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**VALUES OF THE RIEMANN ZETA FUNCTION AND
INTEGRALS INVOLVING $\log(2 \sinh \frac{\theta}{2})$ AND $\log(2 \sin \frac{\theta}{2})$**

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VALUES OF THE RIEMANN ZETA FUNCTION AND
 INTEGRALS INVOLVING $\log\left(2\sinh\frac{\theta}{2}\right)$ AND $\log\left(2\sin\frac{\theta}{2}\right)$

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Integrals involving the functions $\log(2\sinh(\theta/2))$ and $\log(2\sin(\theta/2))$ are studied, particularly their relationship to the values of the Riemann zeta function at integral arguments. For example general formulae are proved which contain the known results

$$\int_0^{\frac{\pi}{3}} \log^2(2\sin(\theta/2))d\theta = 7\pi^3/108,$$

$$\int_0^{\frac{\pi}{3}} \theta \log^2(2\sin(\theta/2))d\theta = 17\pi^4/6480,$$

$$\int_0^{\frac{\pi}{3}} (\log^4(2\sin(\theta/2)) - \frac{3}{2}\theta^2 \log^2(2\sin(\theta/2)))d\theta = 253\pi^5/3240,$$

$$\int_0^{\frac{\pi}{3}} (\theta \log^4(2\sin(\theta/2)) - \frac{\theta^3}{2} \log(2\sin(\theta/2)))d\theta = 313\pi^6/408240,$$

as special cases.

1. Introduction. Since the discovery of the formulae

$$(1.1) \quad \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = 2 \int_0^{\frac{\pi}{3}} \theta \log^2\left(2\sin\frac{\theta}{2}\right)d\theta = \frac{17\pi^4}{2^3 \cdot 3^4 \cdot 5},$$

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} = -2 \int_0^{2\log\tau} \theta \log\left(2\sinh\frac{\theta}{2}\right)d\theta$$

$$= \frac{2}{5}\zeta(3), \text{ where } \tau = \frac{1}{2}(1 + \sqrt{5}),$$

the relationship between the values of the Riemann zeta function and integrals involving $\log\left(2\sin\frac{\theta}{2}\right)$ and $\log\left(2\sinh\frac{\theta}{2}\right)$ has been studied by many authors, see for example [2], [4], [5], [7], [9].

Recently Butzer, Markett and Schmidt [2] made use of central and Stirling numbers to obtain a representation of $\zeta(2m+1)$ by integrals involving $\log\left(2\sin\frac{\theta}{2}\right)$ (see (2.13)). In §2 of this paper, we reprove (2.13) and at the same time prove the analogous formula for $\zeta(2m)$ (see (2.14)). Note that (1.2) is the special case of (2.13) when $m=1$.

In [7], van der Poorten proves (1.1), as well as the formula

$$(1.3) \quad \int_0^{\frac{\pi}{3}} \log^2\left(2\sin\frac{\theta}{2}\right) d\theta = \frac{7\pi^3}{108}$$

and remarks that "It appears that (1.1) and (1.3) are not representative of a much larger class of similar formulas". However in [9] Zucker establishes the two formulae

$$(1.4) \quad \int_0^{\frac{\pi}{3}} \left[\log^4\left(2\sin\frac{\theta}{2}\right) - \frac{3\theta^2}{2} \log^2\left(2\sin\frac{\theta}{2}\right) \right] d\theta = \frac{253\pi^5}{2^3 \cdot 3^4 \cdot 5},$$

$$(1.5) \quad \int_0^{\frac{\pi}{3}} \left[\theta \log^4\left(2\sin\frac{\theta}{2}\right) - \frac{\theta^3}{2} \log\left(2\sin\frac{\theta}{2}\right) \right] d\theta = \frac{313\pi^6}{2^4 \cdot 3^6 \cdot 5 \cdot 7}.$$

In §3, we prove the general formulae

$$(1.6) \quad \int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-2} \frac{(-1)^k \binom{2m-2}{2k}}{(2k+1)2^{2k}} \theta^{2k+1} \log^{2m-2k-2}\left(2\sin\frac{\theta}{2}\right) d\theta$$

$$= \frac{(-1)^m \pi^{2m}}{4m(2m-1)} \left[\left(\frac{1}{6}\right)^{2m-1} - 2\left(1 - \frac{1}{2^{2m-1}}\right) B_{2m} \right],$$

$$(1.7) \quad \int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m}{2k}}{2^{2k}} \theta^{2k} \log^{2m-2k}\left(2\sin\frac{\theta}{2}\right) d\theta$$

$$= \frac{(-1)^m \pi^{2m+1}}{2^{2m+2}} \left[E_{2m} - \frac{1}{(2m+1)3^{2m}} \right],$$

where the B_{2m} and E_{2m} are the Bernoulli and Euler numbers respectively. We remark that (1.1), (1.3), (1.4) and (1.5) are all special

cases of (1.6) and (1.7). Formula (1.6) is basically formula (3.10b) of [2] and formula (1.7) is essentially Theorem 4.1 of [1].

In §4, we establish the relations:

(1.8)

$$\sum_{n=1}^{\infty} \frac{1}{n^{m+2} \binom{2n}{n}} = \frac{(-1)^m 2^m}{m!} \int_0^{\frac{\pi}{3}} \theta \log^m \left(2 \sin \frac{\theta}{2} \right) d\theta,$$

(1.9)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{m+2} \binom{2n}{n}} = \frac{(-1)^m 2^m}{m!} \int_0^{2 \log \tau} \theta \log^m \left(2 \sinh \frac{\theta}{2} \right) d\theta,$$

which generalize the formulae (1.1) and (1.2). The formula (1.8) was given by Zucker [4, (2.5)]. Formula (1.9) can be established in an analogous fashion.

