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R. DANIEL MAULDIN AND MARIUSZ URBANSKI

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CONFORMAL REPELLORS WITH DIMENSION ONE ARE JORDAN CURVES

R. DANIEL MAULDIN AND M. URBAŃSKI

We show that a conformal repellor in $\mathbb{R}^m$ whose Hausdorff and topological dimensions are equal to 1 is a Jordan curve. Moreover, its 1-dimensional Hausdorff measure is finite and it has a tangent at every point.

Introduction. In this note we study the topological and metric structure of conformal repellor $X \subset \mathbb{R}^m$, $m \geq 1$, of topological dimension 1. The definition of conformal repellor is given in the next two sections. We then show the following dichotomy: either the Hausdorff dimension of X exceeds 1 or else X is a Jordan curve (simple closed curve) and its 1-dimensional Hausdorff measure is positive and finite. Moreover, in the latter case X has a tangent at every point - X is smooth. This result generalizes Lemma 3 of [**PUZ**] which is formulated in the plane case ($m = 2$). The proof contained in [**PUZ**] uses the Riemann mapping theorem and can be carried out only in the plane. The proof presented in our paper is different and holds in any dimension. The reader is also encouraged to notice an analogy between our result and a series of other recent papers (see for examples [**B, P, R1, S, U, Z1, Z2**]) which are aimed toward establishing a similar dichotomy. However, to our knowledge, all these results were formulated in the plane case and have as a starting point the assumption that X is a continuum. Then the dichotomy is only that either the Hausdorff dimension of X exceeds 1 or X is a smooth curve.

THE SETTING. Let X be a nonempty compact subset of $\mathbb{R}^m$, U an open set, $X \subset U$ and f a map of U into $\mathbb{R}^m$ of class $C^{1+\alpha}, 0 < \alpha$, such that

(i) $f(X) \subset X$

(ii) there is some $k \geq 1$ such that for each x in X and $0 \neq v \in \mathbb{R}^m$,

$$\|D_x f^k(v)\| > \|v\|.$$

(iii) $\bigcap_{n\geq 0} f^{-n}(U) = X$ (X is a repellor for f).

(iv) if $\emptyset \neq V$ is relatively open with respect to X, then there is some $k \geq 0$ with $f^k(V) = X$. (f is locally eventually onto or topologically exact.)

This is our fundamental setting. Our goal is to determine additional conditions under which X must be a Jordan curve. We make the following conjecture.

CONJECTURE . *If* $\dim_{\text{top}}(X) = 1$, *then either* X *is a smooth Jordan curve or* $HD(X) > 1$.

Towards this end, we shall assume from this point on that

(v) $$\dim_{\text{top}}(X) = 1.$$

Our results are a mixture of dynamical, geometric and topological methods some of which are laid out next.

1. Notations and theorems. Let us recall some facts and theorems we will require. First of all, since X is compact, by condition (ii), we can fix B, $1 > B > 0$, $\lambda > 1$ and a compact neighborhood U of X such that for each x in U, $n \geq 1$, and $0 \neq v \in \mathbb{R}^m$, $\|D_x f^n(v)\| \geq B\lambda^n \|v\|$. We will make considerable use of the following basic theorem which follows by repeatedly applying the inverse function theorem.

THEOREM . *There is some* R, $1 > R > 0$, *such that for each* x *in* X, $B(x,R) \subset U$ *and for all positive integers* n, *there is a homeomorphism* f_x^{-n} *of* $B(f^n(x),R) \to U$ *such that* $f_x^{-n}(f^n(x)) = x$ *and* $f_x^{-n}(f^n(u)) = u$, *if* $u \in B(f^n(x),R)$.

We will say that X *has a strong tangent in the direction* θ *at* x provided for each β, with $0 < \beta < 1$, there is some $0 < r$ such that $X \bigcap B(x,r) \subset S(x,\theta,\beta)$. We will use the following theorem. Since we could not find it stated, we provide a proof.

