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In this paper, we introduce the minimal norm Jordan splittings of quadratic lattices over dyadic local fields. By using these splittings, we prove that orthogonal groups over dyadic local fields are generated by the symmetries and the Eichler transformations of the lattices unless the spinor norms of these groups are entire multiplicative groups of underlying fields.

The generation problem of integral orthogonal groups over local fields was first studied by Kneser (see references in [**K**]). He obtained that orthogonal groups of lattices over nondyadic local fields are generated by the symmetries of the lattices. This can be regarded as an analogy of Cartan- Dieudonne's theorem about generation of orthogonal groups on spaces (see [**L**] or [**O**]). In [**OP1**] and [**OP2**], O'Meara and Pollak studied these integral orthogonal groups over dyadic local fields and obtained that these groups are generated by the symmetries and the Eichler transformations of the lattices when the lattices are modular or 2 is unramified. One of the applications of these results is to study the spinor genus theory of integral quadratic forms over number fields, which essentially depends on the knowledge of the spinor norms of these integral orthogonal groups at each local completion. By using these good generators, Kneser [**K**] was able to determine the spinor norms of integral orthogonal groups over nondyadic local fields explicitly and Hsia [**H**] determined the spinor norms of integral orthogonal groups over dyadic local fields explicitly when the lattices are modular, and Earnest and Hsia [**EH**] computed the spinor norms of integral orthogonal groups explicitly over the dyadic fields in which 2 is unramified.

In this paper we will extend O'Meara-Pollak's results to arbitrary dyadic local fields. More precisely, our main result (Theorem 2.1) shows that orthogonal groups of the lattices are still generated by the symmetries and the Eichler transformations of the lattices unless the integral spinor norms of these groups are the entire multiplicative groups of underlying fields. Therefore, for the purpose of determining the integral spinor norms over arbitrary dyadic local fields, we have solved this generation problem. Some results will also be used in [**HSX**] which gives a full answer to representations by spinor genera over number fields. Our approach is first to modify the local structures by introducing the notion of "minimal norm Jordan splittings" over a dyadic local field and then to combine the techniques from [**OP2**] and [**X**] to obtain the desired results.

NOTATION AND TERMINOLOGY. All unexplained notation and terminology will be from [**O**], [**X**] and [**OP2**]. In particular, F denotes a dyadic local field, ϑ the ring of integers in F, p the maximal ideal of ϑ, U the group of units in ϑ, $e = \operatorname{ord} 2$ the ramification index of 2 in F. π a fixed prime element in F, $D(\ ,\ )$ the quadratic defect function, Δ a fixed unit of quadratic defect 4ϑ, V a regular quadratic space over F associated symmetric bilinear form $B(x, y)$, L a lattice on V, dL the determinant of L, sL the scale of L, nL the norm of L, $O(L)$ the integral orthogonal group of L, $X(L)$ the subgroup generated by the symmetries and Eichler transformations of L, and $\theta(\ ,\ )$ the spinor norm function. We use $[a, b, \dots]$ to denote spaces.

1. Minimal norm Jordan splittings. Since the Jordan splittings of lattices in dyadic local fields are not unique, O. T. O'Meara in [**O1**] obtained a saturated Jordan splitting of which the norm of every Jordan component is maximal. In this section, we establish a Jordan splitting of which the norm of every Jordan component is minimal and hyperbolic components are as much as possible. This kind of splitting plays important role in solving the generation problem of $O(L)$. We call $\pi^r A(0,0)$ a hyperbolic plane and H is denoted as an orthogonal sum of hyperbolic planes (which may sometimes have different scales).

LEMMA 1.1. *Suppose $L = L_1 \perp L_2$ where L_1 is unimodular with* $\operatorname{ord} nL_1 = u_1$ *and L_2 is p^r-modular with* $\operatorname{ord} nL_2 = u_2$, *and $r \geq 1$.*

(1) *If there is a vector $x_2 \in L_2$ such that* $\operatorname{ord} Q(x_2) \equiv u_1 \bmod 2$ *and* $\operatorname{ord} Q(x_2) \leq u_1$, *then $L = \overline{L_1} \perp \overline{L_2}$ where $\overline{L_2}$ is p^r-modular with $n\overline{L_2} = nL_2$ and $\overline{L_1}$ is unimodular with $n\overline{L_1} \subset nL_1$ or $\overline{L_1} \cong H$.*

(2) *If there is a vector $z_1 \in L_1$ such that* $\operatorname{ord} Q(z_1) \equiv u_2 \bmod 2$ *and*

$\operatorname{ord} Q(z_1) \leq (u_2 - 2r)$, *then* $L = \overline{L_1} \perp \overline{L_2}$ *where* $\overline{L_1}$ *is unimodular with* $n\overline{L_1} = nL_1$ *and* $\overline{L_2}$ *is* p^r*-modular with* $n\overline{L_2} \subset nL_2$ *or* $\overline{L_2} \cong H$.

Proof. (1) Without loss of generality, we assume $\operatorname{rank} L_1 = 2$ Write $L_1 \cong A(a, -a^{-1}\delta)$, adapted to a basis $\{x_1, y_1\}$ where a is a norm generator of L_1, $D(1+\delta) = \delta\vartheta$ and $-a^{-1}\delta \in \omega L_1$. Let $k = (u_1 - \operatorname{ord} Q(x_2))/2$, so $-Q(x_1)/Q(\pi^k x_2) \in U$. Put $-Q(x_1)/Q(\pi^k x_2) = \xi^2 + \sigma\pi^d$ with ξ and $\sigma \in U$, $d \geq 1$. Consider a unimodular lattice $\overline{L_1} = \vartheta(x_1 + \xi\pi^k x_2) + \vartheta y_1$ which splits L, we obtain $L = \overline{L_1} \perp \overline{L_2}$.

(i) When $u_1 < e$, then $\operatorname{ord}(-a^{-1}\delta) > u_1$ and $n\overline{L_1} \subset nL_1$.

(ii) When $u_1 = e$ and $L_1 \cong A(2, 2\rho)$, then $-d\overline{L_1} \in U^2$ and $\overline{L_1} \cong A(0,0)$.

Since $n\overline{L_1} \subseteq nL_1 \subseteq nL_2$ and $nL_1 + nL_2 = n\overline{L_1} + n\overline{L_2} = nL$, we have $n\overline{L_2} = nL_2 = nL$.

(2) It follows from applying (1) to $(L^\sharp)^{\pi^r}$. □

The following proposition strengthen [**O**, 91:9 Th.(2)].