2. Representation of $\zeta(n)$ by integrals involving $\log \left(2 \sinh \frac{\theta}{2} \right)$. For $k \geq 1$, $x \geq 1$, we have

$$(2.1) \quad \int_{x^2}^1 \frac{\log^k(t-1)}{t} dt \stackrel{t=e^\theta}{=} - \int_0^{2 \log x} \left[\log \left(2 \sinh \frac{\theta}{2} \right) + \frac{\theta}{2} \right]^k d\theta,$$

since

$$e^\theta - 1 = e^{\frac{\theta}{2}} \left(e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}} \right) = 2e^{\frac{\theta}{2}} \sinh \frac{\theta}{2},$$

$$\log(e^\theta - 1) = \log \left(2 \sinh \frac{\theta}{2} \right) + \frac{\theta}{2}.$$

Similarly, we have

$$(2.2) \quad \int_{\frac{1}{x^2}}^1 \frac{\log^k(1-t)}{t} dt \stackrel{t=e^{-\theta}}{=} \int_0^{2 \log x} \left[\log \left(2 \sinh \frac{\theta}{2} \right) - \frac{\theta}{2} \right]^k d\theta.$$

We set

(2.3)

$$A(\theta) = \log \left(2 \sinh \frac{\theta}{2} \right),$$

$$I(k) = \int_0^1 \frac{\log^k(1-t)}{t} dt = \int_0^1 \frac{\log^k t}{1-t} dt = (-1)^k k! \zeta(k+1).$$

Taking $x = \tau = \frac{1}{2}(1 + \sqrt{5})$ in (2.2) and (2.1) gives respectively

$$(2.4) \quad f(k) = \int_{1/\tau^2}^1 \frac{\log^k(1-t)}{t} dt = \int_0^{2 \log \tau} \left[A(\theta) - \frac{\theta}{2} \right]^k d\theta,$$

$$(2.5) \quad g(k) = \int_{\tau^2}^1 \frac{\log^k(t-1)}{t} dt = - \int_0^{2 \log \tau} \left[A(\theta) + \frac{\theta}{2} \right]^k d\theta.$$

Now we evaluate $f(k)$ and $g(k)$. We have

$$(2.6) \quad f(k) = \int_{1/\tau^2}^1 \frac{\log^k(1-t)}{t} dt \stackrel{u=1-t}{=} \int_0^{1/\tau} \frac{\log^k u}{1-u} du$$

and

$$\begin{aligned} g(k) &= \int_{\tau^2}^1 \frac{\log^k(t-1)}{t} dt \stackrel{u=t-1}{=} - \int_0^\tau \frac{\log^k u}{1+u} du \\ &= \int_0^\tau \left(\frac{1}{1-u} - \frac{1}{1+u} \right) \log^k u du - \int_0^\tau \frac{\log^k u}{1-u} du \\ &= \int_0^\tau \frac{\log^k u}{1-u^2} du^2 - \int_0^\tau \frac{\log^k u}{1-u} du \\ &= \frac{1}{2^k} \int_0^{\tau^2} \frac{\log^k u}{1-u} du - \int_0^\tau \frac{\log^k u}{1-u} du \\ &= \frac{1}{2^k} \left[I(k) - \int_{\tau^2}^1 \frac{\log^k u}{1-u} du \right] - \left[I(k) - \int_\tau^1 \frac{\log^k u}{1-u} du \right], \end{aligned}$$

that is

$$g(k) = \left(\frac{1}{2^k} - 1 \right) I(k) - \frac{1}{2^k} \int_{\tau^2}^1 \frac{\log^k u}{1-u} du + \int_\tau^1 \frac{\log^k u}{1-u} du.$$

In the two integrals, using the substitution $t = \frac{1}{n}$ and noting $\frac{1}{(1-t)t} = \frac{1}{1-t} + \frac{1}{t}$, we have

$$\begin{aligned} -\frac{1}{2^k} \int_{\tau^2}^1 \frac{\log^k u}{1-u} du &= \frac{(-1)^{k+1}}{2^k} \int_{\frac{1}{\tau^2}}^1 \left(\frac{1}{1-t} + \frac{1}{t} \right) \log^k t dt \\ &= \frac{(-1)^{k+1}}{2^k} I(k) + \frac{(-1)^k}{2^k} \int_0^{\frac{1}{\tau^2}} \frac{\log^k t}{1-t} dt \\ &\quad - \frac{2}{k+1} \log^{k+1} \tau \end{aligned}$$

and

$$\begin{aligned} \int_{\tau}^1 \frac{\log^k u}{1-u} du &= (-1)^k \int_{1/\tau}^1 \left(\frac{1}{1-t} + \frac{1}{t} \right) \log^k t dt \\ &= (-1)^k I(k) + (-1)^{k+1} \int_0^{1/\tau} \frac{\log^k t}{1-t} dt \\ &\quad + \frac{1}{k+1} \log^{k+1} \tau. \end{aligned}$$

Hence we have

$$\begin{aligned} g(k) &= \left(\frac{1}{2^k} - 1 \right) \left(1 + (-1)^{k+1} \right) I(k) + (-1)^{k+1} \int_0^{1/\tau} \frac{\log^k t}{1-t} dt \\ &\quad + \frac{(-1)^k}{2^k} \int_0^{1/\tau^2} \frac{\log^k t}{1-t} dt - \frac{\log^{k+1} \tau}{k+1}. \end{aligned}$$