THEOREM 1. *If X is locally arcwise connected at a and X has a tangent θ at a, then X has strong tangent θ at a.*

Proof. Suppose there is some $0 < \beta < 1$ and points x_n in X such that for each n, $|x_n - a - < x_n - a, \theta > \theta| > \beta |x_n - a|$. For each n, let $\alpha_n : [0,1] \to X$ be an arc from a to x_n with $\operatorname{diam}(\alpha_n) \to 0$. For each n, note that since X has tangent θ at a, there is some $t, 0 < t < 1$ such that $\alpha_n(t) \in S(a, \theta, \beta/2)$. Let t_n be the largest number such that $\alpha_n(t_n) \in S(a, \theta, \beta/2)$ and let s_n be the first number larger than t_n such that $\alpha_n(s_n) \in \partial S(a, \theta, \beta)$. Consider y_n a point of the arc α_n from $\alpha_n(t_n)$ to $\alpha_n(s_n)$ at maximum distance from a and take z_n to be a point of this same subarc at minimum distance from a. If $\|z_n - a\| \le \|y_n - a\|/2$, then considering the projection of this subarc on the line through a and y_n,

$$H^1 \left(X \bigcap B(a, \|y_n - a\|) \backslash S(a, \theta, \beta/2) \right) \ge \|y_n - a\|/2.$$

If $\|z_n - a\| > \|y_n - a\|/2$, then, considering the projection of this subarc on the sphere with center a and radius $\|y_n - a\|/2$, we get $H^1 (X \bigcap B(a, \|y_n - a\|) \backslash S(a, \theta, \beta/2)) \ge (\pi/2)(\beta/2)\|y_n - a\|$. Thus, X does not have a tangent at a. □

REMARK. More generally, this theorem remains true if X is only assumed to be connected im kleinen at a.

Let T be the set of points of X which are f-transitive with respect to X; $x \in T$ if and only if $\omega(x)$, the ω-limit set of x is X, where $\omega(x) = \bigcap_{n \ge 1} \operatorname{cl}\{f^p(x) : p \ge n\}$. We recall that T is a dense G_δ subset of X and if μ is an ergodic probability measure on X which gives each nonempty open set positive measure, then $\mu(T) = 1$.

QUESTION. If $0 < H^1(X) < \infty$, does there exist an ergodic measure on X equivalent to $H^1|_X$ which gives positive measure to the nonempty open sets?

REMARK. Note that if $m \ll H^1|_X = \nu$, then $m \circ f^{-1} \ll \nu$ and f is nonsingular with respect to ν. So, we are looking for a positive fixed point of the well-known Perron-Frobenius operator $P : L^1(\nu) \to L^1(\nu)$, where $P(\varphi)(x) = \sum_{y \in f^{-1}(x)} \left(\frac{d\nu \circ f}{d\nu}(y) \right)^{-1} \varphi(y)$.

The difficulty here is that except for the conformal case $\left(\text{where } \frac{d\nu \circ f}{d\nu}(x) = f'(x)\right)$, we do not know whether the distortion of the Radon-Nikodym derivatives, $\frac{d\nu \circ f^n}{d\nu}$, $n \geq 1$, is uniformly bounded away from infinity.

LEMMA 1. *Suppose conditions* (i)-(iv) *hold. If X has a degenerate component, them the component of each transitive point of X is degenerate.*

Proof. Let x be a point of X such that the component of X containing x is $\{x\}$. Let V be an open neighborhood of x with diameter $< R/2$ and such that $\partial_X(V) = \emptyset$. Let z be a transitive point of X and let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers such that for each k, $f^{n_k}(z) \in V$. Then the sets $\{f_z^{-n_k}(V)\}_{k=1}^{\infty}$ form a basis for the topology at z and $\partial_X\left(f_z^{-n_k}(V)\right) = 0$. □

2. The conformal case. In this section, we assume that conditions (i)-(v) hold and, in addition (vi) f is conformal at each point of X.

Our goal is to prove the following theorem.

THEOREM . *Suppose conditions (i) through (vi) hold, then either X is a smooth Jordan curve or $HD(X) > 1$. Moreover, if X is a smooth Jordan curve, then f is topologically equivalent to $z \to z^k$ on the unit circle, where k is the degree of f.*

We will make use of the following fundamental result. Suppose conditions (i)-(iv) hold and condition (vi) holds. If $HD(X) = t$, then $0 < H^t(X) < +\infty$ and there is and ergodic probability measure μ equivalent to H^t which is positive on the nonempty open subsets of X [**R2**]. To prove the theorem, let us assume $HD(X) = 1$. We will make use of the corresponding ergodic measure μ. The theorem will be proved be a series of lemmas.