PROPOSITION 1.1. *Suppose* $L = L_1 \perp L_2 \perp \cdots \perp L_t$ *is a Jordan splitting of* L *with* $sL_i = s_i$; $i = 1, \ldots, t$. *If* $nL^{s_{i_0}} \supset nL^{s_{i_0}+1}$ *and* $nL^{s_{i_0}} \supset (nL)(s_{i_0}s_{i_0-1}^{-1})^2$ *for some* $1 \leq i_0 \leq t$, *then for any Jordan splitting of* L, $L = K_1 \perp K_2 \perp \cdots \perp K_t$, *we have* $nK_{i_0} = nL_{i_0}$.

Proof. It is obvious that $L_j \subseteq L^{s_j}$, so $nL_j \subseteq nL^{s_j}$ for any $1 \leq j \leq t$. Since $nL^{s_{i_0}} \supset nL^{s_{i_0}+1} \supseteq \cdots \supseteq nL^{s_t}$, we have $nL^{s_{i_0}} \supset nL_j$ for all $j > i_0$. Consider

$$nL_j^{s_{i_0}} = (s_{i_0}s_{i_0-1}^{-1})^2 nL_j \subseteq (s_{i_0}s_{i_0-1}^{-1})^2(nL) \subset nL^{s_{i_0}}$$

for all $j < i_0$. Note

$$L^{s_{i_0}} = L_1^{s_{i_0}} \perp \cdots \perp L_{i_0-1}^{s_{i_0}} \perp L_{i_0} \perp L_{i_0+1} \perp \cdots \perp L_t.$$

So

$$nL^{s_{i_0}} = \sum_{j<i_0} nL_j^{s_{i_0}} + nL_{i_0} + \sum_{j>i_0} nL_j = nL_{i_0},$$

and $L^{s_{i_0}}$ is independent of the Jordan splitting of L. □

REMARK 1.1. When $sL_{i_0} = nL_{i_0}$, it can be easily checked that $nL^{s_{i_0}} \supset nL^{s_{i_0+1}}$ and $nL^{s_{i_0}} \supset (nL^{s_{i_0-1}})(s_{i_0}s_{i_0-1}^{-1})^2$. The converse statement is usually not true.

LEMMA 1.2. *Suppose L is a unimodular lattice with $nL \supset 2sL$. Then there exist two sublattices J and M such that $L = J \perp M$ with $nL = nL \supset nM$. Furthermore* rank $J = 1$ *when* rank L *is odd, and* rank $J = 2$ *when* rank L *is even.*

Proof. It follows from [**O**, 93:18]. ☐

THEOREM 1.1. *There exists a Jordan splitting $L = L_1 \perp \cdots \perp L_t$ such that for any Jordan splitting $L = K_1 \perp \cdots \perp K_t$, we have $nK_i \supseteq nL_i$ for all $1 \le i \le t$, and if $K_i \cong H$, then $L_i \cong H$.*

Proof. Put $A_1 = \{(K_1, \dots, K_t) | L = K_1 \perp \cdots \perp K_t$ be a Jordan splitting of L, and $K_1 \cong H\}$.

If this set is empty, we put $A_1 = \{(K_1, \dots, K_t) | L = K_1 \perp \cdots \perp K_t$ be a Jordan splitting of L, and nK_1 is the smallest $\}$.

Put $A_2 = \{(K_1, \dots, K_t) \in A_1 | K_2 \cong H\} \subseteq A_1$.

If this set is empty, we put

$$A_2 = \{(K_1, \dots, K_t) \in A_1 | nK_2\} \text{ is the smallest } \subseteq A_1.$$

$$\dots$$

By induction, put $A_t = \{(K_1, \dots, K_t) \in A_{t-1} | K_t \cong H\}$.

If this set is empty, we put

$$A_t = \{(K_1, \dots, K_t) \in A_{t-1} | nK_t \text{ is the smallest}\}.$$

Let $(L_1, \dots, L_t) \in A_t$, so $L = L_1 \perp \cdots \perp L_t$ is a Jordan splitting. By Lemma 1.1, we have if ord $nL_i \equiv$ ord nL_j mod 2 for some $i < j$, then ord $nL_i <$ ord $nL_j < 2(r_j - r_i)+$ ord nL_i or $L_i \cong H$ when ord $nL_i \ge$ ord nL_j or $L_j \cong H$ when ord $nL_j \ge$ ord $nL_i + 2(r_j - r_i)$. Here $r_k =$ ord s_k and $s_k = sL_k$ for $1 \le k \le t$.

Suppose there is a Jordan splitting of $L = K_1 \perp \cdots \perp K_t$ with $nK_{i_0} \subset nL_{i_0}$ for some $1 \le i_0 \le t$. By [**O**, 91:9 Th.(2)], $sL_{i_0} \supset nL_{i_0} \supset 2sL_{i_0}$ and rank L_{i_0} is even. By Lemma 1.2, $L_{i_0} = J \perp M$ with $nL_{i_0} = nJ \supset nM$ and rank $J = 2$. Write $J = \vartheta x + \vartheta \bar{x}$ where $Q(x)$ is a norm generator of J, ord $Q(x) <$ ord $Q(\bar{x})$ and ord $B(x, \bar{x}) = r_{i_0}$. Put $x = \sum_{i=1}^t a_i y_i$ where y_i is a maximal vector of K_i and $a_i \in \vartheta$ for $i = 1, \dots, t$. Note

$$s_{i_0} = B(x, L) = \sum_{i=1}^{t} B(a_i y_i, K_i) \supseteq B(a_i y_i, K_i)$$

for all $1 \le i \le t$, so ord $a_i \ge r_{i_0} - r_i$ when $i_0 \ge i$ by [**O**, 82:17]. Put $\bar{x} = \sum_{i=1}^t \bar{a}_i \bar{y}_i$ where $\bar{y}_i$ is a maximal vector of K_i and $\bar{a}_i \in \vartheta$. So ord $\bar{a}_i \ge r_{i_0} - r_i$ for all $i \le i_0$ by the same reason. Note

$$\text{ord } B(a_i y_i, \bar{a}_i \bar{y}_i) \ge r_{i_0} + (r_{i_0} - r_i) > r_{i_0}$$

