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CLASS OF HYPERSURFACES AND MAPPINGS**

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The Schwarz reflection principle in one complex variable can be stated as follows. Let M and M' be two real analytic curves in \mathbb{C} and \mathcal{H} a holomorphic function defined on one side of M , extending continuously through M , and mapping M into M' . Then \mathcal{H} has a holomorphic extension across M . We address here the question of extending this classical theorem to higher complex dimensions for some class of hypersurfaces and mappings.

1. Introduction and main results. Let M and M' be two germs of real analytic hypersurfaces at 0 in \mathbb{C}^{n+1} , $n \geq 1$, and \mathcal{H} a holomorphic mapping defined on one side of M , extending smoothly up to M , and mapping M into M' , with $\mathcal{H}(0) = 0$. We say that the reflection principle holds if \mathcal{H} extends holomorphically across M at 0. In the complex plane, by the classical Schwarz reflection principle, the reflection principle holds. The first results in higher dimension were due to H. Lewy [16] and S. Pincuk [18]. They proved independently that the reflection principle holds if M and M' are strictly pseudoconvex, and \mathcal{H} is a diffeomorphism from M to M' . Other results on the reflection principle have been obtained by Baouendi, Jacobowitz and Treves [2], Baouendi and Rothschild [3], [4], [5], Bell [6], Diederich and Forneaess [10], Diederich and Webster [11], as well as by other mathematicians. In [3] and [4] the authors obtain a reflection principle for M and M' germs of real analytic hypersurfaces at 0, of finite type, satisfying an algebraic condition. The mapping they consider is of finite multiplicity. In [5] the authors consider the case of \mathbb{C}^2 and obtain a more general result which allows M and M' to be of infinite type; in fact they obtain a necessary and sufficient condition for the reflection principle to hold. In this paper, we address the question of extending the reflection principle in \mathbb{C}^{n+1} , $n \geq 1$, to a new class of germs of real analytic hypersurfaces allowing them to be of infinite type, and to a new

class of mappings, generalizing the results obtained in [3], [4], and [5].

To make this more precise, we first introduce notation and definitions needed in the sequel. Let M be a germ of a real analytic hypersurface at 0. After a local holomorphic change of coordinates, we can assume that there exists Ω , a sufficiently small open neighborhood of 0 in \mathbb{C}^{n+1} , $n \geq 1$, such that M is given in Ω by

$$(1.1) \quad \text{Im } w = \varphi(z, \bar{z}, \text{Re } w),$$

with $z \in \mathbb{C}^n$, $w \in \mathbb{C}$, φ a real valued convergent power series and $\varphi(z, 0, w) \equiv 0$. Such a choice of coordinates is called normal coordinates.

Let $\Omega^+ = \{(z, w) \in \Omega \mid \text{Im } w > \varphi(z, \bar{z}, \text{Re } w)\}$, and similarly $\overline{\Omega^+} = \{(z, w) \in \Omega \mid \text{Im } w \geq \varphi(z, \bar{z}, \text{Re } w)\}$. Consider a mapping \mathcal{H} holomorphic in Ω^+ , smooth in $\overline{\Omega^+}$, valued in \mathbb{C}^{n+1} and satisfying $\mathcal{H}(M) \subset M'$, where M' is another germ of a real analytic hypersurface at 0 in \mathbb{C}^{n+1} , also given in normal coordinates (z', w') . We shall always assume $\mathcal{H}(0) = 0$. We shall say that such (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle.

Write $\mathcal{H} = (\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n, \mathcal{G}) = (\mathcal{F}, \mathcal{G})$ and denote by $(F_1, F_2, \dots, F_n, G) = (F, G)$ the formal holomorphic Taylor series of the components $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n, \mathcal{G}$ at 0. Let $H = (f_1, \dots, f_n, g)$ be the restriction of \mathcal{H} to M . Recall that M is flat if after a holomorphic change of coordinates in \mathbb{C}^{n+1} , M is given by $\text{Im } w = 0$.

Let M be a germ of a real analytic hypersurface given in normal coordinates by

$$(1.2) \quad \text{Im } w = \varphi(z, \bar{z}, \text{Re } w) = (\text{Re } w)^m \tilde{\varphi}(z, \bar{z}, \text{Re } w),$$

where $\tilde{\varphi}$ is a real valued convergent power series in $z, \bar{z}, \text{Re } w$ such that $\tilde{\varphi}(z, \bar{z}, 0) \neq 0$ and $m \geq 0$. We shall see that m is independent of the choice of normal coordinates. Write

$$(1.3) \quad \tilde{\varphi}(z, \zeta, 0) = \sum_{\alpha} a_{\alpha}(z) \zeta^{\alpha}.$$

Note that $m = 0$ if and only if M is of finite type in the sense of [8], [15]. We introduce the following definition.

DEFINITION 1.4. M is m -essential at 0 if the ideal $(a_{\alpha}(z))$ in the ring of formal power series $\mathbb{C}[[z]]$ generated by all the $a_{\alpha}(z)$ is

of finite codimension, i.e.

$$(1.5) \quad m - \text{ess.type} M = \dim_{\mathbb{C}} \mathbb{C}[[z]] / (a_{\alpha}(z)) < \infty.$$

Note that M is 0-essential at 0 if and only if M is essentially finite in the sense of [2], [3]. Also we shall see that the above definition is independent of the choice of normal coordinates. Recall the following definitions:

Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle. \mathcal{H} is not totally degenerate at 0 if

$$(1.6) \quad \det \left(\frac{\partial F_j}{\partial z_k}(z, 0) \right) \neq 0, \quad j, k = 1, \dots, n.$$

Also, \mathcal{H} is of finite multiplicity at 0 if

$$(1.7) \quad \text{mult.} \mathcal{H} = \dim_{\mathbb{C}} \mathbb{C}[[z]] / (F(z, 0)) < \infty.$$

Note that 1.7 implies 1.6 by standard algebra ([12]). It is known that these two definitions are independent of the choice of normal coordinates ([3]). For germs of real analytic hypersurfaces which are m -essential at 0, we have the following theorems which extend the results obtained in [3] and [4].