Thus

$$\begin{aligned} g(2m) &= - \int_0^{1/\tau} \frac{\log^{2m} t}{1-t} dt + \frac{1}{2^{2m}} \int_0^{1/\tau^2} \frac{\log^{2m} t}{1-t} dt - \frac{\log^{2m+1} \tau}{2m+1}, \\ g(2m-1) &= 2 \left(\frac{1}{2^{2m-1}} - 1 \right) I(2m-1) + \int_0^{1/\tau} \frac{\log^{2m-1} t}{1-t} dt \\ &\quad - \frac{1}{2^{2m-1}} \int_0^{1/\tau^2} \frac{\log^{2m-1} t}{1-t} dt - \frac{\log^{2m} \tau}{2m}. \end{aligned}$$

From (2.6), we obtain

$$(2.7) \quad f(2m) + g(2m) = \frac{1}{2^{2m}} \int_0^{1/\tau^2} \frac{\log^{2m} t}{1-t} dt - \frac{\log^{2m+1} \tau}{2m+1},$$

(2.8)

$$\begin{aligned} f(2m-1) - g(2m-1) &= 2 \left(1 - \frac{1}{2^{2m-1}} \right) I(2m-1) \\ &\quad + \frac{1}{2^{2m-1}} \int_0^{1/\tau^2} \frac{\log^{2m-1} t}{1-t} dt + \frac{\log^{2m} \tau}{2m}. \end{aligned}$$

Next we evaluate the integral $\int_0^{1/\tau^2} \frac{\log^k t}{1-t} dt$. Since

$$\int_0^{1/\tau^2} \frac{\log^k t}{1-t} dt = \int_{1/\tau}^1 \frac{\log^k(1-u)}{u} du = I(k) - \int_0^{1/\tau} \frac{\log^k(1-u)}{u} du,$$

it suffices to evaluate

$$\begin{aligned}
 & \int_0^{1/\tau} \frac{\log^k(1-u)}{u} du \\
 &= \int_0^{1/\tau} \log^k(1-u) d \log u \\
 &= \log^k(1-u) \log u \Big|_0^{1/\tau} + k \int_0^{1/\tau} \frac{\log^{k-1}(1-u) \log u}{1-u} du \\
 &= (-1)^{k+1} 2^k \log^{k+1} \tau + k(-1)^{k+1} \int_0^{2 \log \tau} \theta^{k-1} \left[A(\theta) - \frac{\theta}{2} \right] d\theta \\
 &= \frac{(-1)^{k+1} 2^k}{k+1} \log^{k+1} \tau + k(-1)^{k+1} \int_0^{2 \log \tau} \theta^{k-1} A(\theta) d\theta.
 \end{aligned}$$

Hence we have

(2.9)

$$\begin{aligned}
 \int_0^{1/\tau^2} \frac{\log^{2m} t}{1-t} dt &= I(2m) + \frac{2^{2m} \log^{2m+1} \tau}{2m+1} \\
 &\quad + 2m \int_0^{2 \log \tau} \theta^{2m-1} A(\theta) d\theta,
 \end{aligned}$$

(2.10)

$$\begin{aligned}
 \int_0^{1/\tau^2} \frac{\log^{2m-1} t}{1-t} dt &= I(2m-1) - \frac{2^{2m-1} \log^{2m} \tau}{2m} \\
 &\quad - (2m-1) \int_0^{2 \log \tau} \theta^{2m-2} A(\theta) d\theta.
 \end{aligned}$$

Substituting (2.9), (2.10) into (2.7), (2.8) respectively, we obtain

$$(2.11) \quad f(2m) + g(2m) = \frac{I(2m)}{2^{2m}} + \frac{2m}{2^{2m}} \int_0^{2 \log \tau} \theta^{2m-1} A(\theta) d\theta,$$

(2.12)

$$\begin{aligned}
 f(2m-1) - g(2m-1) &= 2 \left(1 - \frac{1}{2^{2m}} \right) I(2m-1) \\
 &\quad - \frac{2m-1}{2^{2m-1}} \int_0^{2 \log \tau} \theta^{2m-2} A(\theta) d\theta.
 \end{aligned}$$

Combining (2.11), (2.12) with (2.3), (2.4), (2.5) gives

THEOREM 1. For $m = 1, 2, \dots$

(2.13)

$$\begin{aligned} \zeta(2m+1) = & -\frac{1}{(2m-1)!} \int_0^{2 \log \tau} \theta^{2m-1} \log \left(2 \sinh \frac{\theta}{2} \right) d\theta \\ & + \frac{2^{2m}}{(2m)!} \int_0^{2 \log \tau} \left\{ \left[\log \left(2 \sinh \frac{\theta}{2} \right) - \frac{\theta}{2} \right]^{2m} \right. \\ & \left. - \left[\log \left(2 \sinh \frac{\theta}{2} \right) + \frac{\theta}{2} \right]^{2m} \right\} d\theta, \end{aligned}$$

and

(2.14)

$$\begin{aligned} \left(1 - \frac{1}{2^{2m}} \right) \zeta(2m) = & -\frac{1}{2^{2m}(2m-2)!} \int_0^{2 \log \tau} \theta^{2m-2} \log \left(2 \sinh \frac{\theta}{2} \right) d\theta \\ & - \frac{1}{2(2m-1)!} \int_0^{2 \log \tau} \left\{ \left[\log \left(2 \sinh \frac{\theta}{2} \right) - \frac{\theta}{2} \right]^{2m-1} \right. \\ & \left. + \left[\log \left(2 \sinh \frac{\theta}{2} \right) + \frac{\theta}{2} \right]^{2m-1} \right\} d\theta. \end{aligned}$$