for all $i < i_0$, and ord $B(a_i y_i, \bar{a}_i \bar{y}_i) \ge r_i > r_{i_0}$ for all $i > i_0$. Consider $r_{i_0} =$ ord $B(x, \bar{x}) =$ ord $B(a_{i_0} y_{i_0}, \bar{a}_{i_0} \bar{y}_{i_0}) \ge$ ord $a_{i_0} +$ ord $\bar{a}_{i_0} + r_{i_0}$. Therefore ord $a_{i_0} =$ ord $\bar{a}_{i_0} = 0$, ord $B(y_{i_0}, \bar{y}_{i_0}) = r_{i_0}$ and ord $B(y_{i_0}, \bar{x}) = r_{i_0}$. Put $y_{i_0} = \sum_{i=1}^t b_i z_i$ where z_i is a maximal vector of L_i, and $b_i \in \vartheta$. So ord $b_i \ge r_{i_0} - r_i$ for all $i < i_0$ and ord $b_{i_0} = 0$ by the same argument as above. Let $b_{i_0} z_{i_0} = cx + d\bar{x} + w$ with c, $d \in \vartheta$ and $w \in M$; note

$$\begin{aligned} r_{i_0} &= \text{ord } B(y_{i_0}, \bar{x}) = \text{ord } B(b_{i_0} z_{i_0}, \bar{x}) \\ &= \text{ord } B(cx + d\bar{x}, \bar{x}) = \text{ord } (cB(x, \bar{x}) + dQ(\bar{x})) \end{aligned}$$

and ord $Q(\bar{x}) >$ ord $Q(x) > r_{i_0}$. So $r_{i_0} =$ ord $(cB(x, \bar{x})) =$ odr $(c) + r_{i_0}$. Therefore ord $(c) = 0$ and ord $Q(b_{i_0} z_{i_0}) =$ ord $(Q(cx + d\bar{x}) + Q(w)) =$ ord $Q(x)$. Suppose all the vectors in $\{b_i z_i | i \ne i_0,\ 1 \le i \le t\}$ which satisfy ord $Q(b_i z_i) \le$ ord $Q(b_{i_0} z_{i_0})$ are $b_{i_1} z_{i_1}, b_{i_2} z_{i_2}, \dots, b_{i_l} z_{i_l}$.

When $i_k > i_0$ then $nL_{i_k} \supseteq nL_{i_0} \supset 2sL_{i_0} \supset 2sL_{i_k}$. So $L_{i_k} \ne H$ and ord nL_{i_k}+ ord $nL_{i_0} \equiv 1 \bmod 2$ by Lemma 1.1.

When $i_k < i_0$, then

$$\begin{aligned} \text{ord } nL_{i_k} + 2(r_{i_0} - r_{i_k}) &\le \text{ord } Q(b_{i_k} z_{i_k}) \\ &\le \text{ord } Q(b_{i_0} z_{i_0}) = \text{ord } Q(x) = \text{ord } nL_{i_0} < \text{ord } 2sL_{i_0}. \end{aligned}$$

That is ord $nL_{i_k} <$ ord $2 + (r_{i_k} - r_{i_0}) + r_{i_k} <$ ord $2sL_{i_k}$. So $L_{i_k} \ne H$ and ord nL_{i_k}+ ord $nL_{i_0} \equiv 1 \bmod 2$ by Lemma 1.1. Put $N = L_{i_1} \perp \cdots \perp J \perp \cdots \perp L_{i_l}$, for any $k_1 < k_2$; we have ord $nL_{i_{k_1}} \equiv$ ord $nL_{i_{k_2}} \equiv$ ord $nJ + 1 \bmod 2$, and ord $nL_{i_{k_1}} <$ ord $nL_{i_{k_2}} < 2(r_{i_{k_2}} - r_{i_{k_1}})$+ ord $nL_{i_{k_1}}$ and ord $nJ >$ ord nL_{i_k} for all $1 \le k \le l$. Since $nK_{i_0} \subset nL_{i_0}$, ord $Q(y_{i_0}) \ge$ ord $nK_{i_0} >$ ord $nL_{i_0} =$ ord $Q(x)$. Note $Q(y_{i_0}) = \sum_{i=1}^t Q(b_i z_i)$. So ord $Q(x) =$ ord $Q(b_{i_0} z_{i_0}) =$ ord $(\sum_{k=1}^l Q(b_{i_k} z_{i_k}))$. Write $-Q(x) / \sum_{k=1}^l Q(b_{i_k} z_{i_k}) = \xi^2 + \sigma\pi^d$ with ξ, $\sigma \in U$ and $d \ge 1$, then

$$Q\left(x + \xi \sum_{k=1}^l b_{i_k} z_{i_k}\right) = -Q\left(\sum_{k=1}^l b_{i_k} z_{i_k}\right) \sigma\pi^d$$

and

$$\text{ord } Q\left(x + \xi \sum_{k=1}^{l} b_{i_k} z_{i_k}\right) = \text{ord } Q(x) + d > \text{ord } Q(x).$$

Put $\bar{J} = \vartheta(x + \xi \sum_{k=1}^{l} b_{i_k} z_{i_k}) + \vartheta \bar{x}$ which is s_{i_0}-modular. Since ord b_{i_k} = ord $\xi b_{i_k} \geq r_{i_0} - r_{i_k}$ for all $i_k < i_0$, $\bar{J}$ splits N. So we obtain another Jordan splitting of N, $N = \overline{L_{i_1}} \perp \cdots \perp \bar{J} \perp \cdots \perp \overline{L_{i_l}}$.

Since we can check $nN^{s_{i_k}} = nL_{i_k}$ for all $1 \leq k \leq l$, $nN^{s_{i_k}} \supset nN^{s_{i_{k+1}}}$ and $nN^{s_{i_k}} \supset (nN^{s_{i_{k-1}}})(s_{i_k} s_{i_{k-1}}^{-1})^2$ for all $1 \leq k \leq l$. We have $nL_{i_k} = n\overline{L_{i_k}}$ for all $1 \leq k \leq l$ by the above proposition, but $n\bar{J} \subset nJ$. Corresponding to this Jordan splitting of N, we obtain another Jordan splitting of L which contradicts our choice of the Jordan splitting of L.

If $K_{i_0} \cong H$ but $L_{i_0} \neq H$, then $nL_{i_0} = 2sL_{i_0}$ by the above argument. By [**O**, 93:14] we can assume $L_{i_0} \cong \pi^{r_0} A(2, 2\rho)$ adapted to a basis $\{u, \bar{u}\}$ and $K_{i_0} \cong \pi^{r_0} A(0,0)$ adapted to a basis $\{v, \bar{v}\}$. Write $v = \sum_{i=1}^{t} c_i q_i$ where q_i is a maximal vector of L_i and $c_i \in \vartheta$, so ord $c_i \geq (r_{i_0} - r_i)$ for all $i < i_0$ and ord $(c_{i_0}) = 0$. Thus ord $Q(c_{i_0} q_{i_0}) = r_{i_0} + e =$ ord $Q(u)$ by Riehm Domination Principle [**R**]. By the similar arguments as above, we can obtain an new Jordan splitting of L which contradicts our choice of the Jordan splitting of L. □

The Jordan splittings which enjoy the property of Theorem 1.1 are called minimal norm Jordan splittings.