THEOREM 1. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle. Then \mathcal{H} extends holomorphically to a neighborhood of 0 in \mathbb{C}^{n+1} , if one of the following conditions holds:*

- (1) M is m -essential at 0, $G \neq 0$, and \mathcal{H} not totally degenerate at 0.
- (2) M' is m' -essential at 0, $G \neq 0$, and \mathcal{H} of finite multiplicity at 0.
- (3) M' is m' -essential at 0, $G \neq 0$ and \mathcal{H} not totally degenerate at 0.

Write

$$(1.8) \quad G(z) = \sum_{j=k_0}^{\infty} G_j(z) w^j,$$

with k_0 minimal so that $G_{k_0}(z) \neq 0$. We shall see that k_0 is independent of the choice of normal coordinates.

THEOREM 2. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle. If either*

- (1) *M is m -essential at 0, $G \neq 0$, \mathcal{H} is not totally degenerate at 0, or*
- (2) *M' is m' -essential at 0, $G \neq 0$, and \mathcal{H} is of finite multiplicity at 0,*

then

$$(1.9) \quad m - \text{ess. type } M = (\text{mult. } \mathcal{H})(m' - \text{ess. type } M'),$$

with all three integers finite, and

$$(1.10) \quad m - 1 = k_0(m' - 1).$$

Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle. Write

$$(1.11) \quad F(z, w) = \sum_{j=0}^{\infty} F_j^*(z)w^j,$$

where $F_j^*(z) = (F_{j1}^*(z), F_{j2}^*(z), \dots, F_{jn}^*(z))$ are formal power series in z .

Define l to be minimal such that

$$(1.12) \quad F_l^*(z) \neq F_l^*(0).$$

We shall see that l is independent of the choice of normal coordinates if M is not flat, M' is of infinite type and $G \neq 0$. We introduce the following definition.

DEFINITION 1.13. Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M not flat, M' of infinite type, and $G \neq 0$. We say that \mathcal{H} is l -tangentially finite at 0 if

$$(1.14) \quad l - \text{tang. mult. } \mathcal{H} = \dim_{\mathbb{C}} \mathbb{C}[[z]] / (F_l^*(z) - F_l^*(0)) < \infty.$$

Note that \mathcal{H} is 0-tangentially finite if and only if \mathcal{H} is of finite multiplicity. We shall see that the above definition is independent of the choice of normal coordinates. For \mathcal{H} l -tangentially finite, we get the following extension result.

THEOREM 3. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M' of infinite type. Then \mathcal{H} extends holomorphically to a neighborhood of 0 in \mathbb{C}^{n+1} , $n \geq 1$, if any one of the following conditions holds*

- (1) *M is non flat, M' is m' -essential, \mathcal{H} is l -tangentially finite and $G \neq 0$.*
- (2) *M is non flat, M' is m' -essential, $\det \left(\frac{\partial F_{lj}^*}{\partial z_k}(z) \right) \neq 0$ and $G \neq 0$.*

As for mappings of finite multiplicity, there exists a relationship between m -ess.type M and l -tang.mult. \mathcal{H} . Indeed we have:

THEOREM 4. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M m -essential at 0, M' of infinite type, $G \neq 0$, $\det \left(\frac{\partial F_{lj}^*}{\partial z_k}(z) \right) \neq 0$, $j, k = 1, \dots, n$. Then \mathcal{H} is l -tangentially finite and*

$$(l - \text{tang. mult. } \mathcal{H}) \text{ divides } (m - \text{ess. type } M).$$

Note that under the assumptions of Theorem 4, M' need not be m' -essential at 0 as it is shown in the following example.

EXAMPLE 1.15. Consider, in \mathbb{C}^3 , M given by

$$w - \bar{w} = 2i|w|^6(|z_1|^2 + |z_2|^2) + 2i|w|^{18}|z_1|^{16}$$

M' given by

$$w' - \bar{w}' = 2i|w'|^2|z_1'|^{16} + 2i|w'|^4(|z_1'|^2 + |z_2'|^2)$$

$$\mathcal{F}_1(z, w) = z_1w, \mathcal{F}_2(z, w) = z_2w, \mathcal{G}(z, w) = w.$$

Here M is 6-essential but M' is not m' -essential.

REMARK 1.16. Our proof of part (3) of Theorem 1 in the finite type case is different from that given in [3] and [4]. The proof of Theorem 3 for \mathbb{C}^2 is also different from that given in [5].

Section 2 deals with invariants associated to germs of real analytic hypersurfaces and holomorphic maps; we introduce new numerical invariants associated to germs of real analytic hypersurfaces and

holomorphic maps. In Section 3, we give the proofs of Theorem 1 and Theorem 3. The proofs of Theorem 2 and Theorem 4 are given in Section 4.

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2. Invariants associated to germs of real analytic hypersurfaces and holomorphic maps. Let M be a germ of a real analytic hypersurface given in normal coordinates by 1.1. Put $w = s + it$. We have the following lemma:

LEMMA 2.1. *The integer k_0 defined by (1.8) is independent of the choice of normal coordinates.*

This is easily shown by observing from the definition of normal coordinates that we have $G(z, w) = wG^1(z, w)$, with $G^1(z, w)$ another formal power series.