LEMMA 2. *X is a continuum.*

Proof. If each component of X were degenerate, then X would have topological dimension zero. So, some component, C, of X is non-degenerate. We have $\mu(C) > 0$, since $H^1(C) > 0$ Actually, every component of X is non-degenerate. Otherwise, by Lemma 1, T, the set of transitive points is totally disconnected and $C \subset X \backslash T$. But, $\mu(X \backslash T) = 0$.

Since each component of X is non-degenerate and has positive H^1-measure, X has only countably many components and by Baire category theorem, some component has a non-empty interior. Therefore since f is topologically exact X is a continuum. □

QUESTION. Suppose conditions (i)-(v) hold and $HD(X) = 1$. Is it true that X is a continuum?

We recall now that since X is a continuum and $0 < H^1(X) < \infty$, X has finite degree [**EH**]. Such continua are Peano continua and also, X is a regular curve. Indeed, X is a continuum of finite degree, see ([**EH, Wh1**]). We recall that a point x of X has order $\leq n$ with respect to X, $\operatorname{ord}(x) \leq n$, provided there is a neighborhood base of X at x such that the boundary of each of these sets with respect to X has cardinality $\leq n$. Also, $\operatorname{ord}(x) = n$ means $\operatorname{ord}(x) \leq n$ and the order of x is not $\leq n - 1$.

LEMMA 3. *For each $x \in X$, $\operatorname{ord}(x) \geq 2$.*

Proof. Let $x \in X$ and assume $\operatorname{ord}(x) \geq 1$. Since X is a continuum, $\operatorname{ord}(x) = 1$. In particular, there is a relatively open neighborhood V of x with diameter $< R/2$ and such that $\operatorname{card} \partial_X(V) = 1$. Let z be a transitive point of X and let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers such that for each k, $f^{n_k}(z) \in V$. Then the sets $\{f_z^{-n_k}(V)\}_{k=1}^{\infty}$ form a neighborhood basis for the topology at z and $\operatorname{card}\Big(\partial_J(f_z^{-n_k}(V))\Big) = 1$. Thus, the order of X at each transitive point is 1.

On the other hand, X being arcwise connected, contains a set of positive μ measure consisting of points of order at least 2. Thus, some transitive point of X has order at least 2. This contradiction proves the lemma. $\square$

QUESTION. Suppose conditions (1)-(5) hold, $HD(X) = 1$ and X is a continuum. Is it true that X has no end points, i.e., the order of each point of X is ≥ 2?

Fix τ, $0 < \tau < \alpha$. Since f is of class $C^{1+\alpha}$, we can fix some R_1 such that $0 < R_1 \leq R$ and $\forall x \in X \quad \forall z \in B\big(f(x), R_1\big)$

$$(1) \quad \|f_x^{-1}\big(f(x)\big) - f_x^{-1}(z) - D_{f(x)}f_x^{-1}\big(f(x) - z\big)\| \leq \|D_{f(x)}f_x^{-1}\| \cdot \|f(x) - z\|^{1+\tau}.$$

Set

$$K = \prod_{j=0}^{\infty}\left(1 + B^{-\tau}\lambda^{-j\tau}R_1^\tau\right).$$

Of course, $1 < K < \infty$.

Let us make the following notations. Let $x \in X$, $n \geq 1$ and $z \in B\big(f^n(x), R_1\big)$. For $0 \leq k \leq n$ set $x_k = f^{n-k}\big(f_x^{-n}(f^n(x))\big) = f^{n-k}(x)$ and $z_k = f^{n-k}\big(f_x^{-n}(z)\big)$. So, $x_0 = f^n(x)$ and $z_0 = z$. Notice that for $0 \leq k \leq n-1, f(x_{k+1}) = x_k$ and $f(z_{k+1}) = z_k$. Also, note that for each k,

$$(2) \qquad \|x_k - z_k\| \leq \sup_{y \in [x_0, z_0]} \|D_y f_{x_k}^{-k}\| \cdot \|x_0 - z_0\|.$$

We also make the conventions that $1 = \prod_{j=0}^{-1} a_j$ and $\sum_{j=1}^{0} a_j = 0$.

Let $0 < R_2 < \min(B^{-1}R_1, R_1, B\lambda R_1)$.