COROLLARY 1.1. *L can be splitted as $L = L_0 \perp H$ such that L_0 cannot be splitted by any hyperbolic plane and L_0 is determined uniquely by L up to isometry.*

Proof. Suppose L has another splitting $L = \overline{L_0} \perp \bar{H}$ where $\overline{L_0}$ cannot be splitted by any hyperbolic plane, and the type of Jordan splitting of $\overline{L_0}$ is different from that of L_0. Without loss of generality, we assume that the rank of i_0 Jordan component of $\overline{L_0}$ is greater than that of L_0 for some $1 \leq i_0 \leq t$ and $\bar{H}$ does not contain any i_0 hyperbolic component by Cancellation Theorem [**O**, 93:14]. So we can choose a Jordan splitting $\overline{L_0} = J_1 \perp \cdots \perp J_t$ such that $J_{i_0} = M \perp N$ with $nJ_{i_0} = nM \supset nN$, rank $M \leq 2$, rank $N = 2$, and nN is the smallest. Furthermore we assume the Jordan splitting

$R = J_1 \perp \cdots \perp J_{i_0-1} \perp M \perp J_{i_0+1} \perp \cdots \perp J_t$. is the minimal norm Jordan splitting.

Comparing the Jordan splitting of $\overline{L_0}$ with that of L_0, there is a hyperbolic plane $H_{i_0} = \vartheta u + \vartheta \bar{u} \subseteq L$ with $sH_{i_0} = sJ_{i_0}$ and $Q(u) = Q(\bar{u}) = 0$ and $B(u, \bar{u})\vartheta = sH_{i_0}$. So $u = \sum_{i=1}^t b_i z_i$, $\bar{u} = \sum_{i=1}^t \overline{b_i}\overline{z_i}$ where z_i and $\overline{z_i}$ are the maximal vectors of each Jordan component for the Jordan splitting $L = \overline{L_0} \perp H$ with $1 \leq i \leq t$. So ord $b_i \geq (r_{i_0} - r_i)$, ord $\overline{b_i} \geq (r_{i_0} - r_i)$ for all $i < i_0$, and ord $b_{i_0} =$ ord $\overline{b_{i_0}} = 0$. Here $r_i =$ ord sJ_i for all $1 \leq i \leq t$. Write $z_i = p_i + q_i$ with $p_i \in J_i$ and $q_i \in H_i$ where H_i is a suitable hyperbolic component with $sH_i = sJ_i$ or 0 for all i, then ord $Q(b_i q_i) \geq 2$ ord $b_i + e + r_i > e + r_{i_0}$ for all $i \neq i_0$. Consider $z_{i_0} = v_{i_0} + w_{i_0}$ and $\overline{z_{i_0}} = \overline{v_{i_0}} + \overline{w_{i_0}}$ where v_{i_0} and $\overline{v_{i_0}} \in M$, w_{i_0} and $\overline{w_{i_0}} \in N$, then at least one of $Q(v_{i_0})$, $Q(\overline{v_{i_0}})$; or $Q(w_{i_0})$, $Q(\overline{w_{i_0}})$ is a norm generator.

If $Q(v_{i_0})$ is a norm generator of M, then ord $Q(b_{i_0} z_{i_0}) =$ ord $Q(v_{i_0}) <$ ord $Q(w_{i_0}) \leq e + r_{i_0}$, and ord $Q(v_{i_0}) =$ ord $Q(\sum_{i \neq i_0} b_i p_i)$ by $Q(u) = 0$. So we can get the new splitting $R = \overline{J_1} \perp \cdots \perp \bar{M} \perp \cdots \perp \overline{J_t}$ with $n\bar{M} \subset nM$. That is a contradiction.

If $Q(w_{i_0})$ is a norm generator of M, then ord $Q(w_{i_0}) \leq e + r_{i_0}$ and ord $Q(w_{i_0}) =$ ord $Q(\sum_{i \neq i_0} b_i p_i + v_{i_0})$, we can get the new splitting $\overline{L_0} = \overline{J_1} \perp \cdots \perp \overline{J_t}$ with $\overline{J_{i_0}} = \bar{M} \perp \bar{N}$ such that $n\bar{N} \subset nN$. This contradicts our choice.

Therefore L_0 and $\overline{L_0}$ have the same type of Jordan splitting and $L_0 \cong \overline{L_0}$ by Cancellation Theorem [**O**, 93:14]. □

2. Generation and spinor norms of $O(L)$. Suppose $L = L_1 \perp L_2 \perp \cdots \perp L_t$ is a minimal norm Jordan splitting over a dyadic local field F with $r_i =$ ord sL_i, $u_i =$ ord nL_i, for $i = 1, \ldots, t$. $Q(x_i) = \varepsilon_i \pi^{u_i}$ is a norm generator of L_i where $\varepsilon_i \in U$ and $x_i \in L_i$, for $1 \leq i \leq t$.

LEMMA 2.1. *Suppose all the Jordan components are one dimension and there exists i and j with $1 \leq i < j \leq n$ such that $r_j - r_i \leq 2e$ and $D(-\varepsilon_i\varepsilon_j) = p^s$ with $1 \leq s \leq e - (r_j - r_i)/2$. If $0 < |r_k - r_i| \leq 2e$ or $0 < |r_k - r_j| \leq 2e$ for some $1 \leq k \leq n$, then $\theta(O^+(L)) = \dot{F}$.*

Proof. Because of [**X**, Theorem 3.1] we can assume that $r_j - r_i$, $r_k - r_j$ and $r_k - r_i$ are even. Suppose $r_i < r_j < r_k$. The other cases can be done by taking the same arguments.

So $0 < r_k - r_j \le 2e$ and $\theta(O^+(\vartheta x_i \perp \vartheta x_j)) = Q[1, \varepsilon_i\varepsilon_j]$ by [**X**, Prop. 2.3]. By [**H**, Lemma 3] there exists η in U such that $(\eta, -\varepsilon_i\varepsilon_j) = -1$ and $D(\eta) = p^{2e-s}$. Note $D(-1) = p^h$ with $h \ge e$ and $(2e - s) + h \ge e + (r_i - r_j)/2 + e > 2e$. So $(\eta, -1) = 1$ by [**X**, Remark 1]. Therefore $(\eta, -\varepsilon_i\varepsilon_k) = 1$ or $(\eta, -\varepsilon_j\varepsilon_k) = 1$.

When $(\eta, -\varepsilon_j\varepsilon_k) = 1$, write $D(-\varepsilon_j\varepsilon_k) = p^t$.

