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KYUNG BAI LEE

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## MAPS ON INFRA-NILMANIFOLDS

—Rigidity and applications to Fixed-point Theory

KYUNG BAI LEE

**We show that Bieberbach's rigidity theorem for flat manifolds still holds true for any continuous maps on infra-nilmanifolds. Namely, every endomorphism of an almost crystallographic group is semi-conjugate to an affine endomorphism. Applying this result to Fixed-point theory, we obtain a criterion for the Lefschetz number and Nielsen number for a map on infra-nilmanifolds to be equal.**

**0. Infra-nilmanifolds.** Let  $G$  be a connected Lie group. Consider the semi-group  $\text{Endo}(G)$ , the set of all endomorphisms of  $G$ , under the composition as operation. We form the semi-direct product  $G \rtimes \text{Endo}(G)$  and call it  $\text{aff}(G)$ . With the binary operation

$$(a, A)(b, B) = (a \cdot Ab, AB),$$

the set  $\text{aff}(G)$  forms a semi-group with identity  $(e, 1)$ , where  $e \in G$  and  $1 \in \text{Endo}(G)$  are the identity elements. The semi-group  $\text{aff}(G)$  "acts" on  $G$  by

$$(a, A) \cdot x = a \cdot Ax.$$

Note that  $(a, A)$  is not a homeomorphism unless  $A \in \text{Aut}(G)$ . Clearly,  $\text{aff}(G)$  is a subsemi-group of the semi-group of all continuous maps of  $G$  into itself, for  $((a, A)(b, B))x = (a, A)((b, B)x)$  for all  $x \in G$ . We call elements of  $\text{aff}(G)$  *affine endomorphisms*.

Suppose  $G$  is a connected, simply connected, nilpotent Lie group;  $\text{Aff}(G) = G \rtimes \text{Aut}(G)$  is called the group of affine automorphisms of  $G$ . Let  $\pi \subset \text{Aff}(G)$  be a discrete subgroup such that  $\Gamma = \pi \cap G$  has finite index in  $\pi$ . Then  $\pi \backslash G$  is compact if and only if  $\Gamma$  is a lattice of  $G$ . In this case,  $\pi$  is called an *almost crystallographic group*. If moreover,  $\pi$  is torsion-free,  $\pi$  is an *almost Bieberbach group*. Such a group is the fundamental group of an infra-nilmanifold. According to Gromov and Farrell-Hsiang, the class of infra-nilmanifolds coincides with the class of almost flat manifolds.

**1. Generalization of Bieberbach's Theorem.** In 1911, Bieberbach proved that any automorphism of a crystallographic group is conjugation by an element of  $\text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$ . Recently this was generalized to almost crystallographic groups, see [1], [3] and [4].

We shall generalize this result to all homomorphisms (not necessarily isomorphisms). Topologically, this implies that every continuous map on an infra-nilmanifold is homotopic to a map induced by an affine endomorphism on the Lie group level. It can be stated as: every endomorphism of an almost crystallographic group is semi-conjugate to an affine endomorphism.

**THEOREM 1.1.** *Let  $\pi, \pi' \subset \text{Aff}(G)$  be two almost crystallographic groups. Then for any homomorphism  $\theta : \pi \rightarrow \pi'$ , there exists  $g = (d, D) \in \text{aff}(G)$  such that  $\theta(\alpha) \cdot g = g \cdot \alpha$  for all  $\alpha \in \pi$ .*

**COROLLARY 1.2.** *Let  $M = \pi \backslash G$  be an infra-nilmanifold, and  $h : M \rightarrow M$  be any map. Then  $h$  is homotopic to a map induced from an affine endomorphism  $G \rightarrow G$ .*

**COROLLARY 1.3 [3, 4].** *Homotopy equivalent infra-nilmanifolds are affinely diffeomorphic.*

Now we consider the uniqueness problem: How many  $g$ 's are there? Let  $\Phi = \pi / (G \cap \pi) \subset \text{Aut}(G)$  and  $\Phi' = \pi' / (G \cap \pi') \subset \text{Aut}(G)$  be the holonomy groups of  $\pi$  and  $\pi'$ . Let  $\Psi'$  be the image of  $\theta(\pi)$  in  $\Phi'$ . So  $\Phi' \subset \text{Aut}(G)$ . Let  $G^{\Psi'}$  denote the fixed point set of the action. For  $c \in G$ ,  $\mu(c)$  denotes conjugation by  $c$ . Therefore,  $\mu(c)(x) = cxc^{-1}$  for all  $x \in G$ . The group of all inner automorphisms is denoted by  $\text{Inn}(G)$ .