LEMMA 4. *Let $x \in X, n$ be a positive integer and $z \in B\big(f^n(x), R_2\big)$. Then $\forall 0 \leq k \leq n$,*

$$(3) \quad \|x_k - z_k\| \leq \prod_{j=0}^{k-1} \|D_{x_j} f_{x_{j+1}}^{-1}\| \cdot \|x_0 - z_0\| \cdot \prod_{j=0}^{k-1}\left(1 + \|x_j - z_j\|^\tau\right)$$

and

$$(4) \qquad \|x_k - z_k\| \leq K\|D_{x_0} f_{x_k}^{-k}\| \cdot \|x_0 - z_0\|.$$

Proof. Inequality (3) is verified by induction on k. By convention, it holds if $k = 0$. Now, $\|x_1 - z_1\| = \left\| f_{x_1}^{-1}\big(f(x_1)\big) - f_{x_1}^{-1}(z_0)\right\|$. Also, $\|f(x_1) - z_0\| = \|x_0 - z_0\| < R_2 \leq R_1$. So, applying (1), we find

$$\|x_1 - z_1\| \leq \|D_{f(x_1)} f_{x_1}^{-1}\| \cdot \|f(x_1) - z_0\|^{1+\tau} + \left\| D_{f(x_1)} f_{x_1}^{-1}\big(f(x_1) - z_0\big)\right\|.$$

Again, since $x_0 = f(x_1)$,

$$\|x_1 - z_1\| \leq \|D_{x_0} f_{x_1}^{-1}\| \cdot \|x_0 - z_0\|(1 + \|x_0 - z_0\|^\tau).$$

Assume (3) holds for k, $1 \leq k \leq n-1$. Then $\|x_{k+1} - z_{k+1}\| = \left\| f_{x_{k+1}}^{-1}\big(f(x_{k+1})\big) - f_{x_{k+1}}^{-1}(z_k)\right\|$. Also, $\|f(x_{k+1}) - z_k\| = \|x_k - z_k\|$. From (2), we have $\|f(x_{k+1}) - z_k\| \leq B^{-1}\lambda^{-k}\|x_0 - z_0\| \leq B^{-1}\lambda^{-1}R_2 < R_1$. Proceeding as before, we find that

$$\|x_{k+1} - z_{k+1}\| \leq \prod_{j=0}^{k} \|D_{x_j} f_{x_{j+1}}^{-1}\| \cdot \|x_0 - z_0\| \cdot \prod_{j=0}^{k}\Big(1 + \|x_j - z_j\|^\tau\Big).$$

Thus, inequality (3) holds for all k, $0 \leq k \leq n$. To see that (4) holds note that for each j, it follows from (2) that

$$\|x_j - z_j\| \leq B^{-1}\lambda^{-j}\|x_0 - z_0\| \leq B^{-1}\lambda^{-j}R_1.$$

Also, since f is conformal, $\prod_{j=0}^{k-1} \|D_{x_j} f_{x_{j+1}}^{-1}\| = \|D_{x_0} f_{x_k}^{-k}\|$. Inequality (4) now follows from (3). $\square$

LEMMA 5. *For each $\epsilon > 0$, there is some $0 < r(\epsilon) < R_2$ such that the following statement is true:*
$\forall x \in X \quad \forall n \geq 1 \quad \forall z \in B\big(f^n(x), R(\epsilon)\big)$,

(5)
$$\begin{aligned}\left\| f_x^{-n}(z) - f_x^{-n}(f^n(x)) - D_{f^n(x)} f_x^{-n}\big(z - f^n(x)\big)\right\| &\leq \epsilon \left\| D_{f^n(x)} f_x^{-n}\big(z - f^n(x)\big)\right\| \\ &= \epsilon \|D_{f^n(x)} f_x^{-n}\| \cdot \left\|\big(z - f^n(x)\big)\right\|.\end{aligned}$$

Proof. Fix $0 < \beta, \gamma$ such hat $\beta + \gamma < \tau$. Let $\epsilon > 0$. Choose $R(\epsilon)$ such that $0 < R(\epsilon) < R_2$, $KB^{-\beta}R(\epsilon)^\beta \leq 1$ and $R(\epsilon)^\gamma\Big(1 +$

$B^{-\gamma}\sum_{j=1}^{\infty}\lambda^{-j\gamma}\Big) < \epsilon$. Now, let $x \in X$ and $n \geq 1$. Suppose $z \in B\big(f^n(x), R(\epsilon)\big)$. From (1) we have

$$\text{(6)}\quad \|z_1 - x_1 - D_{x_0} f_{x_1}^{-1}(z_0 - x_0)\| \leq \|D_{x_0} f_{x_1}^{-1}\| \cdot \|x_0 - z_0\| \cdot \|x_0 - z_0\|^{\beta+\gamma}.$$