If $1 \le t \le e - (r_k - r_j)/2$, then $\eta \in \theta(O^+(\vartheta x_j \perp \vartheta x_k))$ by [**X**, Prop. 2.3]. If $(3e - (r_k - r_j)/2)/2 \ge t > e - (r_k - r_j)/2$, note $2e - s \ge e + (r_j - r_i)/2 \ge t - e + (r_k - r_j)/2$. Then $\eta \in \theta(O^+(\vartheta x_j \perp \vartheta x_k))$.

If $t > (3e - (r_k - r_j)/2)/2$,note $2e - s \ge e + (r_j - r_i)/2 \ge e - [e/2 - (r_k - r_j)/4]$. Then $\eta \in \theta(O^+(\vartheta x_j \perp \vartheta x_k))$. Therefore $\theta(O^+(L)) = \theta(O^+(\vartheta x_i \perp \vartheta x_j))\theta(O^+(\vartheta x_j \perp \vartheta x_k)) = \dot{F}$.

When $(\eta, -\varepsilon_i\varepsilon_k) = 1$, the result follows from the same arguments as above if $r_k - r_i \le 2e$. So we assume $4e \ge r_k - r_i > 2e$. Write $D(-\varepsilon_k\varepsilon_i) = p^d$.

If $1 \le d \le 2e - (r_k - r_i)/2$, then

$$2e - s \ge e + (r_j - r_i)/2 \ge (r_k - r_i)/2 \ge (r_k - r_i) - 2e + d.$$

So $\eta \in \theta(O^+(\vartheta x_i \perp \vartheta x_k))$.

If $d > 2e - (r_k - r_i)/2$, note $2e - s \ge (r_k - r_i)/2$; then $\eta \in \theta(O^+(\vartheta x_i \perp \vartheta x_k))$. Therefore $\theta(O^+(L)) = \theta(O^+(\vartheta x_i \perp \vartheta x_j))\theta(O^+(\vartheta x_i \perp \vartheta x_k)) = \dot{F}$. □

LEMMA 2.2. *If* $L_{i_0} \cong \pi^{r_{i_0}} A(\varepsilon_{i_0}\pi^{u_{i_0} - r_{i_0}}, -\varepsilon_{i_0}^{-1}\pi^{-u_{i_0} + r_{i_0}}\delta_{i_0})$ *adapted to a basis* $\{x_{i_0}, y_{i_0}\}$ *with* $D(1 + \delta_{i_0}) = \delta_{i_0}\vartheta$ *for some* $1 \le i_0 \le t$.

(1) *When* $\operatorname{ord}\delta_{i_0} < u_{i_0} + e - r_{i_0}$, *and* $u_k + u_{i_0} \equiv 0 \bmod 2$, *and* $u_k + \operatorname{ord} Q(y_{i_0}) - 2r_k \le 2e + 1$ *with some* $k < i_0$ *or* $u_k + \operatorname{ord} Q(y_{i_0}) - 2r_{i_0} \le 2e + 1$ *with some* $k > i_0$, *then* $\theta(O^+(L)) = \dot{F}$.

(2) *When* $u_{i_0} + u_k \equiv 1 \bmod 2$, $u_k + u_{i_0} - 2r_{i_0} \le 2e + 1$ *with some* $k > i_0$ *or* $u_k + u_{i_0} - 2r_k \le 2e + 1$ *with some* $k < i_0$, *then* $\theta(O^+(L)) = \dot{F}$.

(3) *When* $u_{i_0} + u_k \equiv 0 \bmod 2$, $L_{i_0} \ne \pi^{r_{i_0}} A(0, 0)$, $D(-\varepsilon_{i_0}\varepsilon_k) = p^t$, $t \le e - (u_k + u_{i_0} - 2r_{i_0})/2$ *with some* $k > i_0$ *or* $t \le e - (u_k + u_{i_0} - 2r_k)/2$ *with some* $k < i_0$, *then* $\theta(O^+(L)) = \dot{F}$.

Proof. (1) Put $K = \vartheta y_{i_0} \perp \vartheta x_k$. Since $\operatorname{ord} Q(x_k) + \operatorname{ord} Q(y_{i_0}) \equiv 1$, it can be checked that $\tau_z \in O(L_k \perp L_{i_0}) \subseteq O(L)$ for any maximal vector z of K. Therefore $\theta(O^+(L)) \supseteq Q[1, \dot{\varepsilon}_{i_0}\varepsilon_k\pi]$ which does not contain Δ, but Δ is in $\theta(O^+(L_{i_0}))$ by [**H**]. Thus $\theta(O^+(L)) = \dot{F}$.

(2) It follows from the same arguments as the above case (1).

(3) Without loss of generality, we assume $k > i_0$. By Lemma 1.1, we know $u_k - 2r_k + 2r_{i_0} < u_{i_0} < u_k$. Put $K = \vartheta x_{i_0} \perp \vartheta x_k$. Since $1 \le t < e$, we have $D(-\varepsilon_{i_0}\varepsilon_k) = D(\varepsilon_{i_0}\varepsilon_k)$ and $u_k + u_{i_0} - 2r_{i_0} \le 2e$. It can be checked that $\tau_z \in O(L_{i_0} \perp L_k) \subseteq O(L)$ for any maximal vector z of K. Therefore $\theta(O^+(L)) \supseteq Q[1, \dot{\varepsilon}_{i_0}\varepsilon_k]$. By [**H**, Lemma 3] there exists η in U such that $(\eta, -\varepsilon_{i_0}\varepsilon_k) = -1$ with $D(\eta) = p^{2e-t}$.

(i) If $\operatorname{ord}\delta_{i_0} \ge u_{i_0} + e - r_{i_0}$, then $2e - t \ge e + (u_k + u_{i_0} - 2r_{i_0})/2$. So $\eta \in \theta(O^+(L_{i_0}))$ by [**H**, Prop. B], [**X**, Remark 1] and [**H**, Lemma 2]. Thus $\theta(O^+(L)) = \dot{F}$.