As previously remarked (2.13) is due to Butzer, Market and Schmidt [2], while (2.14) appears to be new. We note that (2.13) can be rewritten as

$$\begin{aligned} (2.15) \quad \int_0^{2 \log \tau} \sum_{k=1}^{m-1} \frac{\binom{2m-1}{2k-2}}{(2k-1)2^{2k-2}} \theta^{2k-1} \log^{2m-2k+1} \left(2 \sinh \frac{\theta}{2} \right) d\theta \\ + \frac{5}{2^{2m}} \int_0^{2 \log \tau} \theta^{2m-1} \log \left(2 \sinh \frac{\theta}{2} \right) d\theta \\ = -\frac{(2m-1)!}{2^{2m}} \zeta(2m+1), \end{aligned}$$

and (2.14) as

$$\begin{aligned} (2.16) \quad \int_0^{2 \log \tau} \sum_{k=0}^{m-1} \frac{\binom{2m-1}{2k}}{2^{2k}} \theta^{2k} \log^{2m-2k-1} \left(2 \sinh \frac{\theta}{2} \right) d\theta \\ + \frac{2m-1}{2^{2m}} \int_0^{2 \log \tau} \theta^{2m-2} \log \left(2 \sinh \frac{\theta}{2} \right) d\theta \\ = \left(\frac{1}{2^{2m}} - 1 \right) (2m-1)! \zeta(2m) \\ = \frac{(-1)^m}{4m} \pi^{2m} (2^{2m} - 1) B_{2m}, \end{aligned}$$

since

$$(2.17) \quad \zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!}.$$

Taking $m = 1, 2$ in (2.15), we have

$$(2.18) \quad \int_0^{2 \log \tau} \theta \log \left(2 \sinh \frac{\theta}{2} \right) d\theta = -\frac{1}{5}\zeta(3),$$

$$(2.19) \quad \int_0^{2 \log \tau} \left[\theta \log^3 \left(2 \sinh \frac{\theta}{2} \right) + \frac{5}{16}\theta^3 \log \left(2 \sinh \frac{\theta}{2} \right) \right] d\theta = -\frac{3}{8}\zeta(5).$$

Taking $m = 1, 2$ in (2.16) gives

$$(2.20) \quad \int_0^{2 \log \tau} \log \left(2 \sinh \frac{\theta}{2} \right) d\theta = -\frac{\pi^2}{10},$$

$$(2.21) \quad \int_0^{2 \log \tau} \left[\log^3 \left(2 \sinh \frac{\theta}{2} \right) + \frac{15}{16}\theta^2 \log \left(2 \sinh \frac{\theta}{2} \right) \right] d\theta = -\frac{\pi^4}{16}.$$

3. Representation of $\zeta(n)$ by integrals involving $\log \left(2 \sin \frac{\theta}{2} \right)$. We set

$$B = B(\theta) = \log \left(2 \sin \frac{\theta}{2} \right),$$

so that

$$\log i(1 - e^{i\theta}) = B(\theta) + \frac{i\theta}{2}.$$

We consider the integral

$$(3.1) \quad J(k) = \int_1^\omega \frac{[\log i(1-u)]^k}{u} du \stackrel{u=e^{i\theta}}{=} i \int_0^{\frac{\pi}{3}} \left(B + \frac{i\theta}{2} \right)^k d\theta$$

where the first integral is along the arc of the unit circle $|u| = 1$ from $u = 1$ to $u = \omega = e^{\pi i/3}$ in a counter-clockwise direction. Making the substitution $t = i(1-u)$, $u = i(t-i)$, $du = idt$, we obtain

$$(3.2) \quad J(k) = \int_0^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i} dt = \int_0^1 \frac{\log^k t}{t-i} dt + \int_1^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i} dt.$$

We now evaluate the two integrals on the right side of (3.2). We have

$$\begin{aligned} \int_0^1 \frac{\log^k t}{t-i} dt &= \int_0^1 \frac{t+i}{t^2+1} \log^k t dt \\ &= \frac{1}{2} \int_0^1 \frac{\log^k t}{1+t^2} dt^2 + i \int_0^1 \frac{\log^k t}{1+t^2} dt. \end{aligned}$$

Since

$$\int_0^1 \frac{\log^k t}{1+t^2} dt = (-1)^k k! S(k+1),$$

where $S(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$ ($\text{Re}(s) > 0$) is a particular Dirichlet L -function (see (4.4) in [8]) and

$$\begin{aligned} \int_0^1 \frac{\log^k t}{1+t^2} dt^2 &= \frac{1}{2^k} \int_0^1 \frac{\log^k t}{1+t} dt \\ &= \frac{1}{2^k} \left[\int_0^1 \left(\frac{1}{1+t} - \frac{1}{1-t} \right) \log^k t dt + \int_0^1 \frac{\log^k t}{1-t} dt \right] \\ &= \frac{1}{2^k} \left[I(k) - \int_0^1 \frac{\log^k t}{1-t^2} dt^2 \right] = \frac{1}{2^k} \left(1 - \frac{1}{2^k} \right) I(k), \end{aligned}$$

we have

$$(3.3) \quad \int_0^1 \frac{\log^k t}{t-i} dt = \frac{1}{2^k} \left(1 - \frac{1}{2^k} \right) I(k) + i(-1)^k k! S(k+1).$$