For $0 \leq k \leq n$, set $\Delta_k = \|z_k - x_k - D_{x_0} f_{x_k}^{-k}(z_0 - x_0)\|$. So, $\Delta_0 = 0$ and if $0 < k \leq n$,

$$\text{(7)}\quad \Delta_k \leq \|D_{x_{k-1}} f_{x_k}^{-1}(z_{k-1} - x_{k-1}) - D_{x_0} f_{x_k}^{-k}(z_0 - x_0)\| + \|z_k - x_k - D_{x_{k-1}} f_{x_k}^{-1}(z_{k-1} - x_{k-1})\|.$$

Applying the chain rule to the first term and (6) to the second term,

$$\Delta_k \leq \|D_{x_{k-1}} f_{x_k}^{-1}\| \cdot \|z_{k-1} - x_{k-1} - D_{x_0} f_{x_{k-1}}^{-(k-1)}(z_0 - x_0)\|$$
$$+\|D_{x_{k-1}} f_{x_k}^{-1}\| \cdot \|z_{k-1} - x_{k-1}\| \cdot \|z_{k-1} - x_{k-1}\|^{\beta+\gamma}.$$

By conformality, $\|D_{x_{k-1}} f_{x_k}^{-1}\| \cdot \|D_{x_0} f_{x_k}^{-(k-1)}\| = \|D_{x_0} f_{x_k}^{-1}\|$, so applying (4) to $\|z_{k-1} - x_{k-1}\|$, we have

$$\Delta_k \leq \|D_{x_{k-1}} f_{x_k}^{-1}\|\Delta_{k-1} + K\|D_{x_0} f_{x_k}^{-k}\| \cdot \|z_0 - x_0\| \cdot \|z_{k-1} - x_{k-1}\|^{\beta+\gamma}.$$

By the conditions placed on $R(\epsilon)$, $K\|z_{k-1} - x_{k-1}\|^{\beta} \leq 1$, so

$$\text{(8)}\quad \Delta_k \leq \|D_{x_{k-1}} f_{x_k}^{-1}\|\Delta_{k-1} + \|D_{x_0} f_{x_k}^{-k}\| \cdot \|(z_0 - x_0)\| \cdot \|z_{k-1} - x_{k-1}\|^{\gamma}.$$

□

CLAIM . *For* $1 \leq k \leq n$,

$$\text{(9)}\qquad \Delta_k \leq \|D_{x_0} f_{x_k}^{-k}\| \cdot \|z_0 - x_0\|^{1+\gamma} \cdot \left(1 + B^{-\gamma}\sum_{j=1}^{k-1}\lambda^{-j\gamma}\right).$$

Proof of claim. Since $\Delta_0 = 0$, by (8), $\Delta_1 \leq \|D_{x_0} f_{x_1}^{-1}\| \cdot \|z_0 - x_0\|^{1+\gamma}$. Assume $1 \leq k < n$ and (9) holds, then by (8), conformality and the fact that $\|z_k - x_k\|^{\gamma} \leq B^{-\gamma}\lambda^{-k\gamma}$,

$$\Delta_{k+1} \leq \|D_{x_0} f_{x_{k+1}}^{-(k+1)}\| \cdot \|z_0 - x_0\|^{1+\gamma} \cdot \left(1 + B^{-\gamma}\sum_{j=1}^{k}\lambda^{-j\gamma}\right).$$

In particular,

$$\begin{aligned}\Delta_n &= \left\| f_x^{-n}(z) - f_x^{-n}\big(f^n(x)\big) - D_{f^n(x)} f_x^{-n}\big(z - f^n(x)\big) \right\| \\ &\le \|D_{x_0} f_{x_n}^{-n}\| \cdot \|z_0 - x_0\| \cdot \|z_0 - x_0\|^\gamma \cdot \left(1 + B^{-\gamma} \sum_{j=1}^{\infty} \lambda^{-j\gamma}\right). \\ &\le \epsilon \|D_{f^n(x)} f_x^{-n}\| \cdot \|z - f^n(x)\|.\end{aligned}$$