**PROPOSITION 1.4 (Uniqueness).** *With the same notation as above, suppose  $\theta(\alpha) \cdot g = g \cdot \alpha$  for all  $\alpha \in \pi$ . Then  $\theta(\alpha) \cdot \gamma = \gamma \cdot \alpha$  for all  $\alpha \in \pi$  if and only if  $\gamma = \xi \cdot g$ , where  $\xi = (c, \mu(c^{-1}))$ , for  $c \in G^{\Psi'}$ . Therefore,  $D$  is unique up to  $\text{Inn}(G)$ . If  $\theta$  is an isomorphism, then  $c \in G^{\Phi'}$ . In particular, if  $\pi$  is a Bieberbach group with  $H^1(\pi; \mathbb{R}) = 0$  and  $\theta$  is an isomorphism, then such a  $g$  is unique.*

**EXAMPLE 1.5.** The subgroup  $\Gamma = \pi \cap G$  of an almost crystallographic group  $\pi$  is characteristic, but not fully invariant. The

homomorphism  $\theta$  in the Theorem 1.1 may not map the maximal normal nilpotent subgroup  $\Gamma$  of  $\pi$  into that of  $\pi'$ . This causes a lot of trouble. Let  $\pi$  be an orientable 4-dimensional Bieberbach group with holonomy group  $\mathbb{Z}_2$ . More precisely,  $\pi \subset \mathbb{R}^4 \rtimes O(4) = E(4) \subset \text{Aff}(\mathbb{R}^4)$  is generated by  $(e_1, I), (e_2, I), (e_3, I), (e_4, I)$  and  $(a, A)$ , where  $a = (1/2, 0, 0, 0)^t$ , and  $A$  is diagonal matrix with diagonal entries 1,  $-1, -1$  and 1. Note that  $(a, A)^2 = (e_1, I)$ . The subgroup generated by  $(e_1, I), (e_2, I), (e_3, I)$ , and  $(a, A)$  forms a 3-dimensional Bieberbach group  $\mathcal{G}_2$ , and  $\pi = \mathcal{G}_2 \times \mathbb{Z}$ . Consider the endomorphism  $\theta : \pi \rightarrow \pi$  which is the composite  $\pi \rightarrow \mathbb{Z} \rightarrow \pi$ , where the first map is the projection onto  $\mathbb{Z} = \langle (e_4, I) \rangle$  and the second map sends  $(e_4, I)$  to  $(a, A)$ . Thus the homomorphism  $\theta$  does not map the maximal normal abelian subgroup  $\mathbb{Z}^4$  (generated by the 4 translations) into itself. Such a  $\mathbb{Z}^4$  is characteristic but not fully invariant in  $\pi$ . Let

$$d = \begin{bmatrix} x \\ 0 \\ 0 \\ y \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and let  $g = (d, D)$ . Then it is easy to see  $\theta(\alpha) \cdot g = g \cdot \alpha$  for all  $\alpha \in \pi$ .

According to the Proposition 1.4, the element  $g = (d, D)$  is the most general form. The matrix  $D$  is uniquely determined and the translation part  $d$  can vary only in two dimensions.

*Proof of Theorem 1.1.* Let  $\Gamma = \pi \cap G, \Gamma' = \pi' \cap G$ . As the example shows, the characteristic subgroup  $\Gamma$  may not go into  $\Gamma'$  by the homomorphism  $\theta$ . Let  $\Lambda = \Gamma \cap \theta^{-1}(\Gamma')$ . Then  $\Lambda$  is a normal subgroup of  $\pi$  and has a finite index. Let  $Q = \pi/\Lambda$ .