PROPOSITION 2.2. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M' of infinite type, and $G \not\equiv 0$. Then the following is true:*

- (1) $G_{k_0}(z) \equiv G_{k_0}(0) \in \mathbb{R} - \{0\}$.
- (2) *If \mathcal{H} is not totally degenerate, then*

$$m - 1 = k_0(m' - 1),$$

where m and m' are defined by (1.2).

We have the following corollary:

COROLLARY 2.3. *The integer m defined by (1.2) is independent of the choice of normal coordinates.*

The proof is immediate by using (2), since $k_0 = 1$ in this case.

Proof of Proposition 2.2. Applying Proposition 3.16 of [4], we obtain that M is of infinite type. Let M given by 1.2 and M' given by $t' = s'^{m'}\psi(z', \bar{z}', s')$. Write $G(z, w) = w^{k_0}G^{k_0}(z, w)$, with $k_0 \geq 1$.

As $\mathcal{H}(M) \subset M'$, we get

$$(2.4) \quad \begin{aligned} & (s + is^m \tilde{\varphi}(z, \bar{z}, s))^{k_0} G^{k_0} - (s - is^m \tilde{\varphi}(z, \bar{z}, s))^{k_0} \overline{G^{k_0}} \\ & \equiv 2i \left(\frac{(s + is^m \tilde{\varphi}(z, \bar{z}, s))^{k_0} G^{k_0} + (s - is^m \tilde{\varphi}(z, \bar{z}, s))^{k_0} \overline{G^{k_0}}}{2} \right)^{m'} \\ & \quad \cdot \psi \left(F, \bar{F}, \frac{G + \bar{G}}{2} \right), \end{aligned}$$

where

$$G^{k_0} = G^{k_0}(z, s + is^m \tilde{\varphi}(z, \bar{z}, s)) \text{ and } \bar{G}^{k_0} = \bar{G}^{k_0}(\bar{z}, s - is^m \tilde{\varphi}(z, \bar{z}, s)).$$

Using the binomial formula, we can rewrite 2.4 as

$$(2.5) \quad \begin{aligned} & s^{k_0} (G^{k_0} - \bar{G}^{k_0}) + cs^{m+k_0-1} \tilde{\varphi} [G^{k_0} + \bar{G}^{k_0} + \alpha(z, \bar{z}, s)] \\ & = c' s^{k_0 m'} [G^{k_0} + \bar{G}^{k_0} + \beta(z, \bar{z}, s)]^{m'} \psi \left(F, \bar{F}, \frac{G + \bar{G}}{2} \right), \end{aligned}$$

with c, c' constants $\neq 0$.

$$(2.6) \quad \alpha(0, \bar{z}, s) \equiv \alpha(z, 0, s) \equiv 0, \quad \beta(0, \bar{z}, s) \equiv \beta(z, 0, s) \equiv 0.$$

Dividing 2.5 by s^{k_0} , and putting $s = 0, \bar{z} = 0$, we obtain (1). In order to prove (2), we first assume that $m + k_0 - 1 < k_0 m'$. Differentiating 2.5 $m + k_0 - 1$ times with respect to s and putting $s = 0$, we get

$$\begin{aligned} & \frac{\partial^{m-1} G^{k_0}}{\partial w^{m-1}}(z, 0) - \frac{\overline{\partial^{m-1} G^{k_0}}}{\partial w^{m-1}}(\bar{z}, 0) \\ & \quad + C \tilde{\varphi} (G^{k_0}(z, 0) + \overline{G^{k_0}}(\bar{z}, 0) + \alpha(z, \bar{z}, 0)) \equiv 0, \end{aligned}$$

with C constant $\neq 0$.

Using (2.6) and the fact that we work in normal coordinates, we get that $\tilde{\varphi}(z, \bar{z}, 0) \equiv 0$, which is impossible by 1.2.

Suppose that $m + k_0 - 1 > k_0 m'$. Differentiating 2.5 $k_0 m'$ times with respect to s and putting $s = 0$, we get

$$\begin{aligned} & \frac{\partial^{k_0 m' - k_0} G^{k_0}}{\partial w^{k_0 m' - k_0}}(z, 0) - \frac{\overline{\partial^{k_0 m' - k_0} G^{k_0}}}{\partial w^{k_0 m' - k_0}}(\bar{z}, 0) \\ & \quad \equiv C' (G^{k_0}(z, 0) + \overline{G^{k_0}}(\bar{z}, 0) \\ & \quad \quad + \beta(z, \bar{z}, 0))^{m'} \psi(F(z, 0), \bar{F}(\bar{z}, 0), 0), \end{aligned}$$

with C' a constant $\neq 0$.

Using 2.6 and the fact that we work in normal coordinates, we get that

$$(2.7) \quad \psi(F(z, 0), \bar{F}(\bar{z}, 0), 0) \equiv 0.$$

Since \mathcal{H} is not totally degenerate by assumption, it is easily shown, differentiating 2.7 with respect to z_k , $k = 1, \dots, n$, and using Cramer's rule, that 2.7 is impossible. Hence we get the desired equation (2) of Proposition 2.2.