□

LEMMA 6. $\exists \epsilon_0 > 0 \quad \forall \epsilon_0 > \epsilon > 0 \quad \forall x \in X \quad \forall n \ge 1 \quad \forall u \in R^m \quad 0 < \|u\| < R(\epsilon)$

(10)
$$f_x^{-n}\Big([f^n(x), f^n(x) + u]\Big) \subset S\Big(x, D_{f^n(x)} f_x^{-n}(u), 2\epsilon, B^{-1}\lambda^{-n} R(\epsilon)\Big).$$

Proof. Fix $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then $\epsilon/1 - \epsilon < 2\epsilon$. Let $y = f_x^{-n}(z)$, where $z \in \big(f^n(x), f^n(x) + u\big)$. By (2), $\|y - x\| = \|f_x^{-n}(z) - x\| \le B^{-1}\lambda^{-n}\|z - f^n(x)\|$. It remains to show that $y \in S\Big(x, D_{f^n(x)} f_x^{-n}\big(z - f^n(x)\big), 2\epsilon\Big)$. Or, it remains to show,

$$\left|\sin \measuredangle\Big(y - x, D_{f^n(x)} f_x^{-n}\big(z - f^n(x)\big)\Big)\right| \le 2\epsilon.$$

From (5), we have

$$\left\| f_x^{-n}(z) - f_x^{-n}\big(f^n(x)\big) \right\| \ge (1 - \epsilon) \left\| D_{f^n(x)} f_x^{-n}\big(z - f^n(x)\big) \right\|.$$

So , using the fact that $|\sin \measuredangle(a.b)| \le \|b - a\| / \min\big(\|a\|, \|b\|\big)$, we have

$$\left|\sin \measuredangle\Big(y - x, D_{f^n(x)} f_x^{-n}\big(z - f^n(x)\big)\Big)\right| \le \frac{\epsilon}{1 - \epsilon} < 2\epsilon.$$

□

LEMMA 7. *If X has a strong tangent at x, then X has a strong tangent at every point y of $\omega(x)$.*

Proof. Let $u \in R^m$ be a unit tangent vector for X at x. By compactness of X and S_{m-1}, there exists some v with $\|v\| = 1$

and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} f^{n_k}(x) = y$ and, setting $u_k = D_x f^{n_k}(u)$, $\lim_{n\to\infty} u_k/\|u_k\| = v$. Fix $\delta > 0$ and let $q \geq 1$ be so large that $S_c(x, u, \delta/4, r) \bigcap X = \emptyset$ for every $0 < r \leq \lambda^{-q}$. We claim that

$$X \bigcap S^c\Big(y, v, \delta, R(\delta/8)\Big) = \emptyset. \tag{11}$$

By way of contradiction, suppose $z \in X \bigcap S^c\Big(y, v, \delta, R(\delta/8)\Big)$. Then if n_k is large enough, $z \in X \bigcap S^c\Big(f^{n_k}(x), u_k, \delta/2, R(\delta/8)\Big)$. Now by Lemma 6

$$f_x^{-n_k}(z) \in S\Big(x, D_{f^{n_k}(x)} f_x^{-n_k}\Big(z - f^{n_k}(x)\Big), \delta/4, B^{-1}\lambda^{-n_k} R(\delta/8)\Big).$$

By conformality,

$$\measuredangle\Big(D_{f^{n_k}(x)} f_x^{-n_k}\Big(z - f^{n_k}(x)\Big), u\Big) = \measuredangle\Big(z - f^{n_k}(x), u_k\Big).$$

This implies that $f_x^{-n_k}(z) \in X \bigcap S^c\Big(x, u, \delta/4, B^{-1}\lambda^{-n_k} R(\delta/8)\Big)$. This contradiction proves (11) which shows that X has strong tangent v at the point y. □

COROLLARY . *The continuum X has a strong tangent at every point.*

Proof. Since X is a regular 1-set, X has a tangent and and therefore a strong tangent at H^1-a.e. point. Thus, X has a strong tangent at some transitive point and therefore at all of its points. □

LEMMA 8. *There are not three points $y_1, y_2, y \in X$ such that $y_1, y_2 \in B\Big(y, R(\pi/16)\Big)$ and $\pi/4 < |\measuredangle(y_1, y, y_2)| < 3\pi/4$.*

Proof. Suppose that three such points exist. Then there is some $\epsilon > 0$ such that for every $z \in B(y, \epsilon)$

$$y_1, y_2 \in B\Big(z, R(\pi/16)\Big) \text{ and } \pi/4 < |\measuredangle(y_1, z, y_2)| < 3\pi/4. \tag{12}$$

