

By Gauss formula,

$$
(Rm(g))_{ijkl} = (Rm(g(r)))_{ijkl} + B_{ik}(r)B_{jl}(r) - B_{il}(r)B_{jk}(r).
$$

Therefore

$$
\begin{aligned}
&\int_\eta^\rho \left(\int_{S(x,r)} \left|R_{ijkl}(g(r)) - \frac{1}{r^2}(g_{ik}(r)g_{jl}(r)\right.\right. \\
&\qquad\qquad \left.\left. - g_{il}(r)g_{jk}(r))\right|^{fracn4} dg(r)\right) dr \\
&\leq C(H, n, \eta, \rho) \left(\int_{B(x,\rho)} |Rm(g)|^{\frac{n}{2}} \, dg\right)^{\frac{1}{2}} \\
&\qquad + C(H, n, \eta, \rho) \left(\max_{\eta \leq r \leq \rho} \int_{S(x,r)} \left|B(r) + \frac{1}{r} g(r)\right|^{\frac{n}{2}} dg(r)\right)^{\frac{1}{2}} \\
&\qquad + C(H, n, \eta, \rho) \left(\max_{\eta \leq r \leq \rho} \int_{S(x,r)} \left|B(r) + \frac{1}{r} g(r)\right|^{\frac{n}{2}} dg(r)\right).
\end{aligned}
$$

Observe that

$$\int_{T_x(\eta,\rho)} \left| R(r) - \frac{(n-1)(n-2)}{r^2} \right|^{\frac{n}{4}} dg$$
$$\leq C(H, K, i_0, n, \eta, \rho) \left( \int_{B(x,\rho)} |Rm(g)|^{\frac{n}{2}} dg \right)^{\frac{1}{2}} .$$

Hence (3.4) follows immediately. □

PROPOSITION 3.5. *For* $0 < \eta < \rho \leq \frac{i_0}{4}$, *let* $(M_k, g_k) \in \mathcal{L}'$, $x_k \in M_k$ *with* $dist(x_k, 0) \in \left(\frac{\eta}{4}, \frac{\eta}{2}\right)$. *Assume*

$$\eta_k = \max_{\eta \leq r \leq \rho} \int_{S(x,r)} \left| B(x_k, r) + \frac{1}{r} g_k(r) \right|^{\frac{n}{2}} dg_k(r) \to 0$$

*and*

$$\mu_k = \int_{B(x_k,\rho)} |Rm(g_k)|^{\frac{n}{2}} dg_k \to 0 \quad as \quad k \to \infty.$$

*Then there exists a diffeomorphism* $\phi_k : S(1) \to S(x_k, \rho)$ *for each* $k = 1, 2, 3, \cdots$, s.t.

$$\int_{S(1)} |\phi_k^* g_k(r) - r^2\, d\theta^2|^{\frac{n}{2}}\, d\theta \to 0$$

*uniformly for* $\eta \leq r \leq \rho$, *where* $S(1)$ *is the Euclidean unit sphere, and*

$$|\phi_k^* g_k(\rho) - \rho^2\, d\theta^2|_{C^0} \to 0 \quad as \quad k \to \infty.$$

*Proof.* Proposition 1.10 and Theorem 1.13 enable us to carry out the arguments in [**12**] (cf. 5.18, 5.21, and 5.25). □

**4.** In this section we prove the existence of a controllable harmonic coordinate system under the smallness condition of the $L^{n/2}$-norm of curvature tensor.

PROPOSITION 4.1. *For any* $\eta \in (0,1)$, *there exists* $\epsilon = \epsilon(H, n, i_0, \eta) > 0$ s.t. *if* $(M, g) \in \mathcal{L}$ *satisfies* $\int_M |Rm(g)|^{\frac{n}{2}} dg \leq$

$\epsilon$, *then there exists a diffeomorphism*

$$F = (h^1, h^2, \cdots, h^n): \quad T\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \to T\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \subset \mathbb{R}^n$$

*having the following properties:*

(a) $\Delta = 0$;

(b) $F^{-1}\left(T\left(1 + \frac{\eta}{4}, \frac{\eta}{4} + \eta\right)\right) \supset T(1-\eta, 2\eta)$ *and the image of* $F \supset T\left(1 + \frac{\eta}{4}, \frac{5\eta}{4}\right)$;