Consider the homomorphism  $\Lambda \rightarrow \Gamma' \hookrightarrow G$ , where the first map is the restriction of  $\theta$ . Since  $\Lambda$  is a lattice of  $G$ , by Mal'cev's work, any such a homomorphism extends uniquely to a continuous homomorphism  $C : G \rightarrow G$ , cf. [5, 2.11]. Thus,  $\theta|_\Lambda = C|_\Lambda$ , where  $C \in \text{Endo}(G)$ ; and hence,  $\theta(z, 1) = (Cz, 1)$  for all  $z \in \Lambda$  (more precisely,  $(z, 1) \in \Lambda$ ).

Let us denote the composite homomorphism  $\pi \rightarrow \pi' \rightarrow G \rtimes \text{Aut}(G) \rightarrow \text{Aut}(G)$  by  $\bar{\theta}$ ; and define a map  $f : \pi \rightarrow G$  by

$$(1) \quad \theta(w, K) = (Cw \cdot f(w, K), \bar{\theta}(w, K)).$$

For any  $(z, 1) \in \Lambda$  and  $(w, K) \in \pi$ , apply  $\theta$  to both sides of  $(w, K)(z, 1)(w, K)^{-1} = (w \cdot Kz \cdot w^{-1}, 1)$  to get  $Cw \cdot f(w, K) \cdot \bar{\theta}(w, K)(Cz) \cdot f(w, K)^{-1} \cdot (Cw)^{-1} = \theta(w \cdot Kz \cdot w^{-1})$ . However,  $w \cdot Kz \cdot w^{-1} \in \Lambda$  since  $\Lambda$  is normal in  $\pi$ , and the latter term equals to  $C(w \cdot Kz \cdot w^{-1}) = Cw \cdot CKz \cdot (Cw)^{-1}$  since  $C : G \rightarrow G$  is a homomorphism. From this we have

$$(2) \quad \bar{\theta}(w, K)(Cz) = f(w, K)^{-1} \cdot CKz \cdot f(w, K).$$

This is true for all  $z \in \Lambda$ . Note that  $\bar{\theta}(w, K)$  and  $K$  are automorphisms of the Lie group  $G$ ; and  $C : G \rightarrow G$  is an endomorphism. By the uniqueness of extension of a homomorphism  $\Lambda \rightarrow G$  to an endomorphism  $G \rightarrow G$ , as mentioned above, *the equality (2) holds true for all  $z \in G$* . It is also easy to see that  $f(zw, K) = f(w, K)$  for all  $z \in \Lambda$  so that  $f : \pi \rightarrow G$  does not depend on  $\Lambda$ . Thus,  $f$  factors through  $Q = \pi/\Lambda$ . Moreover,  $\bar{\theta} : \pi \rightarrow \text{Aut}(G)$  also factors through  $Q$  since  $\Lambda$  maps trivially into  $\text{Aut}(G)$ . We still use the notation  $(w, K)$  to denote elements of  $Q$  and  $\bar{\theta}$  to denote the induced map  $Q \rightarrow \text{Aut}(G)$ .

*CLAIM.* *With the  $Q$ -structure on  $G$  via  $\bar{\theta} : Q \rightarrow \text{Aut}(G)$ ,  $f \in Z^1(Q; G)$ ; that is  $f : Q \rightarrow G$  is a crossed homomorphism.*

*Proof.* We shall show

$$f((w, K) \cdot (w', K')) = f(w, K) \cdot \bar{\theta}(w, K)f(w', K')$$

for all  $(w, K), (w', K') \in \pi$ . (Note that we are using the elements of  $\pi$  to denote the elements of  $Q$ .) Apply  $\theta$  to both sides of  $(w, K)(w', K') = (w \cdot Kw', KK')$  to get  $Cw \cdot f(w, K) \cdot \bar{\theta}(w, K)[Cw' \cdot f(w', K')] = C(w \cdot Kw') \cdot f((w, K)(w', K'))$ . From this it follows that

$$\begin{aligned} f((w, K)(w', K')) &= (CKw')^{-1} \cdot f(w, K) \\ &\quad \cdot \bar{\theta}(w, K)(Cw') \cdot \bar{\theta}(w, K)f(w', K'). \end{aligned}$$

From (2) we have  $\bar{\theta}(w, K)Cw' = f(w, K)^{-1} \cdot CKw' \cdot f(w, K)$  so that  $f((w, K) \cdot (w', K')) = f(w, K) \cdot \bar{\theta}(w, K)f(w', K')$ .  $\square$