Now, let x be a transitive point and $n_1, n_2, n_3, \ldots$ an increasing sequence of positive integers such that for all k, $f^{n_k}(x) \in X \bigcap B(y, \epsilon)$. Applying Lemma 6 and (11), we see that for every $k \geq 1$,

$$\pi/8 < |\measuredangle\Big(f_x^{-n_k}(y_1), x, f_x^{-n_k}(y_2)\Big)| < 7\pi/8.$$

Since $f_x^{-n_k}(y_1)$ and $f_x^{-n_k}(y_2)$ both converge to x, this implies that X does not have a strong tangent at x. This contradicts the corollary and proves the lemma. □

LEMMA 9. *Let $x \in X$. If γ is an arc from $x_1 \neq x$ to $x_2 \neq x \in X \bigcap B\big(x, R(\pi/16)\big)$, then* $|\measuredangle(x_1, x, x_2)| \geq 3\pi/4$

Proof. Suppose that $|\measuredangle(x_1, x, x_2)| < 3\pi/4$. Then by Lemma 8, $|\measuredangle(x_1, x, x_2)| \leq \pi/4$. So, either $|\measuredangle(x, x_1, x_2)| > \pi/4$ or $|\measuredangle(x, x_2, x_1)| > \pi/4$. Let us assume $|\measuredangle(x, x_1, x_2)| > \pi/4$. Again, by Lemma 8, $|\measuredangle(x, x_1, x_2)| \geq 3\pi/4$. Thus, $|\measuredangle(x, x_2, x_1)| < \pi/4$. As the point z moves along the arcγ from x to x_2, $|\measuredangle(z, x_1, x_2)|$ varies continuously from $|\measuredangle(x, x_1, x_2)|$ to 0. Since $|\measuredangle(z, x_2, x_1)|$ and $|\measuredangle(x_1, z, x_2)|$ also vary continuously, it follows from Lemma 8 that both are bounded above by $\pi/4$. Thus, we would find some z such that the angles of the triangle with vertices x_1, z and x_2 would not sum to π. This contradiction proves the lemma. □

LEMMA 10. *For every $x \in X$,* $\mathrm{ord}(x) \leq 2$.

Proof. Suppose that there is some $y \in X$ with $\mathrm{ord}(y) \geq 3$. Since X is locally connected, it follows from Menger's, "n-Beinsatz" theorem ([**K**, p.277], [**M**, p.213]) that there are three arcs γ_1, γ_2 and $\gamma_3 \subset X$ all having y as a common endpoint and which are otherwise pairwise disjoint. Choose $0 < \delta < R(\pi/16)/2$ so small that each arc γ_i contains a point at distance δ from y. Also, let y_i, $i = 1, 2, 3$ be the first point on γ_i in the order starting at y at distance δ from y. Let $\hat{\gamma}_1$ be the subarc of γ_i from y to y_i. Thus

$$\big(\hat{\gamma}_1 \cup \hat{\gamma}_2\big) \bigcup \big(\hat{\gamma}_1 \cup \hat{\gamma}_3\big) \bigcup \big(\hat{\gamma}_2 \cup \hat{\gamma}_3\big) \subset B\big(y, (\pi/16)/2\big).$$

Applying Lemma 9, we get

$$|\measuredangle(y_1, y, y_2)|, |\measuredangle(y_1, y, y_3)|, |\measuredangle(y_2, y, y_3)| \geq 3\pi/4. \tag{13}$$

But, the inequalities $|\measuredangle(y_1, y, y_2)| \geq 3\pi/4$ and $|\measuredangle(y_1, y, y_3)| \geq 3\pi/4$ imply that $|\measuredangle(y_2, y, y_2)| \leq \pi/4 + \pi/4 = \pi/2$ which contradicts (13) and finishes the proof. □

Now, to complete the proof of the theorem, we first quote the fact (see [**K**, p.294]) that the only continua which have order 2 at each point are simple closed curves. Let k be the degree of f. That f acting on X is topologically equivalent to $z \to z^k$ follows from [**Wh2**, p.184].

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UNIVERSITY OF NORTH TEXAS
DENTON, TEXAS 76203
E-mail address: Mauldin@unt.edu
E-mail address: Urbanski@unt.edu

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