For the second integral in (3.2), we have

$$\begin{aligned} \int_1^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i} dt &= \int_1^{\omega^{\frac{1}{2}}} \frac{t+i}{1+t^2} \log^k t dt \\ &\stackrel{t=e^{\frac{i\theta}{2}}}{=} \int_0^{\frac{\pi}{3}} \frac{\left(e^{\frac{i\theta}{2}} + i \right)}{1+e^{i\theta}} \left(\frac{i\theta}{2} \right)^k \frac{i}{2} e^{\frac{i\theta}{2}} d\theta \\ &= \frac{i^{k+1}}{2^{k+2}} \int_0^{\frac{\pi}{3}} \frac{\left[\cos \frac{\theta}{2} + i \left(1 + \sin \frac{\theta}{2} \right) \right]}{\cos \frac{\theta}{2}} \theta^k d\theta \\ &= \frac{i^{k+1}}{2^{k+2}} \left[\frac{1}{k+1} \left(\frac{\pi}{3} \right)^{k+1} + i \int_0^{\frac{\pi}{3}} \frac{1 + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \theta^k d\theta \right] \\ &= \frac{1}{2(k+1)} \left(\frac{\pi}{6} \right)^{k+1} i^{k+1} - \frac{i^k}{2^{k+2}} \int_0^{\frac{\pi}{3}} \frac{\theta^k \cos \frac{\theta}{2}}{1 - \sin \frac{\theta}{2}} d\theta. \end{aligned}$$

Since

$$\begin{aligned} \frac{-1}{2} \int_0^{\frac{\pi}{3}} \frac{\theta^k \cos \frac{\theta}{2}}{1 - \sin \frac{\theta}{2}} d\theta &= \int_0^{\frac{\pi}{3}} \theta^k d \log \left(1 - \sin \frac{\theta}{2} \right) \\ &= \theta^k \log \left(1 - \sin \frac{\theta}{2} \right) \Big|_0^{\frac{\pi}{3}} \\ &\quad - k \int_0^{\frac{\pi}{3}} \theta^{k-1} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \\ &= - \left(\frac{\pi}{3} \right)^k \log 2 - k \int_0^{\frac{\pi}{3}} \theta^{k-1} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta, \end{aligned}$$

we have

$$(3.4) \quad \int_1^{\omega^{\frac{1}{2}}} \frac{\log^k t}{t-i} dt = \frac{1}{2(k+1)} \left(\frac{\pi i}{6} \right)^{k+1} - \frac{i^k}{2^{k+1}} \left[\left(\frac{\pi}{3} \right)^k \log 2 + k \int_0^{\frac{\pi}{3}} \theta^{k-1} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \right].$$

From (3.1)–(3.4), we obtain

$$(3.5) \quad \begin{aligned} &i \int_0^{\frac{\pi}{3}} \left(B + \frac{i\theta}{2} \right)^k d\theta \\ &= \frac{1}{2^{k+1}} \left(1 - \frac{1}{2^k} \right) (-1)^k k! \zeta(k+1) \\ &\quad + i(-1)^k k! S(k+1) + \frac{1}{2(k+1)} \left(\frac{\pi i}{6} \right)^{k+1} \\ &\quad - \frac{i^k}{2} \left[\left(\frac{\pi}{6} \right)^k \log 2 + \frac{k}{2^k} \int_0^{\frac{\pi}{3}} \theta^{k-1} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \right]. \end{aligned}$$

Taking $k = 2m - 1$ and $k = 2m$ in (3.5) gives respectively

$$(3.6) \quad \begin{aligned} &i \int_0^{\frac{\pi}{3}} \left(B + \frac{i\theta}{2} \right)^{2m-1} d\theta \\ &= - \frac{1}{2^{2m}} \left(1 - \frac{1}{2^{2m-1}} \right) (2m-1)! \zeta(2m) + \frac{(-1)^m}{4m} \left(\frac{\pi}{6} \right)^{2m} \\ &\quad + i \left[- (2m-1)! S(2m) + \frac{(-1)^m}{2} \left(\frac{\pi}{6} \right)^{2m-1} \log 2 \right. \\ &\quad \left. + \frac{(-1)^m (2m-1)}{2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-2} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \right], \end{aligned}$$

$$\begin{aligned}
 (3.7) \quad & i \int_0^{\frac{\pi}{3}} \left(B + \frac{i\theta}{2} \right)^{2m} d\theta \\
 &= \frac{1}{2^{2m+1}} \left(1 - \frac{1}{2^{2m}} \right) (2m)! \zeta(2m+1) \\
 &+ \frac{(-1)^{m+1}}{2} \left[\left(\frac{\pi}{6} \right)^{2m} \log 2 + \frac{2m}{2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-1} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \right] \\
 &+ i \left[(2m)! S(2m+1) + \frac{(-1)^m}{2(2m+1)} \left(\frac{\pi}{6} \right)^{2m+1} \right].
 \end{aligned}$$

Taking the real part of (3.6) yields

$$\begin{aligned}
 (3.8) \quad & \left(1 - \frac{1}{2^{2m-1}} \right) \zeta(2m) \\
 &= \frac{(-1)^m}{2(2m)!} \left(\frac{\pi}{3} \right)^{2m} + \frac{2^{2m}}{(2m-1)!} \int_0^{\frac{\pi}{3}} \operatorname{Im} \left(B + \frac{i\theta}{2} \right)^{2m-1} d\theta.
 \end{aligned}$$