In [4], it was proved that  $H^1(Q; G) = 0$  whenever  $Q$  is a finite group and  $G$  is a connected and simply connected nilpotent Lie

group. The proof uses induction on the nilpotency of  $G$  together with the fact that  $H^1(Q; G) = 0$  for a finite group  $Q$  and a real vector group  $G$ . This means that any crossed homomorphism is “principal”. In other words, there exists  $d \in G$  such that

$$(3) \quad f(w, K) = d \cdot \bar{\theta}(w, K)(d^{-1}).$$

Let  $D = \mu(d^{-1}) \circ C$  and  $g = (d, D) \in \text{aff}(G)$ , and we check that  $\theta$  is “conjugation” by  $g$ . Using (1), (2) and (3), one can show  $\bar{\theta}(w, K) \circ \mu(d^{-1}) \circ C = \mu(d^{-1}) \circ C \circ K$ . Thus, for any  $(w, K) \in \pi$ ,

$$\begin{aligned} & \theta(w, K) \cdot (d, D) \\ &= (Cw \cdot f(w, K), \bar{\theta}(w, K)) \cdot (d, \mu(d^{-1}) \circ C) \\ &= (Cw \cdot f(w, K) \cdot \bar{\theta}(w, K)(d), \bar{\theta}(w, K) \circ \mu(d^{-1}) \circ C) \\ &= (Cw \cdot d \cdot \bar{\theta}(w, K)(d^{-1}) \cdot \bar{\theta}(w, K)(d), \bar{\theta}(w, K) \circ \mu(d^{-1}) \circ C) \\ &= (Cw \cdot d, \mu(d^{-1}) \circ C \circ K) \\ &= (d, D) \cdot (w, K). \end{aligned}$$

This finishes the proof of theorem. □

*Proof of Corollary 1.2.* We start with the homomorphism  $h_{\#} : \pi_1(M) \rightarrow \pi_1(M)$ , induced from  $h$ , as our  $\theta$  in the Theorem 1.1 and obtain  $\tilde{g} = (d, D)$  satisfying

$$h_{\#}(\alpha) \circ \tilde{g} = \tilde{g} \circ \alpha.$$

Let  $g : M \rightarrow M$  be the induced map. Then  $h_{\#} = g_{\#}$ . Since any two continuous maps on a closed aspherical manifold inducing the same homomorphism on the fundamental group (up to conjugation by an element of the fundamental group) are homotopic to each other,  $h$  is homotopic to  $g$ . This completes the proof of the corollary. □

*Proof of Proposition 1.4.* Let  $g = (d, D)$ ,  $\gamma = (c, C)$ . Since  $\theta(\alpha) \cdot g = g \cdot \alpha$  holds when  $\alpha = (z, 1) \in \Lambda$ , we have  $Dz = d^{-1}z'd$ , where  $\theta(z, 1) = (z', 1)$ . Similarly,  $Cz = c^{-1}z'c$ . Thus  $Cz = \mu(c^{-1}d)Dz$  for all  $z \in \Lambda$ . Since  $\Lambda$  is a lattice, this equality holds on  $G$ . Consequently,  $C = \mu(c^{-1}d)D$ . Now  $\gamma = (c, C) = (c, \mu(c^{-1}d)D) = (d^{-1}c, \mu(c^{-1}d))(d, D) = (h, \mu(h^{-1}))(d, D)$ , if we

let  $h = d^{-1}c$ . Set  $\xi = (h, \mu(h^{-1}))$ . Then  $\gamma = \xi \cdot g$ . Now we shall observe that  $h \in G^{\Psi'}$ . Let  $\theta(\alpha) = (b, B)$ . Then  $\theta(\alpha)\xi g = \theta(\alpha)\gamma = \gamma\alpha = \xi g\alpha = \xi\theta(\alpha)g$  yields  $Bh = h$  for all  $(b, B) = \theta(\alpha)$ . Clearly then  $B \in \Psi'$  by definition. For a Bieberbach group  $\pi$ , note that  $\text{rank } H^1(\pi; \mathbb{Z}) = \dim G^{\Phi}$ .  $\square$