(c) $|h^{ij} - \delta^{ij}|_{C^0} < \frac{\eta^2}{100n}$ *on* $T\left(1 + \frac{\eta}{4}, \frac{5\eta}{4}\right)$; *where* $h^{ij} =< \nabla h^i, \nabla h^j >$;

(d) $|dh^{ij}|_{C^0} \leq C(H, n, \eta)$ *for some* $\alpha \in (0,1)$ *on* $T\left(1 + \frac{\eta}{4}, \frac{5\eta}{4}\right)$;

(e) $||F|^2 - r^2| \leq \frac{\eta}{100n}$ , *where* $|F|^2 = \sum_i (h^i)^2$, $r = dist(x, 0)$;

(f) $\|d^2 h^{ij}\|_{L^q} \leq C(H, n, \eta)$ *on* $T\left(1 + \frac{\eta}{4}, \frac{5\eta}{4}\right)$ *for some* $q > n$.

*Proof.* Suppose for $k = 1, 2, \cdots$, $(M_g, g_k) \in \mathcal{L}$ with $\int_{M_k} |Rm(g_k)|^{\frac{n}{2}} \leq \frac{1}{k}$.

Proposition 3.1 implies that $\exists y_k \in T\left(\frac{\eta}{2}, \frac{\eta}{4}\right)$ *s.t.*

$$\begin{aligned}
\eta_k &= \max_{\eta \leq r \leq 1} \int_{S_k(y_k, r)} \left| B_k(y_k, r) + \frac{1}{r} g_k(y_k, r) \right|^{\frac{n}{2}} dg_k(y_k, r) \\
&\leq C\left(H, n, i_0, \eta, \frac{1}{\eta}\right) \int_{B_2} |Rm(g_k)|^{\frac{n}{2}} dg_k \\
&\leq Ck^{-1}.
\end{aligned}$$

Proposition 3.5 implies that there exists $\phi_k : S_1 \to S_k(y_k) \approx S_1$ *s.t.*

$$\int_{T(1,\eta)} |\phi_k^* g_k - g_0|^{\frac{n}{2}} dg_0 < Ck^{-1},$$

where $\phi_k$ has been extended trivially to $T(1, \eta)$, $g_0$ is the flat metric on $B_1$. In the Euclidean coordinates $x = (x^1, \cdots, x^n)$, $g_0 = \delta_{ij}$.

Next we solve the Dirichlet problem

$$\begin{cases} \Delta F = 0 & \text{in } T(1, \eta) \\ F = x & \text{on } \partial T(1, \eta). \end{cases}$$

By Proposition 1.10, we can show (as in [**14**])

$$\int_{T(1,\eta)} |\nabla F - \nabla x|_g^2 \, dg \le \frac{1}{k} C\left(H, n, \frac{1}{\eta}, \eta, i_0\right).$$

By a standard argument involving DeGiorgi-Nash-Moser iteration, it follows that $F$ is the desired diffeomorphism. □

THEOREM 4.2. *For each $M_k, g_k \in \mathcal{L}$, there exists, for $l = 1, 2, \cdots$, open sets $F_k(l) \subset M_k$ s.t. $F_k(l+1) \supset F_k(l)$ and $F_k(l) \cup B(l^{-1}) = M_k$. There also exists a diffeomorphism $\phi_k(l)$ for each pair of $k$ and $l$ : $\phi_k(l) : T(1, l^{-1}) \subset \mathbb{R}^n \to F_k(l)$ such that $\phi_k(l)^* g_k$ converges in $C^{1,\alpha}$ norm to some $C^{1,\alpha}$ metric $g_l'$ on $T(1, l^{-1}) \subset \mathbb{R}^n$.*

*Proof.* By rescaling, we can assume that $g_k$ satisfies

$$\int_{M_k} |Rm(g_k)|^{\frac{n}{2}} \, dg_k \le \epsilon$$

where $\epsilon > 0$ is given by Proposition 4.1. Therefore we have harmonic coordinates

$$h^k : T_k\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \subset M_k \to D(\eta) = T\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \subset \mathbb{R}^n,$$

satisfying (a)-(f) of 4.1. Taking $\eta = l^{-1}$, by the Hölder estimate (d), we have, for each $l = 1, 2, \cdots$, a subsequence of $(M_k, g_k)$, denoted by $g_k(l)$, *s.t.* $g_k(l)$ converges in the $C^2$-norm on $T_k\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right) \subset M$ to a $C^{1,\alpha}$ metric $g_l'$ on $D(l)$. We can then take

$$F_k(l) = T_k\left(1 + \frac{\eta}{2}, \frac{3\eta}{2}\right), \quad \eta = \frac{1}{l}.$$