Since

$$\begin{aligned}
 \operatorname{Im} \left(B + \frac{i\theta}{2} \right)^{2m-1} &= \sum_{k=0}^{m-1} (-1)^k \binom{2m-1}{2k+1} \left(\frac{\theta}{2} \right)^{2k+1} B^{2m-2k-2} \\
 &= \frac{2m-1}{2} \sum_{k=0}^{m-2} \frac{(-1)^k \binom{2m-2}{2k}}{(2k+1)2^{2k}} \theta^{2k+1} B^{2m-2k-2} \\
 &\quad + \frac{(-1)^{m-1} \theta^{2m-1}}{2^{2m-1}},
 \end{aligned}$$

we have

$$\begin{aligned}
 (3.9) \quad & \left(1 - \frac{1}{2^{2m-1}} \right) \zeta(2m) \\
 &= \frac{(-1)^{m-1} \pi^{2m}}{2(2m)! 3^{2m-1}} - \frac{2^{2m-1}}{(2m-2)!} \int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-2} \frac{(-1)^k \binom{2m-2}{2k}}{(2k+1)2^{2k}} \theta^{2k+1} B^{2m-2k-2} d\theta.
 \end{aligned}$$

Recalling the formula (2.17), (3.9) gives the following result.

THEOREM 2. For $m \geq 2$,

(3.10)

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-2} \frac{(-1)^k \binom{2m-2}{2k}}{(2k+1)2^{2k}} \theta^{2k+1} \log^{2m-2k-2} \left(2 \sin \frac{\theta}{2} \right) d\theta \\ &= \frac{(-1)^m \pi^{2m}}{4m(2m-1)} \left[\left(\frac{1}{6} \right)^{2m-1} - 2 \left(1 - \frac{1}{2^{2m-1}} \right) B_{2m} \right]. \end{aligned}$$

Taking $m = 2, 3$ in (3.10) we obtain

$$(3.11) \quad \int_0^{\frac{\pi}{3}} \theta \log^2 \left(2 \sin \frac{\theta}{2} \right) d\theta = \frac{17\pi^4}{2^4 \cdot 3^4 \cdot 5},$$

(3.12)

$$\int_0^{\frac{\pi}{3}} \left[\theta \log^4 \left(2 \sin \frac{\theta}{2} \right) - \frac{\theta^3}{2} \log \left(2 \sin \frac{\theta}{2} \right) \right] d\theta = \frac{313\pi^6}{2^4 \cdot 3^6 \cdot 5 \cdot 7}.$$

Taking the imaginary part of (3.7) yields

(3.13)

$$\int_0^{\frac{\pi}{3}} \operatorname{Re} \left(B + \frac{i\theta}{2} \right)^{2m} d\theta = (2m)! S(2m+1) + \frac{(-1)^m}{2(2m+1)} \left(\frac{\pi}{6} \right)^{2m+1}.$$

Since

$$\operatorname{Re} \left(B + \frac{i\theta}{2} \right)^{2m} = \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m}{2k}}{2^{2k}} \theta^{2k} B^{2m-2k} + \frac{(-1)^m \theta^{2m}}{2^{2m}}$$

and

$$(3.14) \quad (2m)! S(2m+1) = \frac{(-1)^m}{2} \left(\frac{\pi}{2} \right)^{2m+1} E_{2m},$$

where the E_{2m} are the Euler numbers, we have from (3.13)

THEOREM 3. For $m \geq 1$,

$$(3.15) \quad \begin{aligned} & \int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m}{2k}}{2^{2k}} \theta^{2k} \log^{2m-2k} \left(2 \sin \frac{\theta}{2} \right) d\theta \\ &= \frac{(-1)^m \pi^{2m+1}}{2^{2m+2}} \left[E_{2m} - \frac{1}{(2m+1)3^{2m}} \right]. \end{aligned}$$

Taking $m = 1, 2, 3$ in (3.15), we have

$$(3.16) \quad \int_0^{\frac{\pi}{3}} \log^2 \left(2 \sin \frac{\theta}{2} \right) d\theta = \frac{7\pi^3}{2^2 \cdot 3^3},$$

$$(3.17) \quad \int_0^{\frac{\pi}{3}} \left[\log^4 \left(2 \sin \frac{\theta}{2} \right) - \frac{3}{2} \theta^2 \log^2 \left(2 \sin \frac{\theta}{2} \right) \right] d\theta = \frac{253\pi^5}{2^3 \cdot 3^4 \cdot 5},$$

$$(3.18) \quad \int_0^{\frac{\pi}{3}} \left[\log^6 \left(2 \sin \frac{\theta}{2} \right) - \frac{15}{4} \theta^2 \log^4 \left(2 \sin \frac{\theta}{2} \right) + \frac{15}{16} \theta^4 \log^2 \left(2 \sin \frac{\theta}{2} \right) \right] d\theta \\ = \frac{77821\pi^7}{2^6 \cdot 3^6 \cdot 7}.$$

Taking the imaginary part of (3.6), we obtain

$$(3.19) \quad \int_0^{\frac{\pi}{3}} \operatorname{Re} \left(B + \frac{i\theta}{2} \right)^{2m-1} d\theta \\ = -(2m-1)! S(2m) + \frac{(-1)^m}{2} \left(\frac{\pi}{6} \right)^{2m-1} \log 2 \\ + \frac{(-1)^m (2m-1)}{2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-2} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta.$$