We have the following proposition:

PROPOSITION 2.8. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M' of infinite type, $G \not\equiv 0$ and \mathcal{H} not totally degenerate. Then $G_{k_0}, G_{k_0+1}, \dots, G_{k_0+m-1}$ are constant and real.*

Proof. Dividing 2.5 by s^{k_0} , we get

$$(2.9) \quad (G^{k_0} - \bar{G}^{k_0}) + cs^{m-1}\tilde{\varphi}[G^{k_0} + \bar{G}^{k_0} + \alpha(z, \bar{z}, s)] \\ = c's^{m-1}[G^{k_0} + \bar{G}^{k_0} + \beta(z, \bar{z}, s)]^{m'}\psi\left(F, \bar{F}, \frac{G + \bar{G}}{2}\right).$$

Differentiating 2.9 j times with respect to s , $j \leq m - 1$, and putting $s = 0$, we get

$$G^{k_0}(z, 0) - \bar{G}^{k_0}(\bar{z}, 0) \equiv 0 \\ \vdots \\ \frac{\partial^{m-1}G^{k_0}}{\partial w^{m-1}}(z, 0) - \frac{\overline{\partial^{m-1}G^{k_0}}}{\partial w^{m-1}}(\bar{z}, 0) \equiv 0,$$

as we work in normal coordinates. Putting $\bar{z} = 0$ in these equations, we get the desired conclusion. \square

COROLLARY 2.10. *If \mathcal{H} is a local biholomorphism at 0, and M' is of infinite type, then G_1, \dots, G_m are constant and real.*

We have the following proposition:

PROPOSITION 2.11. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M m -essential, $G \not\equiv 0$ and \mathcal{H} not totally degenerate. Then M' is m' -essential, \mathcal{H} is of finite multiplicity and*

$$(2.12) \quad m - \text{ess. type } M = (\text{mult. } \mathcal{H})(m' - \text{ess. type } M').$$

Before giving the proof, we state the following corollary:

COROLLARY 2.13. *Definition 1.4 and the number defined by 1.5 is independent of the choice of normal coordinates.*

The proof of the Corollary is immediate from Proposition 2.11. Note that, unlike the finite type case, the conditions M m -essential and $G \neq 0$ are not enough to guarantee \mathcal{H} not totally degenerate, as it is shown in the following example (considered in [5] for another purpose).

EXAMPLE 2.14. Consider in \mathbb{C}^2 $\mathcal{F}(z, w) = (1 + z)w$, $\mathcal{G}(z, w) = -z(1 + z)w^3$, M' given by $w' - \bar{w}' = z' \bar{z}'^2 - \bar{z}' z'^2$, and M given by $t = s\psi(z, \bar{z})$, with $\psi(z, 0) \equiv \psi(0, \bar{z}) \equiv 0$, and ψ chosen such that $\mathcal{H} = (\mathcal{F}, \mathcal{G})$ maps M into M' . Here, we have M is 1-essential, $G \neq 0$, M' is 0-essential, but \mathcal{H} is totally degenerate.

Proof of Proposition 2.11. The case $m = 0$ has been considered in [3] and [4]. Assume $m > 0$. By Proposition 3.28 in [4], we have $m' > 0$. Differentiating 2.5 $m + k_0 - 1 = k_0 m'$ times with respect to s , and putting $s = 0$, we obtain

$$(2.15) \quad \begin{aligned} C\tilde{\varphi}(z, \bar{z}, 0)(G^{k_0}(z, 0) + \overline{G^{k_0}}(\bar{z}, 0) + \alpha(z, \bar{z}, 0)) \\ \equiv C'(G^{k_0}(z, 0) + \overline{G^{k_0}}(\bar{z}, 0) \\ + \beta(z, \bar{z}, 0))^{m'} \psi(F(z, 0), \overline{F}(\bar{z}, 0), 0), \quad C, C' \text{ constants } \neq 0. \end{aligned}$$

Using 2.6 and (1) of proposition 2.2, we can rewrite 2.15 as

$$h(z, \bar{z})\varphi(z, \bar{z}, 0) \equiv \psi(F(z, 0), \overline{F}(\bar{z}, 0), 0),$$

where $h(z, \bar{z})$ is a formal power series with $h(0) \neq 0$. Inspecting the proof of Theorem 3 in [3], which uses tools of commutative algebra, we conclude that \mathcal{H} is of finite multiplicity, M' is m' -essential and that 2.12 holds. This completes the proof of Proposition 2.11.

We denote by L_k , $k = 1, \dots, n$, the antiholomorphic vector fields tangent to M given by

$$(2.16) \quad L_k = \frac{\partial}{\partial \bar{z}_k} - i \frac{\varphi_{\bar{z}_k}}{1 + i\varphi_s} \frac{\partial}{\partial s}.$$

Let l be defined by 1.12. We have the following propositions:

PROPOSITION 2.17. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with $G \neq 0$ and M' of infinite type at 0. Then we have:*

$$(2.18) \quad m > 2l,$$

$$(2.19) \quad G_{k_0+j}(z) \equiv G_{k_0+j}(0), \quad 0 \leq j \leq l.$$

Proof. The proof is similar to that of 2.5 and 2.6 in [5], and is left to the reader. \square

PROPOSITION 2.20. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with $G \neq 0$, M not flat and M' of infinite type at 0. Then*

$$(2.21) \quad \begin{aligned} f(z, \bar{z}, s) &= s^p \tilde{f}_0(z, \bar{z}, s), \\ \bar{L}_j f(z, \bar{z}, s) &= s^l \tilde{f}_j(z, \bar{z}, s), \quad j = 1, \dots, n, \end{aligned}$$

where p is minimal such that $F_p^*(z) \neq 0$, $\tilde{f}_0 = (\tilde{f}_{01}, \tilde{f}_{02}, \dots, \tilde{f}_{0n})$, with \tilde{f}_0 smooth, and $\tilde{f}_j = (\tilde{f}_{j1}, \tilde{f}_{j2}, \dots, \tilde{f}_{jn})$, with \tilde{f}_j smooth.