(ii) If $\operatorname{ord}\delta_{i_0} < u_{i_0} + e - r_{i_0}$, we only need to consider $u_k +$ ord $Q(y_{i_0}) - 2r_{i_0} > 2e + 1$ with $k > i_0$. Note

$$\begin{aligned}
2e - t + &u_{i_0} - r_{i_0} + \operatorname{ord} Q(y_{i_0}) - r_{i_0}\\
&\ge e + (u_k + \operatorname{ord} Q(y_{i_0}) - 2r_{i_0})/2\\
&\quad + u_{i_0}/2 + u_{i_0} + \operatorname{ord} Q(y_{i_0})/2 - 2r_{i_0}\\
&> e + e + u_{i_0}/2 + u_{i_0} + u_{i_0}/2 - 2r_{i_0} \ge 2e.
\end{aligned}$$

Then $\eta \in \theta(O^+(L_{i_0}))$ by [**X**, Remark 1]. Thus $\theta(O^+(L)) = \dot{F}$. □

LEMMA 2.3. *If* $\operatorname{rank} L_i \ge 3$ *and* $\operatorname{ord} nL_i + \operatorname{ord} wL_i \equiv 1 \mod 2$ *for some* $1 \le i \le t$, *then* $\theta(O^+(L)) = \dot{F}$.

Proof. It follows from [**H**, Prop. A]. □

LEMMA 2.4. *If* $\operatorname{rank} L_i = \operatorname{rank} L_j = 1$ *and* $\operatorname{rank} L_k = 2$ *for some* $i > j$ *and* k, $0 < u_i - u_j \le 2e+1$ *and* $u_i - u_j$ *is odd, then* $\theta(O^+(L)) = \dot{F}$.

Proof. Since Δ is not in $\theta(O^+(L_i \perp L_j)) = Q[1, \dot{\varepsilon}_i\varepsilon_j\pi]$ by [**X**, Prop. 2.2 i] and [**X**, Prop. 2.3 i] and $\Delta \in \theta(O^+(L_k))$ by [**H**]; therefore $\theta(O^+(L)) = \dot{F}$. □

LEMMA 2.5. *Suppose* $\operatorname{rank} L_i = \operatorname{rank} L_j = 1$ *and for some* $i > j$ *with* $0 < u_i - u_j \le 2e$ *and* $u_i - u_j$ *is even and* $D(-\varepsilon_i\varepsilon_j) = p^t$ *with* $t \le e - (u_i - u_j)/2$. *If there is* $L_k \cong \pi^{r_k} A(\varepsilon_k \pi^{u_k - r_k}, -\varepsilon_k^{-1}\pi^{-u_k + r_k}\delta_k)$ *with* $\operatorname{ord}\delta_k \ge u_k - r_k + e$ *for some* $1 \le k \le t$, *then* $\theta(O^+(L)) = \dot{F}$.

Proof. By [**H**, Lemma 3] and [**X**, Prop. 2.3], there exists $\eta \bar{\in} \theta(O^+ (L_i \perp L_j)) = Q[1, \dot{\varepsilon}_i\varepsilon_j]$ with $D(\eta) = p^{2e-t}$ and $2e - t \ge e + (u_i -$

$u_j)/2 \geq e+1$. So $\eta \in \theta(O^+(L_k))$ by [**H**, Prop. B], [**X**, Remark 1] and [**H**, Lemma 2]. Therefore $\theta(O^+(L)) = \dot{F}$. □

LEMMA 2.6. *If* $\operatorname{rank} L_i = \operatorname{rank} L_j = 1$ *and* $\operatorname{rank} L_k = 2$, $0 < u_j - u_i \leq 2e$ *and* $u_j - u_i$ *is even for some* $k > j > i$, $D(-\varepsilon_i\varepsilon_j) = \dot{p}^t$ *with* $t \leq e - (u_j - u_i)/2$, *and* $u_k - u_i \leq 2e$, *then* $\theta(O^+(L)) = \dot{F}$.

Proof. Put $L_k \cong \pi^{r_k} A(\varepsilon_k \pi^{u_k - r_k}, -\varepsilon_k^{-1}\pi^{-u_k+r_k}\delta_k)$. By Lemma 2.2 and Lemma 2.5, we can assume $u_k - u_i$ is even and ord $\delta_k < u_k - r_k + e$ and ord $\delta_k + 2r_k - u_k - u_j > 2e+1$. So $r_k - u_i > r_k - u_j > e+1$. It can be checked that any $\tau_z \in O(\vartheta x_j \perp \vartheta x_k)$ is also in $O(L_j \perp L_k)$. So $O(\vartheta x_j \perp \vartheta x_k) \subseteq O(L_j \perp L_k)$. By the same reason, $O(\vartheta x_i \perp \vartheta x_k) \subseteq O(L_i \perp L_k)$. By the proof of Lemma 2.1, we obtain $\theta(O^+(\vartheta x_i \perp \vartheta x_j))\theta(O^+(\vartheta x_i \perp \vartheta x_k))\theta(O^+(\vartheta x_j \perp \vartheta x_k)) = \dot{F}$. Thus $\theta(O^+(L)) = \dot{F}$. □

Before obtaining our main result, we first establish the following Witt- type result.

PROPOSITION 2.1. *Suppose* L *cannot be splitted by any hyperbolic plane and* $\theta(O^+(L)) \neq \dot{F}$. *If* $\sigma L_1 \subseteq L$ *for some* $\sigma \in O(V)$, *then there is* τ *a product of symmetries in* $O(L)$ *such that* $\tau\sigma|_{L_1} = 1$.

Proof. When $e = 1$, it has been done in [**OP1**]. We assume $e > 1$ and $r_1 = 0$. By Lemma 2.3 and [**O**, 93:18], we know all the Jordan components are one or two dimensions and none of them is hyperbolic plane. □

When rank $L_1 = 2$, write $L_1 \cong A(\varepsilon_1\pi^{u_1}, -\varepsilon_1^{-1}\pi^{-u_1}\delta_1)$ adapted to a basis $\{x_1, y_1\}$ with $D(1+\delta_1) = \delta_1\vartheta$. Put $\sigma x_1 = ax_1 + by_1 + z$ where a and b are in ϑ, $z \in L_2 \perp \cdots \perp L_t$.

(1) ord $Q(y_1) \geq e$.

When $u_k + u_1 \equiv 1 \mod 2$ for some $2 \leq k \leq t$, then

$$u_k - u_1 \geq 2e + 3 - 2u_1 \geq 3,\ r_k \geq u_k - e \geq e - u_1 + 3 \geq 3$$

by Lemma 2.2(2).

When $u_k + u_1 \equiv 0 \mod 2$ for some $2 \leq k \leq t$, then $u_k \geq u_1 + 2$ and $r_k > (u_k - u_1)/2 \geq 1$ by Lemma 1.1.

So ord $Q(z)-$ ord $Q(x_1) \geq 2$. Note $Q(x_1) = a^2Q(x_1) + 2ab + b^2Q(y_1) + Q(z)$, $Q(\sigma x_1 - x_1) = 2((1-a)Q(x_1) - b)$.