**2. Application to Fixed-point theory.** Let  $M$  be a closed manifold and let  $f : M \rightarrow M$  be a continuous map. The *Lefschetz number*  $L(f)$  of  $f$  is defined by

$$L(f) := \sum_k \text{trace}\{(f_*)_k : H_k(M; \mathbb{Q}) \rightarrow H_k(M; \mathbb{Q})\}.$$

To define the *Nielsen number*  $N(f)$  of  $f$ , we define an equivalence relation on  $\text{Fix}(f)$  as follows: For  $x_0, x_1 \in \text{Fix}(f)$ ,  $x_0 \sim x_1$  if and only if there exists a path  $c$  from  $x_0$  to  $x_1$  such that  $c$  is homotopic to  $f \circ c$  relative to the end points. An equivalence class of this relation is called a *fixed point class* (=FPC) of  $f$ . To each FPC  $F$ , one can assign an integer  $\text{ind}(f, F)$ . A FPC  $F$  is called *essential* if  $\text{ind}(f, F) \neq 0$ . Now,

$$N(f) := \text{the number of essential fixed point classes.}$$

These two numbers give information on the existence of fixed point sets. If  $L(f) \neq 0$ , every self-map  $g$  of  $M$  homotopic to  $f$  has a non-empty fixed point set. The Nielsen number is a lower bound for the number of components of the fixed point set of all maps homotopic to  $f$ . Even though  $N(f)$  gives more information than  $L(f)$  does, it is harder to calculate. If  $M$  is an infra-nilmanifold, and  $f$  is homotopically periodic, then it is known that  $L(f) = N(f)$ .

**LEMMA 2.1.** *Let  $B \in \text{GL}(n, \mathbb{R})$  with a finite order. Then  $\det(I - B) \geq 0$ .*

*Proof.* Since  $B$  has finite order, it can be conjugated into the orthogonal group  $O(n)$ . Since all eigenvalues are roots of unity, there exists  $P \in \text{GL}(n, \mathbb{R})$  such that  $PBP^{-1}$  is a block diagonal matrix, with each block being a  $1 \times 1$ , or, a  $2 \times 2$ -matrix. All  $1 \times 1$  blocks must be  $D = [\pm 1]$ , and hence  $\det(I - D) = 0$  or  $2$ . For a

$2 \times 2$  block, it is of the form  $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ . Consequently, each  $2 \times 2$ -block  $D$  has the property that  $\det(I - D) = (1 - \cos t)^2 + \sin^2 t = 2(1 - \cos t) \geq 0$ .  $\square$

**THEOREM 2.2.** *Let  $f : M \rightarrow M$  be a continuous map on an infra-nilmanifold  $M = \pi \backslash G$ . Let  $g = (d, D) \in \text{aff}(G)$  be a homotopy lift of  $f$  by Corollary 1.2. Then  $L(f) = N(f)$  (resp.,  $L(f) = -N(f)$ ) if and only if  $\det(I - D_* A_*) \geq 0$  (resp.,  $\det(I - D_* A_*) \leq 0$ ) for all  $A \in \Phi$ , the holonomy group of  $M$ .*

*Proof.* Since  $L(f)$  and  $N(f)$  are homotopy invariants, we may assume that  $f = g$ . Let  $\Gamma = \pi \cap G$ , and let  $\Lambda = \Gamma \cap f_{\#}^{-1} f_{\#}(\Gamma \cap f_{\#}^{-1}(\Gamma))$ . Then  $\Gamma$  is a normal subgroup of  $\pi$ , of finite index. Moreover,  $f_{\#} : \pi \rightarrow \pi$  maps  $\Lambda$  into itself. Therefore,  $f$  induces a map on the finite-sheeted regular covering space  $\Lambda \backslash G$  of  $\pi \backslash G$ .

Let  $\tilde{f}$  be a lift of  $f$  to  $\Gamma \backslash G$ . Then

$$L(f) = \frac{1}{[\pi : \Lambda]} \sum \text{ind}(f, p_{\Lambda} \text{Fix}(\alpha \tilde{f}))$$

$$N(f) = \frac{1}{[\pi : \Lambda]} \sum |\text{ind}(f, p_{\Lambda} \text{Fix}(\alpha \tilde{f}))|$$

where the sum ranges over all  $\alpha \in \pi/\Lambda$ . See, [2, III 2.12]. However, each  $\alpha \tilde{f}$  is a map on the nilmanifold  $\Lambda \backslash G$ , and hence  $\text{ind}(f, p_{\Lambda} \text{Fix}(\alpha \tilde{f}))$  is determined by  $\det(I - (\alpha f)_*)$ . It is not hard to see that, for any  $\alpha \in \text{Inn}(G)$ ,  $\alpha_*$  has eigenvalue only 1. Therefore, it is enough to look at elements with non-trivial holonomy. Now the hypothesis guarantees that  $\det(I - (\alpha f)_*) = \det(I - D_* A_*) \geq 0$  or  $\leq 0$  always. Consequently,  $L(f) = N(f)$  or  $L(f) = -N(f)$ .