Since

$$\operatorname{Re} \left(B + \frac{i\theta}{2} \right)^{2m-1} = \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m-1}{2k}}{2^{2k}} \theta^{2k} B^{2m-2k-1},$$

we have

$$(3.20) \quad S(2m) = \frac{(-1)^m}{2(2m-1)!} \left(\frac{\pi}{6} \right)^{2m-1} \log 2 \\ + \frac{(-1)^m}{(2m-2)! 2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-2} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \\ + \frac{1}{(2m-1)!} \int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m-1}{2k}}{2^{2k}} \theta^{2k} \\ \cdot \log^{2m-2k-1} \left(2 \sin \frac{\theta}{2} \right) d\theta.$$

Taking $m = 1$ in (3.20) we obtain

$$S(2) = -\frac{\pi}{12} \log 2 - \frac{1}{4} \int_0^{\frac{\pi}{3}} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta - \int_0^{\frac{\pi}{3}} \log \left(2 \sin \frac{\theta}{2} \right) d\theta,$$

where $S(2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965\dots$ is Catalan's constant.

Taking the real part of (3.7), we have

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \operatorname{Im} \left(B + \frac{i\theta}{2} \right)^{2m} d\theta &= \frac{1}{2^{2m+1}} \left(\frac{1}{2^{2m}} - 1 \right) (2m)! \zeta(2m+1) \\ &+ \frac{(-1)^m}{2} \left[\left(\frac{\pi}{6} \right)^{2m} \log 2 + \frac{2m}{2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-1} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \right]. \end{aligned}$$

In view of

$$\operatorname{Im} \left(B + \frac{i\theta}{2} \right)^{2m} = m \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m-1}{2k}}{(2k+1)2^{2k}} \theta^{2k+1} B^{2m-2k-1},$$

we obtain

(3.21)

$$\begin{aligned} &\int_0^{\frac{\pi}{3}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2m-1}{2k}}{(2k+1)2^{2k}} \theta^{2k+1} \log^{2m-2k-1} \left(2 \sin \frac{\theta}{2} \right) d\theta \\ &+ \frac{(-1)^{m-1}}{2^{2m}} \int_0^{\frac{\pi}{3}} \theta^{2m-1} \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \\ &= \frac{1}{2^{2m}} \left(\frac{1}{2^{2m}} - 1 \right) (2m-1)! \zeta(2m+1) + \frac{(-1)^m}{2m} \left(\frac{\pi}{6} \right)^{2m} \log 2. \end{aligned}$$

Taking $m = 1, 2$ in (3.21), we obtain

(3.22)

$$\int_0^{\frac{\pi}{3}} \theta \left[\log \left(2 \sin \frac{\theta}{2} \right) + \frac{1}{4} \log \left(1 - \sin \frac{\theta}{2} \right) \right] d\theta = -\frac{3}{16} \zeta(3) - \frac{\pi^2}{72} \log 2,$$

(3.23)

$$\begin{aligned} &\int_0^{\frac{\pi}{3}} \left[\theta \log^3 \left(2 \sin \frac{\theta}{2} \right) - \frac{1}{4} \theta^3 \log \left(2 \sin \frac{\theta}{2} \right) \right] d\theta \\ &\quad - \frac{1}{16} \int_0^{\frac{\pi}{3}} \theta^3 \log \left(1 - \sin \frac{\theta}{2} \right) d\theta \\ &= -\frac{45}{128} \zeta(5) + \frac{1}{4} \left(\frac{\pi}{6} \right)^4 \log 2. \end{aligned}$$

4. Relations between integrals involving $\log\left(2\sin\frac{\theta}{2}\right)$ and $\log\left(2\sinh\frac{\theta}{2}\right)$ and certain series. The power series expansion of $\frac{\arcsin x}{\sqrt{1-x^2}}$ is given by

$$\frac{\arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{2^{2n} x^{2n-1}}{2n \binom{2n}{n}}, \quad |x| < 1.$$

Integrating and differentiating this equality, we have

$$(4.1) \quad (\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2n^2 \binom{2n}{n}},$$

$$(4.2) \quad \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n}} = \frac{x^2}{1-x^2} + \frac{x \arcsin x}{(1-x^2)^{3/2}}.$$

Next, we apply the method of constructing polylogarithms to the function

$$K_0(x) = \frac{2x \arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n \binom{2n}{n}}.$$

We set

$$K_1(x) = \int_0^x \frac{K_0(x)}{x} dx = (\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2n^2 \binom{2n}{n}},$$

$$K_2(x) = \int_0^x \frac{K_1(x)}{x} dx = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2^2 n^3 \binom{2n}{n}}, \quad \text{etc.}$$

In general, we have

$$(4.3) \quad K_m(x) = \int_0^x \frac{K_{m-1}(x)}{x} dx = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2^m n^{m+1} \binom{2n}{n}}, \quad m = 1, 2, 3, \dots$$

Taking $x = \frac{1}{2}$ in (4.3) gives

$$\sum_{n=1}^{\infty} \frac{1}{2^m n^{m+1} \binom{2n}{n}} = K_m\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{K_{m-1}(x)}{x} dx.$$