Furthermore, there exists an index j_0 such that

$$(2.22) \quad \bar{L}_{j_0} f(z, \bar{z}, s) = s^l \tilde{f}_{j_0}(z, \bar{z}, s),$$

with $\tilde{f}_{j_0}(z, \bar{z}, 0) \neq 0$.

Proof. The proof is similar to that of Theorem 3 in [5] and is left to the reader. \square

PROPOSITION 2.23. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M not flat, M' of infinite type at 0 and $G \neq 0$. Then the number l is independent of the choice of normal coordinates.*

Proof. Let $H = (f_1, f_2, \dots, f_n, g)$ and consider H^* , the pushforward of tangent vectors from M to M' . We have for $i = 1, \dots, n$,

$$H^*(L_{iz,s}) = \sum_{j=1}^n c_{ij}(z, \bar{z}, s) L'_{jH(z, \bar{z}, s)},$$

with c_{ij} smooth. Write $(c_{ij}(z, \bar{z}, s)) = s_\alpha (\tilde{c}_{ij}(z, \bar{z}, s))$ with \tilde{c}_{ij} smooth, $(\tilde{c}_{ij}(z, \bar{z}, 0)) \neq (0)$. Using standard tools of linear algebra, it is easily shown that α is independent of the choice of normal coordinates. Using the chain rule, it is easy to show that $(c_{ij}(z, \bar{z}, s)) = (L_i f_j(z, \bar{z}, s))$. Hence, by 2.22, we conclude that the number l is independent of the choice of normal coordinates. \square

REMARK 2.24. It is easily shown that the number l is also an invariant if M is of finite type and $G \neq 0$. It would be interesting to know whether l is again a biholomorphic invariant in the case M of infinite type, M' of finite type and $G \neq 0$. It should be noted as shown in Theorem 2 in [5] that l is an invariant in \mathbb{C}^2 for this case. Also, if $k_0 = \infty$, i.e. $G \equiv 0$, then l is not a biholomorphic invariant, even for the \mathbb{C}^2 case, as it is shown in Remark 2.30 in [5].

We have the following proposition:

PROPOSITION 2.25. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, where M is not flat, M' of infinite type at 0 and $G \neq 0$. Then the number defined by (1.14) is independent of the choice of normal coordinates.*

Proof. We have $F_l^*(z) = l! \frac{\partial^l F}{\partial w^l}(z, 0)$. The case $l = 0$ has been considered in [3]. Let $l \geq 1$. Consider $\theta : (z, w) \rightarrow (\tilde{z}, \tilde{w})$ a holomorphic change of normal coordinates in the source, and let $\tilde{F} = F \circ \theta^{-1}$. By Proposition 2.23, we have to compare

$$\frac{\partial^l F}{\partial w^l}(z, 0) \text{ and } \frac{\partial^l \tilde{F}}{\partial \tilde{w}^l}(\tilde{z}, 0).$$

We have

$$\begin{aligned} (2.26) \quad \frac{\partial^l F}{\partial w^l}(z, 0) &= \frac{\partial^{l-1}}{\partial w^{l-1}} \left(\frac{\partial \tilde{F}(\tilde{z}, \tilde{w})}{\partial w} \right) (z, 0) \\ &= \frac{\partial^{l-1}}{\partial w^{l-1}} \left(\frac{\partial \tilde{F}}{\partial \tilde{w}}(\tilde{z}, \tilde{w}) \frac{\partial \tilde{w}}{\partial w}(z, w) \right. \\ &\quad \left. + \sum_{j=1}^n \frac{\partial \tilde{F}}{\partial \tilde{z}_j}(\tilde{z}, \tilde{w}) \frac{\partial \tilde{z}_j}{\partial w}(z, w) \right) (z, 0). \end{aligned}$$

By definition of l , we have that 2.26 is of the form $\frac{\partial^l \tilde{F}}{\partial \tilde{w}^l}(\tilde{z}, 0)$ · $\left(\frac{\partial \tilde{w}}{\partial w}(z, 0)\right)^l$ + a sum of terms which are product of $\frac{\partial^k \tilde{F}}{\partial \tilde{w}^k}(\tilde{z}, 0)$ and $\frac{\partial^\alpha \tilde{w}}{\partial w^\alpha}(z, 0)$, $1 \leq \alpha \leq l$, $1 \leq k \leq l - 1$. Using Corollary 2.10 and 2.18, we conclude that

$$\frac{\partial^l F}{\partial w^l}(z, 0) - \frac{\partial^l F}{\partial w^l}(0, 0) = C \left(\frac{\partial^l \tilde{F}}{\partial \tilde{w}^l}(\tilde{z}, 0) - \frac{\partial^l \tilde{F}}{\partial \tilde{w}^l}(0, 0) \right),$$

with C constant $\neq 0$. This completes the proof for a holomorphic change of normal coordinates in the source.