By passing to a subsequence if necessary, we can make $F_k(l+1) \supset F_k(l)$. □

THEOREM 4.3. *Let $g'$ be a metric on $M' \cong B_1 | \{0\}$ defined by $g'(x) = g_l'(x)$ if $x \in F_k(l)$. Then $g'$ can be extended as a $C^0$ metric on $B_1$.*

*Proof.* Theorem 2.1 says that the diameter of a small geodesic sphere around 0 is small. Hence 0 is the only possible singularity. To show that 0 is a removable singular point, let, for fixed $N = 1, 2, \cdots$,

$$C(\rho, N) = \left\{x \in M' | \frac{\rho}{N} < d(x, 0) < 2\rho\right\}.$$

By Theorem 4.2, a subsequence $(M_k, g_k)$ converges to $M'$ away from 0. Thus for each $\rho, \exists k = k(\rho), \exists$ a submanifold $C_k(\rho, N) \subset (M_k, g_k), \exists y_\rho \in C_k(\rho, N)$ *s.t.* $y_\rho \to x_\rho \in C(\rho, N)$ (with $dist(x_\rho, 0) = \rho$), and such that

$$\left| \int_{C_k(\rho,N)} |Rm(g_k)|^{\frac{n}{2}} \, dg_k - \int_{C(\rho,N)} |RM(g')|^{\frac{n}{2}} \, dg' \right| \leq \rho^2,$$

and

$$\left\| \left(\frac{1}{\rho} C(\rho, N), x_\rho\right) - \left(\frac{1}{\rho} C_k(\rho, N), y_k\right) \right\|_{C^{1,\alpha}} < \rho.$$

By (0.5),

$$\int_{C(\rho,N)} |RM(g')|^{\frac{n}{2}} \, dg' \to 0 \quad \text{as} \quad \rho \to 0.$$

Consequently,

$$\int_{C_k(\rho,N)} |RM(g_k)|^{\frac{n}{2}} \, dg_k \to 0 \quad \text{as} \quad \rho \to 0.$$

Therefore, from the zero pinching theorem of [**12**], it follows that $\left(\frac{1}{\rho} C_k(\rho, N), y_\rho\right)$ converges to a flat manifold $D_N$ in $C^{1,\alpha}$-norm as $\rho \to 0$. Thus $\left(\frac{1}{\rho} C(\rho, N), x_\rho\right)$ converges to $(D_N, e_N)$ in $C^{1,\alpha}$-norm. The direct union of $(D_N, e_N)$ has to be $(U(0), e)$ where 0 is the isolated singular point, $e$ is a unit vector in $|BbbR^n$, and $U(0)$ is a simply connected flat manifold since $\frac{1}{\rho} C(\rho, N)$ is the $C^{1,\alpha}$ limit of simply connected manifolds $\frac{1}{\rho} C_k(\rho, N)$. Hence $U(0) \cong B(2) - \{0\}$. Letting $N \to \infty$ have that $\left(\frac{1}{\rho} C(\rho, 0), x_\rho\right)$ converges to $\{B(2) - \{0\}, e\}$ in $C^{1,\alpha}$-norm. It follows that $g'$ can extend to a $C^0$ metric on $M'$, diffeomorphic to $B_1 \subset \mathbb{R}^n$. □

REMARK. In the case $(M_k, g_k) \in \mathcal{L}'$, we use Proposition 3.5 directly in place of Proposition 4.1 and Theorem 4.2. This, combined with Theorem 4.3, proves Theorem (0.8).

REMARK. Let $O$ be the set of compact orbifolds with finitely many singular points, satisfying (0.3)-(0.6). Let $\Gamma$ be the group acting on these orbifolds. We can lift a neighbourhood of each singular point via $\Gamma$ to $B^n$. It then follows from Theorem (0.7) that $O$ has the same compactness property.

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