Then, using integration by parts ($m - 1$) times, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{2^m n^{m+1} \binom{2n}{n}} &= \int_0^{\frac{1}{2}} K_{m-1}(x) d \log(2x) \\ &= - \int_0^{\frac{1}{2}} \log(2x) \frac{K_{m-2}(x)}{x} dx \\ &= - \frac{1}{2!} \int_0^{\frac{1}{2}} K_{m-2}(x) d \log^2(2x) = \frac{1}{2!} \int_0^{\frac{1}{2}} \log^2(2x) \frac{K_{m-3}(x)}{x} dx \\ &= \dots = \frac{(-1)^{m-1}}{(m-1)!} \int_0^{\frac{1}{2}} \log^{m-1}(2x) \frac{2 \arcsin x}{\sqrt{1-x^2}} dx \left(x = \sin \frac{\theta}{2} \right) \\ &= \frac{(-1)^{m-1}}{2(m-1)!} \int_0^{\frac{\pi}{3}} \theta \log^{m-1} \left(2 \sin \frac{\theta}{2} \right) d\theta, \end{aligned}$$

that is

$$(4.4) \quad \sum_{n=1}^{\infty} \frac{1}{n^{m+2} \binom{2n}{n}} = \frac{(-1)^m 2^m}{m!} \int_0^{\frac{\pi}{3}} \theta \log^m \left(2 \sin \frac{\theta}{2} \right) d\theta, \quad m = 0, 1, 2, \dots$$

Taking $x = \frac{1}{2}$ in (4.2), (4.3) and $m = 0, 1, 2$ in (4.4), we obtain

$$(4.5) \quad \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\sqrt{3}}{27} \pi,$$

$$(4.6) \quad \sum_{n=1}^{\infty} \frac{1}{n \binom{2n}{n}} = \frac{\sqrt{3}}{9} \pi,$$

$$(4.7) \quad \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{\pi^2}{18},$$

$$(4.8) \quad \sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = -2 \int_0^{\frac{\pi}{3}} \theta \log \left(2 \sin \frac{\theta}{2} \right) d\theta,$$

$$(4.9) \quad \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = 2 \int_0^{\frac{\pi}{3}} \theta \log^2 \left(2 \sin \frac{\theta}{2} \right) d\theta = \frac{17\pi^4}{2^3 \cdot 3^4 \cdot 5},$$

$$(4.10) \quad \sum_{n=1}^{\infty} \frac{1}{n^5 \binom{2n}{n}} = -\frac{4}{3} \int_0^{\frac{\pi}{3}} \theta \log^3 \left(2 \sin \frac{\theta}{2} \right) d\theta.$$

Changing x into ix in (4.1) (4.2) and (4.3) with $m = 0$, we have

$$(4.11) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{\binom{2n}{n}} = \frac{x}{1+x^2} + \frac{x \sinh^{-1} x}{(1+x^2)^{3/2}},$$

$$(4.12) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{n \binom{2n}{n}} = \frac{2x \sinh^{-1} x}{\sqrt{1+x^2}},$$

$$(4.13) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{2n^2 \binom{2n}{n}} = (\sinh^{-1} x)^2.$$

Now, taking $x = \frac{1}{2}$ in (4.11), (4.12), (4.13), we deduce (as $\sinh^{-1} \frac{1}{2} = \log \tau$)

$$(4.14) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4}{5\sqrt{5}} \log \tau,$$

$$(4.15) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \binom{2n}{n}} = \frac{1}{\sqrt{5}} \log \tau,$$

$$(4.16) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 \binom{2n}{n}} = 2 \log^2 \tau.$$

Analogous to the construction of $K_m(x)$, we set

$$F_0(x) = \frac{2x \sinh^{-1} x}{\sqrt{1+x^2}}, \quad F_1(x) = \int_0^x \frac{F_0(x)}{x} dx = (\sinh^{-1} x)^2,$$

and, from (4.11) and (4.12), we have

$$(4.17) \quad \begin{aligned} F_m(x) &= \int_0^x \frac{F_{m-1}(x)}{x} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^{2n}}{2^m n^{m+1} \binom{2n}{n}}, \quad m = 1, 2, 3, \dots \end{aligned}$$

After integration by parts, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{m+2} \binom{2n}{n}} = \frac{(-1)^m 2^{m+1}}{m!} \int_0^{\frac{1}{2}} \log^m(2x) \frac{F_0(x)}{x} dx,$$

that is

$$(4.18) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{m+2} \binom{2n}{n}} = \frac{(-1)^m 2^m}{m!} \int_0^{2 \log \tau} \theta \log^m \left(2 \sinh \frac{\theta}{2} \right) d\theta, \\ m = 0, 1, 2, \dots$$

Taking $m = 1$ in (4.18) and appealing to (2.18), we have the well-known formula

$$(4.19) \quad \zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Taking $m = 3$ in (4.18) yields

$$(4.20) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5 \binom{2n}{n}} = -\frac{4}{3} \int_0^{2 \log \tau} \theta \log^3 \left(2 \sinh \frac{\theta}{2} \right) d\theta$$

and substituting into (2.18) gives

$$(4.21) \quad \zeta(5) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5 \binom{2n}{n}} - \frac{5}{6} \int_0^{2 \log \tau} \theta^3 \log \left(2 \sinh \frac{\theta}{2} \right) d\theta.$$

Since

$$\log \left(2 \sinh \frac{\theta}{2} \right) = \log \theta - \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n)}{n(2\pi)^{2n}} \theta^{2n}, \quad 0 < \theta < \pi/2,$$

from (4.21) we obtain

$$(4.22) \quad \zeta(5) = -\frac{5}{6} \left(4 \log \log \tau + 4 \log 2 - 1 \right) \log^4 \tau + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5 \binom{2n}{n}} \\ - \frac{20}{3} \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n)}{n(n+2)\pi^{2n}} \log^{2n+4} \tau.$$

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