If ord $b = 0$ and ord $Q(x_1) =$ ord $Q(y_1) = e$, then $\tau_{\sigma x_1 - x_1} \in O(L)$. Otherwise, $a \equiv 1 \bmod p$ and we assume ord $b \leq 1$ because we can consider $\tau_{\pi^{[(e-u_1)/2]}x_1+y_1}\sigma(x_1)$ instead of σx_1 if necessary and $\tau_{\pi^{[(e-u_1)/2]}x_1+y_1} \in O(L)$.

(i) $u_1 \geq 1$ or ord $b = 0$. Then $\tau_{\sigma x_1 - x_1} \in O(L)$.

(ii) $u_1 = 0$ and ord $b = 1$ and $u_2 \geq 3$. Since $u_k \geq u_2 \geq 3$ for all $k \geq 3$ by Lemma 1.1 and Lemma 2.2(2), ord $Q(z) \geq 3$ and ord $(1 - a^2) \geq 3$. Therefore ord $(1 - a) \geq 2$ and $\tau_{\sigma x_1 - x_1} \in O(L)$.

(iii) $u_1 = 0$ and ord $b = 1$ and $u_2 = 2$. By the above arguments we only need to consider e is odd and ord $(1 - a) = 1$. Note $D(-\varepsilon_1\varepsilon_2) = p^t$ with $t > e - (u_1 + u_2)/2$ by Lemma 2.2(3). Write $a = 1 + l\pi$ with $l \in U$ and $-\varepsilon_1\varepsilon_2 = \xi^2 + \lambda\pi^t$ with $\xi,\ \lambda \in U$.

Let

$$\eta = \xi + \pi^{(e-1)/2} \in U,\ \delta = \varepsilon_2(h^2 - \lambda\pi^{t-e+1} + 2\pi^{-e}\xi h\pi^{(e-1)/2+1}) \in U.$$

We have $\varepsilon_1\varepsilon_2^2 + \varepsilon_2\eta^2 = \delta\pi^{e-1}$ and $\tau_{\varepsilon_2\pi x_1+\eta x_2}$ is in $O(L)$. Write $\tau_{\varepsilon_2\pi x_1+\eta x_2}\sigma x_1 = a'x_1 + b'y_1 + z'$ with $a' \equiv a(1 - 2\varepsilon_2^2\varepsilon_1\pi^{-e}\delta^{-1}\pi) \bmod p^2$ and $z' \in L_2 \perp \cdots \perp L_t$. Note ord $(1 - a') \geq 2$ by a direct computation. Therefore $\tau_{\sigma' x_1 - x_1} \in O(L)$ with $\sigma' = \tau_{\varepsilon_2\pi x_1+\eta x_2}\sigma$ by the same argument as above.

(2) ord $Q(y_1) < e$.

When $u_1 + u_k \equiv 1 \bmod 2$ for some $2 \leq k \leq t$, then $u_1 + u_k \geq 2e+3$ by Lemma 2.2(2) and $r_k \geq u_k - e \geq e + 3 - u_1$.

When $u_1 + u_k \equiv 0 \bmod 2$ for some $2 \leq k \leq t$, then ord $Q(y_1) + u_k \geq 2e + 3$ by Lemma 2.2(1) and $r_k \geq u_k - e \geq e + 3-$ ord $Q(y_1)$.

So ord $Q(z) \geq 2e+3-$ ord $Q(y_1)$ and $a \equiv 1 \bmod p$. We can assume ord $b \leq e-$ ord $Q(y_1)$ because we consider $\tau_{2x_1+y_1}\sigma(x_1)$ instead of σx_1 if necessary and $\tau_{2x_1+y_1} \in O(L)$. We claim ord $b = e-$ ord $Q(y_1)$. If ord $b < e-$ ord $Q(y_1)$, then ord $(b^2Q(y_1)) <$ ord $2ab <$ ord $Q(z)$ and ord $((1 - a^2)Q(x_1)) =$ ord $(b^2Q(y_1)) < 2e$. Therefore ord $(1 - a) =$ ord $(1 + a) < e$ and ord $Q(x_1) \equiv$ ord $Q(y_1) \bmod 2$. It is a contradiction since ord $Q(y_1) < e$. So we have ord $((1-a^2)Q(x_1)) =$ ord $(2ab + b^2Q(y_1) + Q(z)) \geq 2e-$ ord $Q(y_1)$.

If ord $(1 - a) < e$, then ord $(1 + a) =$ ord $(1 - a)$ and

$$\operatorname{ord}((1 - a)Q(x_1)) \geq e - (\operatorname{ord} Q(y_1) - \operatorname{ord} Q(x_1))/2 > e - \operatorname{ord} Q(y_1)$$

and

$$\operatorname{ord}(1-a) \geq e - (\operatorname{ord} Q(y_1) + \operatorname{ord} Q(x_1))/2 > e - \operatorname{ord} Q(y_1).$$

Therefore $\tau_{\sigma x_1 - x_1} \in O(L)$.

If ord $(1-a) \geq e$, then $\tau_{\sigma x_1 - x_1} \in O(L)$.

Now we can assume $\sigma x_1 = x_1$, $\sigma y_1 = \alpha x_1 + \beta y_1 + w$ by the above arguments. Here α, $\beta \in \vartheta$, $w \in L_2 \perp \cdots \perp L_t$. So we have $1 = \alpha Q(x_1) + \beta$ and $Q(\sigma y_1 - y_1) = 2\alpha(Q(x_1)Q(y_1) - 1)$.

If ord $\alpha \leq r_2$, then $\tau_{\sigma y_1 - y_1} \in O(L)$.

If ord $\alpha > r_2$ and $u_1 + u_2 \geq 2e$, then $r_2 \geq u_2 - e \geq e - u_1$. Put $u = x_1 - Q(x_1)y_1$, so $\tau_u(x_1) = x_1$ and $\tau_u\sigma(y_1) = \alpha' x_1 + \beta' y_1 + w'$ with ord $\alpha' = e - u_1 < r_2$ and $\tau_u \in O(L)$. Therefore $\tau_{\tau_u \sigma y_1 - y_1} \in O(L)$.