Consider $\kappa^{-1} : (z', w') \rightarrow (\tilde{z}', \tilde{w}')$ a holomorphic change of normal coordinates in the target space, and let $\tilde{H} = \kappa^{-1} \circ H$. We have $F_j(z, w) = \kappa_j \circ \tilde{H}(z, w)$, where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n+1})$. As $l \geq 1$, we have

$$(2.27) \quad \tilde{H}(z, 0) = \tilde{H}(0, 0) = 0.$$

We get

$$(2.28) \quad \begin{aligned} \frac{\partial^l F_j}{\partial w^l}(z, 0) &= \frac{\partial^l (\kappa_j \circ \tilde{H})}{\partial w^l}(z, 0) \\ &= \frac{\partial^{l-1}}{\partial w^{l-1}} \left(\sum_{k=1}^n \frac{\partial \kappa_j}{\partial \tilde{z}'_k}(\tilde{H}(z, w)) \frac{\partial \tilde{F}_k}{\partial w} \right. \\ &\quad \left. + \frac{\partial \kappa_j}{\partial \tilde{w}'}(\tilde{H}(z, w)) \frac{\partial \tilde{G}}{\partial w} \right) (z, 0). \end{aligned}$$

Using the definition of l , 2.19 and 2.27, we obtain that 2.28 is of the form

$$C' + \sum_{k=1}^n \frac{\partial \kappa_j}{\partial \tilde{z}'_k}(0) \frac{\partial^l \tilde{F}_k}{\partial w^l}(z, 0),$$

with C' constant. Therefore

$$\frac{\partial^l F_j}{\partial w^l}(z, 0) - \frac{\partial^l F_j}{\partial w^l}(0, 0) = \left(\frac{\partial \kappa_j}{\partial \tilde{z}'_k}(0) \right) \left(\frac{\partial^l \tilde{F}_k}{\partial w^l}(z, 0) - \frac{\partial^l \tilde{F}_k}{\partial w^l}(0, 0) \right).$$

As we work in normal coordinates, $\det \left(\frac{\partial \kappa_j}{\partial \tilde{z}'_k}(0) \right) \neq 0$. This completes the proof in the case of a holomorphic change of normal coordinates in the target. \square

3. Proof of Theorem 1 and Theorem 3. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$. Define L^α to be $L^\alpha = L_1^{\alpha_1} L_2^{\alpha_2} \dots L_n^{\alpha_n}$. We have the following propositions:

PROPOSITION 3.1. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle. Then the following is true:*

- (1) *If \mathcal{H} is not totally degenerate at 0, then there exists a multi-index α_0 such that*

$$L^{\alpha_0}(\det L_j \bar{f}_k)(0) \neq 0, \quad j, k = 1, \dots, n.$$

- (2) *If M is not flat, M' is of infinite type at 0, $G \not\equiv 0$ and $\det \left(\frac{\partial F_{ij}^*}{\partial z_k}(z) \right) \neq 0$, then there exists a multi-index β_0 such that*

$$(3.2) \quad L^{\beta_0} \left(\det \bar{f}_{jk} \right) (0) \neq 0, \quad j, k = 1, \dots, n,$$

where \bar{f}_{jk} is given by (2.21). Furthermore, if

$$(3.3) \quad D(z, \bar{z}, s) = \det L_j \bar{f}_k(z, \bar{z}, s),$$

then for every multi-index α ,

$$(3.4) \quad L^\alpha D(z, \bar{z}, s) = s^{n_l} D_\alpha(z, \bar{z}, s),$$

with $D_\alpha(z, \bar{z}, s)$ smooth and $D_{\beta_0}(0) \neq 0$.

Proof of (1). See Proposition 3.18 of [3].

Proof of (2). We can assume $l \geq 1$. Let M be given in normal coordinates by 1.2. By 2.21, we have

$$(3.5) \quad \bar{f}_k(z, \bar{z}, s) = s^p \bar{f}_{0k}(z, \bar{z}, s),$$

with \bar{f}_{0k} smooth, $k = 1, \dots, n$. We claim that $s^{p-l} L_j \bar{f}_{0k}$ is smooth and

$$(3.6) \quad \bar{f}_{jk}(z, \bar{z}, 0) = \left(\frac{L_j \bar{f}_{0k}}{s^{l-p}} \right) (z, \bar{z}, 0).$$

Indeed, using 2.18, 2.21, and 3.5, we obtain

$$(3.7) \quad \begin{aligned} L_j \bar{f}_k(z, \bar{z}, s) &= s^l \overline{f_{jk}}(z, \bar{z}, s) \\ &= s^p L_j \overline{f_{0k}}(z, \bar{z}, s) + s^{2l} h(z, \bar{z}, s), \end{aligned}$$

with $h(z, \bar{z}, s)$ smooth. Therefore, dividing 3.7 by s^l , we get that $s^{p-l} L_j \overline{f_{0k}}$ is smooth; putting $s = 0$ in 3.7, we get 3.6. Hence the claim is proved. On the other hand, as $m \geq 1$, we have

$$(3.8) \quad \begin{aligned} [L^\alpha(\det \overline{f_{jk}})](0) &= [L^\alpha(\det \overline{f_{jk}}(z, \bar{z}, 0))](0) \\ &= [D^\alpha(\det \overline{f_{jk}}(z, \bar{z}, 0))](0), \end{aligned}$$

where

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial \bar{z}_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial \bar{z}_n^{\alpha_n}}.$$