Conversely, suppose  $L(f) = N(f)$  (resp.  $L(f) = -N(f)$ ). Let  $\alpha = (a, A) \in \pi$ . If  $\text{Fix}(g \circ \alpha) = \emptyset$ , then clearly  $\det(I - D_* A_*) = 0$ . Otherwise,  $\text{Fix}(g \circ \alpha)$  is isolated, and the indices at these fixed points are  $\det(I - D_* A_*)$ . By the formula above relating  $L(f)$ ,  $N(f)$  with the ones on covering spaces, all  $\det(I - D_* A_*)$  must have the same sign. This proves the theorem.  $\square$

**COROLLARY 2.3 [3].** *Let  $f : M \rightarrow M$  be a homotopically periodic map on an infra-nilmanifold. Then  $N(f) = L(f)$ .*

*Proof.* Here is an argument which is completely different from the one in [3]. Let  $\Gamma = \pi \cap G$ , and  $\Phi = \pi/\Gamma$ , the holonomy group.



Let  $g = (d, D) \in G \rtimes \text{Aut}(G)$  be a homotopy lift of  $f$  to  $G$ . Let  $E$  be the lifting group of the action of  $\langle g \rangle$  to  $G$ . That is,  $E$  is generated by  $\pi$  and  $g$ . Then  $E/\Gamma$  is a finite group generated by  $\Phi$  and  $D$ . For every  $A \in \Phi$ ,  $DA$  lies in  $E/\Gamma$ , and has a finite order. By Lemma 2.1,  $\det(I - DA) \geq 0$  for all  $A \in \Phi$ . By Theorem 2.2,  $L(f) = N(f)$ . □

Let  $S$  be a connected, simply connected solvable Lie group and  $H$  be a closed subgroup of  $S$ . The coset space  $H \backslash S$  is called a solvmanifold.

**COROLLARY 2.4 [7].** *Let  $f : M \rightarrow M$  be a homotopically periodic map on an infra-solvmanifold. Then  $N(f) = L(f)$ .*

*Proof.* In [5], the statement for solvmanifolds was proved. We needed a subgroup invariant under  $f_{\#}$ . To achieve this, a new model space  $M'$  which is homotopy equivalent to  $M$ , together with a map  $f' : M' \rightarrow M'$  corresponding to  $f$  was constructed. The new space  $M'$  is a fiber bundle over a torus with fiber a nilmanifold; and  $f'$  is fiber-preserving. Moreover, we found a fully invariant subgroup  $\Lambda$  of  $\pi$  of finite index (so, is invariant under  $f'_{\#}$ ). Now we can apply the same argument as in the proof of Theorem 2.2. □

**EXAMPLE 2.5.** Let  $\pi$  be an orientable 3-dimensional Bieberbach group with holonomy group  $\mathbb{Z}_2$ . More precisely,  $\pi \subset \mathbb{R}^3 \rtimes O(3) = E(3)$  is generated by  $(e_1, I), (e_2, I), (e_3, I)$  and  $(a, A)$ , where  $a = (1/2, 0, 0)^t$ ,  $A$  is a diagonal matrix with diagonal entries 1,  $-1$  and  $-1$ . Note that  $(a, A)^2 = (e_1, I)$ . Let  $M = \mathbb{R}^3/\pi$  be the flat manifold. Consider the endomorphism  $\theta : \pi \rightarrow \pi$  which is defined by the conjugation by  $g = (d, D)$ , where

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Let  $f : M \rightarrow M$  be the map induced from  $g$ . There are only two conjugacy classes of  $g$ ; namely,  $g$  and  $\alpha g$ .  $\text{Fix}(g) = (0, 0, 0)^t$  and  $\text{Fix}(\alpha g) = (1/4, 0, 0)^t$ . Since  $\det(I - D) = \det(I - AD) = +2$ ,  $L(f) = N(f) = 2$ .