If ord $\alpha > r_2$ and $u_1 + u_2 < 2e$, then $u_1 \equiv u_2 \bmod 2$ and $D(-\varepsilon_1\varepsilon_2) = p^t$ with $t > e - (u_2 + u_1)2$ by Lemma 2.2(3). Write $-\varepsilon_1\varepsilon_2 = \xi^2 + \lambda\pi^t$ with ξ, $\lambda \in U$ and

$$\eta = \xi + \pi^{[e/2-(u_1+u_2)/4]} \in \vartheta,$$
$$\delta = \varepsilon_2(1 + 2\xi\pi^{-[e/2-(u_1+u_2)/4]} - \lambda\pi^{t-2[e/2-(u_1+u_2)/4]}) \in U.$$

So $\varepsilon_1\varepsilon_2^2 + \varepsilon_2\eta^2 = \delta\pi^{2[e/2-(u_1+u_2)/4]}$. Put

$$u = \pi^{(u_2-u_1)/2}\varepsilon_2 x_1 - \pi^{(u_2-u_1)/2}\varepsilon_2 Q(x_1)y_1 + \eta x_2 \in L.$$

Then $\tau_u \in O(L)$ whenever ord $Q(y_1) \geq e$; or ord $Q(y_1) < e$ but u_2+ ord $Q(y_1) > 2e+1$ by Lemma 2.2(1). Note $\tau_u x_1 = x_1$, $\tau_u\sigma y_1 = \alpha' x_1 + \beta' y_1 + w'$ with

$$\begin{aligned}\operatorname{ord}\alpha' &= e + u_2 - u_1 - (u_2 + 2[e/2 - (u_1+u_2)/4]) \\ &\leq (u_2 - u_1)/2 + 1 \leq r_2\end{aligned}$$

by Lemma 1.1 and $w' \in L_2 \perp \cdots \perp L_t$. Therefore $\tau_{\tau_u\sigma y_1 - y_1} \in O(L)$. We have $\tau_{\sigma y_1 - y_1}\sigma|_{L_1} = 1$ or $\tau_{\tau_u\sigma y_1 - y_1}\tau_u\sigma|_{L_1} = 1$.

When rank $L_1 = 1$, write $\sigma x_1 = a x_1 + z$ with $a \in \vartheta$ and $z \in L_2 \perp \cdots \perp L_t$. So $Q(\sigma x_1 \pm x_1) = 2(1 \pm a)Q(x_1)$. Since $(1+a)+(1-a) = 2$, ord $(1-a) \leq e$ or ord $(1+a) \leq e$. Note $\tau_{\sigma x_1 \pm x_1} \in O(L)$ whenever ord $(1 \mp a) \leq r_2$ We only need to consider the following cases by Lemma 2.2.

(1) $u_2 \equiv 1 \bmod 2$ and $u_2 < e$ and rank $L_2 = 1$.

By Lemma 2.4 and [**X**, Theorem 3.1], we have rank $L_3 = 1$ and $r_3 = u_3 > 2e$. Write $\sigma x_1 = ax_1 + bx_2 + w$ with $b \in \vartheta$ and $w \in L_3 \perp \cdots \perp L_t$. We can assume ord $b + u_2 < e$. So ord $(1-a^2) = 2$ ord $b + u_2 < 2e$. Therefore ord $(1-a) =$ ord $(1+a) =$ ord $b + u_2/2 <$ ord $b + u_2 < r_3$ and $\tau_{\sigma x_1 - x_1} \in O(L)$.

(2) $u_2 \equiv 0 \bmod 2$ and $D(-\varepsilon_1\varepsilon_2) = p^t$ with $t > e - u_2/2$ and ord $(1-a) > r_2$. Write $-\varepsilon_1\varepsilon_2 = \xi^2 + \lambda\pi^t$ with $\xi,\ \lambda \in U$ and $\eta = \xi + \pi^{[e/2-u_2/4]} \in \vartheta$ and $\delta = \varepsilon_2(1+2\xi\pi^{-[e/2-u_2/4]} - \lambda\pi^{t-2[e/2-u_2/4]}) \in U$.

So $\varepsilon_1\varepsilon_2^2 + \varepsilon_2\eta^2 = \delta\pi^{2[e/2-u_2/4]}$. Put $u = \pi^{u_2/2}\varepsilon_2 x_1 + \eta x_2 \in L$; then $\tau_u \in O(L)$. Consider $\tau_u\sigma x_1 = a'x_1 + z'$ with

$$a' = a - 2\varepsilon_1\varepsilon_2^2 a\delta^{-1}\pi^{-2[e/2-u_2/4]} - 2\eta\varepsilon_2\delta^{-1}B(x_2,z)\pi^{-u_2/2-2[e/2-u_2/4]}$$

and $z' \in L_2 \perp \cdots \perp L_t$. Note ord $(1-a') = e - 2[e/2 - u_2/4] \leq u_2/2 + 1 \leq r_2$ by Lemma 1.1. So $\tau_{\tau_u\sigma x_1 - x_1} \in O(L)$.

(3) $u_2 \equiv 0 \bmod 2$ and $D(-\varepsilon_1\varepsilon_2) = p^t$ with $t \leq e - u_2/2$.

By Lemma 2.2, we have rank $L_2 = 1$. Write $\sigma x_1 = ax_1 + bx_2 + w$ with $w \in L_3 \perp \cdots \perp L_t$, we only need to consider ord $b + u_2 < e$.

If $u_k + u_2 \equiv \bmod 2$ for some $3 \leq k \leq t$, then $u_k - u_2 > 2e + 1$ by [**X**, Theorem 3.1] and Lemma 2.2(2) and $r_k \geq u_k - e > e + 1 + u_2$. If $u_k + u_2 \equiv 0 \bmod 2$ for some $3 \leq k \leq t$, then $u_k > 2e$ by Lemma 2.1 and Lemma 2.6 and $r_k \geq u_k - e > e$.

Since ord $(1-a^2) = 2$ ord $b + u_2 < 2e$,

$$\operatorname{ord}(1-a) = \operatorname{ord}(1+a) = \operatorname{ord} b + u_2/2 < \operatorname{ord} b + u_2 < e < r_k.$$

for $k = 3, \ldots, t$. Therefore $\tau_{\sigma x_1 - x_1} \in O(L)$.

COROLLARY 2.1. *If $\sigma x_1 \in L$ for some $\sigma \in O(V)$, then there is τ a product of symmetries in $O(L)$ such that $\tau\sigma x_1 = x_1$.*

Proof. It follows from the proof of Proposition 2.1. □

REMARK 2.1. The assumptions in Proposition 2.1 cannot be removed.

THEOREM 2.1. *If $\theta(O^+(L)) \neq \dot{F}$, then $O(L) = X(L)$.*

Proof. It follows from Proposition 2.1 and [**OP2**, 2.5] and induction on rank L. □

REMARK 2.2. In fact we have proved a slightly stronger result. If L does not satisfy the hypotheses of the above lemmas (Lemma 2.1, 2.2, 2.3, 2.4, 2.5, 2.6) and [**X**, Theorem 3.1], then Proposition 2.1 and Corollary 2.1 and Theorem 2.1 are still true.

REMARK 2.3. [**EH1**, Prop. 2.1] can follow from Remark 2.2.

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