Using 2.18, we have, for $\bar{w} = s - is^{m'} \tilde{\varphi}(z, \bar{z}, s)$,

$$\begin{aligned} \overline{f_{jk}}(z, \bar{z}, 0) &= \left(\frac{L_j \overline{f_{0k}}}{s^{l-p}} \right) (z, \bar{z}, 0) \\ &\sim \left(s^{p-l} [L_j (s^{-p} (\bar{F}_{pk}^* \bar{w}^p + \dots + \bar{F}_{lk}^* \bar{w}^l + \dots))] \right) (z, \bar{z}, 0) \\ &\equiv \left(p s^{p-l} \bar{F}_{pk}^* (1 - is^{m-1} \tilde{\varphi})^{p-1} (-is^{m-1} \tilde{\varphi}_{\bar{z}_j}) \right) (z, \bar{z}, 0) + \dots \\ &\quad + \left(l \bar{F}_{lk}^* (1 - is^{m-1} \tilde{\varphi})^{l-1} (-is^{m-1} \tilde{\varphi}_{\bar{z}_j}) \right) \\ &\quad + \frac{\partial \bar{F}_{lk}^*}{\partial \bar{z}_j}(\bar{z}) (1 - is^{m-1} \tilde{\varphi})^l \Big) (z, \bar{z}, 0) \equiv \frac{\partial \bar{F}_{lk}^*}{\partial \bar{z}_j}(\bar{z}). \end{aligned}$$

Therefore,

$$(3.9) \quad L^\alpha(\det \overline{f_{jk}})(0) = \left[D^\alpha \left(\det \frac{\partial \bar{F}_{lk}^*}{\partial \bar{z}_j} \right) \right] (0).$$

By assumption, we have

$$(3.10) \quad \det \left(\frac{\partial \bar{F}_{lk}^*}{\partial \bar{z}_j}(\bar{z}) \right) \neq 0.$$

Using 3.9 and 3.10, we get the desired conclusion 3.2. On the other hand,

$$D(z, \bar{z}, s) = \det L_j \bar{f}_k(z, \bar{z}, s) = \det s^l \bar{f}_{jk}(z, \bar{z}, s) = s^{nl} (\det \bar{f}_{jk})(z, \bar{z}, s).$$

Hence, we have

$$\begin{aligned} L^\alpha D(z, \bar{z}, s) &= L^\alpha (s^{nl} (\det \bar{f}_{jk}))(z, \bar{z}, s) \\ &= s^{nl} (L^\alpha (\det \bar{f}_{jk})(z, \bar{z}, s) + sk(z, \bar{z}, s)), \end{aligned}$$

with $k(z, \bar{z}, s)$ smooth. Using 3.2, we get the desired conclusion 3.4.

Let M given in normal coordinates by 1.2. Solving for \bar{w} in 1.2, it is easily shown that M is also given by

$$(3.11) \quad \bar{w} = Q(z, \bar{z}, w) = w + w^m S(z, \bar{z}, w) = w + \sum_{j=m}^{\infty} R_j(z, \bar{z}) w^j,$$

with $R_j(z, 0) \equiv R_j(0, \bar{z}) \equiv 0$.

Let k_0 be defined by 1.9 and g be the CR function obtained from \mathcal{G} by restriction to M . We need the following lemmas:

LEMMA 3.12. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, where M' is of infinite type at 0 and $G \not\equiv 0$. Then we have*

$$(3.13) \quad g(z, \bar{z}, s) = s^{k_0} g_1(z, \bar{z}, s),$$

with $g_1(z, \bar{z}, s)$ smooth and $g_1(0) \neq 0$.

Proof. See Proof of Theorem 3 in [5]. □

LEMMA 3.14. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, where M' is of infinite type and $G \not\equiv 0$. If $h(z, \bar{z}, s + it)$ is holomorphic in $R = \{s + it \text{ such that } |s| < r, 0 < t < r\}$, C^∞ in $R \cup (-r, r)$ for $|z| < \epsilon$, if $\frac{h}{g}(z, \bar{z}, s)$ is $C^\infty(-r, r)$ for $|z| < \epsilon$, then $\frac{h}{g}(z, \bar{z}, s)$ extends to R , for $|z| < \epsilon$.*

Using Lemma 3.12, the proof of Lemma 3.14 is similar to that of Corollary 4.8 in [5]. We shall say that a function $t(z, \bar{z}, s)$ extends

down (resp. up) if $s \rightarrow t(z, \bar{z}, s)$ extends continuously to a function $s + it \rightarrow T(z, \bar{z}, s + it)$, holomorphically for small $t < 0$ (resp. small $t > 0$), uniformly in z .

We have the following proposition:

PROPOSITION 3.15. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle. If either*

- (1) \mathcal{H} is not totally degenerate at 0, or
- (2) M is not flat, M' is of infinite type at 0 given by (3.11), $G \neq 0$ and

$$\det \left(\frac{\partial F_{lj}^*}{\partial z_k}(z) \right) \neq 0,$$

then $R_{m'\zeta'^\alpha}(f, \bar{f})$ satisfies the following equation:

$$(3.16) \quad R_{m'\zeta'^\alpha}(f, \bar{f}) + K_\alpha(f, u) = 0,$$

where α is any multi-index, with $|\alpha| \geq 1$, K_α is holomorphic at $(0, u(0))$, u is a set of functions which extend down, and $K_\alpha(Z, u(0)) \equiv 0$, $Z \in \mathbb{C}^n$.