The Lefschetz number can be calculated from homology groups also.

- (1)  $H_0(M; \mathbb{R}) = \mathbb{R}$ ;  $f_*$  is the identity map.

(2)  $H_1(M; \mathbb{R}) = \mathbb{R}$ , which is generated by the element  $(e_1, I)$ .  $f_*$  is multiplication by 3 (the (1,1)-entry of  $D$ ).

(3)  $H_2(M; \mathbb{R}) = \mathbb{R}$ ;  $f_*$  is multiplication by  $\det \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = -2$ .

(4)  $H_3(M; \mathbb{R}) = \mathbb{R}$ ;  $f_*$  is multiplication by  $\det(D) = -6$ .

Therefore,  $L(f) = \sum (-1)^i \text{trace } f_*^i = 1 - 3 + (-2) - (-6) = 2$ . Note that  $f$  has infinite period, and this example is not covered by Corollary 2.3.

EXAMPLE 2.6. Let  $\pi$  be same as in Example 2.5. This time  $g = (d, D)$ , is given by

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Let  $f : M \rightarrow M$  be the map induced from  $g$ . There are six conjugacy classes of  $g$ ; namely,  $g$  and  $\alpha g, \alpha t_1 g, \alpha t_1^2 g, \alpha t_1^3 g,$  and  $\alpha t_1^4 g$ . Each class has exactly one fixed point. Clearly,  $\det(I - D) = +2$  and  $\det(I - AD) = -10$ . Therefore, the first fixed point has index +1 and the rest have index -1. Consequently,  $L(f) = -4$ , while  $N(f) = 6$ .

### REFERENCES

- [1] L. Auslander, *Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups*, Annals of Math., **71** (1960), 579-590.
- [2] B. J. Jiang, *Lectures on Nielsen fixed point theory*, Contemporary Mathematics 14, A.M.S., vol 14, 1983.
- [3] S. Kwasik and K. B. Lee, *Nielsen numbers of homotopically periodic maps on infra-nilmanifolds*, J. London Math. Soc., (2) **38** (1988), 544-554.
- [4] Y. Kamishima, K. B. Lee and F. Raymond, *The Seifert construction and its applications to infranilmanifolds*, Quarterly J. Math. (Oxford), **34** (1983), 433-452.
- [5] K. B. Lee, *Nielsen Numbers of periodic maps on Solmanifolds*, Proc. Amer. Math. Soc., **116** (1992), 575-579.
- [6] K. B. Lee and F. Raymond, *Rigidity of almost crystallographic groups*, Contemp. Math. Amer. Math. Soc., **44** (1985), 73-78.
- [7] C. McCord, *Nielsen numbers of homotopically periodic maps on infra-solvmanifolds*, Proc. Amer. Math. Soc., **120** (1994), 311-315.
- [8] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, 1972.

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UNIVERSITY OF OKLAHOMA

NORMAN, OK 73019

*E-mail address:* `kblee@dstn3.math.uoknor.edu`

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### EDITORS

Sun-Yung Alice Chang (Managing Editor)  
University of California  
Los Angeles, CA 90095-1555  
pacific@math.ucla.edu

F. Michael Christ  
University of California  
Los Angeles, CA 90095-1555  
christ@math.ucla.edu

Thomas Enright  
University of California  
San Diego, La Jolla, CA 92093  
tenright@ucsd.edu

Nicholas Ercolani  
University of Arizona  
Tucson, AZ 85721  
ercolani@math.arizona.edu

Robert Finn  
Stanford University  
Stanford, CA 94305  
finn@gauss.stanford.edu

Vaughan F. R. Jones  
University of California  
Berkeley, CA 94720  
vfr@math.berkeley.edu

Steven Kerckhoff  
Stanford University  
Stanford, CA 94305  
spk@gauss.stanford.edu

Martin Scharlemann  
University of California  
Santa Barbara, CA 93106  
mgscharl@math.ucsb.edu

Gang Tian  
Courant Institute  
New York University  
New York, NY 10012-1100  
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V. S. Varadarajan  
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