Proof. We shall prove Proposition 3.15 in the case (2). As $\mathcal{H}(M) \subset M'$, we have

$$(3.17) \quad \bar{g} = g + g^{m'} S(f, \bar{f}, g).$$

First, consider the case $|\alpha| = 1$. Applying L_j , $j = 1, \dots, n$, to 3.17, and using the fact that $L_j g = L_j f_k = 0$, $k = 1, \dots, n$, we get

$$(3.18) \quad L_j \bar{g} = \sum_{k=1}^n g^{m'} S_{\zeta'_k}^l(f, \bar{f}, g) L_j f_k.$$

Considering the n equations 3.18 with unknown $g^{m'} S_{\zeta'_k}^l(f, \bar{f}, g)$ and using Cramer's rule, we obtain

$$(3.19) \quad Dg^{m'} S_{\zeta'_k}^l(f, \bar{f}, g) = h_k,$$

where h_k extends down, and D given by 3.3. Choose β_0 of minimal length satisfying 3.4. Taking the complex conjugate of 3.17, and raising to the m' th power, we obtain

$$(3.20) \quad g^{m'} = \bar{g}^{m'} (1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'}.$$

Replacing 3.20 in 3.19, and using Lemma 3.14, we obtain

$$(3.21) \quad \begin{aligned} D(1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k}(f, \bar{f}, g) \\ = \frac{h_k}{\bar{g}^{m'}} = \tilde{h}_k, \quad \tilde{h}_k \text{ extending down.} \end{aligned}$$

Applying L^{β_0} to both sides of 3.21, we get

$$(3.22) \quad \begin{aligned} (L^{\beta_0} D)(1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k}(f, \bar{f}, g) \\ + \sum_{\gamma_1 + \gamma_2 = \beta_0, \gamma_2 \neq 0} (L^{\gamma_1} D) \left(L^{\gamma_2} \left((1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k}(f, \bar{f}, g) \right) \right) \\ = L^{\beta_0} \tilde{h}_k. \end{aligned}$$

Dividing 3.22 by $L^{\beta_0} D = s^{nl} D_{\beta_0}$, we get

$$(3.23) \quad \begin{aligned} (1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k}(f, \bar{f}, g) \\ + \sum_{\gamma_1 + \gamma_2 = \beta_0, \gamma_2 \neq 0} \left(\frac{L^{\gamma_1} D}{s^{nl} D_{\beta_0}} \right) \left(L^{\gamma_2} \left((1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k}(f, \bar{f}, g) \right) \right) \\ = \frac{L^{\beta_0} \tilde{h}_k}{s^{nl} D_{\beta_0}}. \end{aligned}$$

Taking the complex conjugate of 3.17 and using 3.11, we get

$$(3.24) \quad S_{\zeta'_k}(f, \bar{f}, g) = R_{m' \zeta'_k}(f, \bar{f}) + \bar{g} T_k(f, \bar{f}, \bar{g}),$$

with T_k holomorphic near 0. Since we work in normal coordinates, we have

$$(3.25) \quad b(f, \bar{f}, g) = \frac{1}{(1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'}},$$

with b holomorphic near 0. Multiplying 3.23 by 3.25, and making use of 3.24, we obtain

$$(3.26) \quad R_{m' \zeta'_k}(f, \bar{f}) \\ = -b(f, \bar{f}, \bar{g}) \left[\sum_{\gamma_1 + \gamma_2 = \beta_0, \gamma_2 \neq 0} \left(\frac{L^{\gamma_1} D}{s^{nl} D_{\beta_0}} \right) \cdot \left(L^{\gamma_2} \left((1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k}(f, \bar{f}, g) \right) \right) \right] \\ + b(f, \bar{f}, \bar{g}) \left[\frac{L^{\beta_0} \tilde{h}_k}{s^{nl} D_{\beta_0}} \right] - \bar{g} T_k(f, \bar{f}, g).$$

Put

$$u = \left\{ \frac{L^{\gamma_1} D}{s^{nl} D_{\beta_0}}, \bar{f}, \bar{g}, L^{\alpha} \bar{f}, \mathbb{L}^{\beta} \bar{g}, \frac{L^{\beta_0} \tilde{h}_k}{s^{nl} D_{\beta_0}}, |\alpha|, |\beta| \leq \beta_0 \right\}.$$

By 3.23,

$$\frac{L^{\beta_0} \tilde{h}_k}{s^{nl} D_{\beta_0}}$$

is smooth and hence extends down by Lemma 3.14. Hence, u is a set of functions which extend down. Using 3.23, the minimality of β_0 , we obtain that

$$(3.27) \quad \frac{L^{\beta_0} \tilde{h}_k}{s^{nl} D_{\beta_0}}(0) = 0.$$

Using the minimality of β_0 , 3.26, and 3.27, we obtain the desired conclusion 3.16 for $|\alpha| = 1$. Consider the case $|\alpha| = 2$. Dividing 3.19 by D , and applying L_j , $j = 1, \dots, n$, to both sides of the obtained equation, we get

$$L_j(g^{m'} S_{\zeta'_k}(f, \bar{f}, g)) = L_j \frac{h_k}{D} = \frac{h_{kj}^*}{D^2},$$

with h_{kj}^* smooth, extending down. Therefore

$$g^{m'} \sum_{p=1}^n S_{\zeta'_k \zeta'_p}(f, \bar{f}, g) L_j \bar{f}_p = \frac{h_{kj}^*}{D^2}.$$

Hence, by Cramer's rule, $Dg^{m'} S_{\zeta'_k \zeta'_p}(f, \bar{f}, g) = \frac{a_{kp}}{D^2}$, with a_{kp} smooth, extending down. Using 3.20, and Lemma 3.14, we obtain

$$(3.28) \quad D^3(1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k \zeta'_p}(f, \bar{f}, g) = \tilde{a}_{kp},$$

with \tilde{a}_{kp} extending down. Define

$$B^0 = \{ \beta \in \mathbb{Z}^n \text{ of minimal length such that } D_\beta(0) \neq 0 \},$$

with D_β defined by (3.4)

Define

$$B^{12\dots j} = \{ \beta = (\beta_1, \dots, \beta_n) \in B^{12\dots j-1} \text{ such that } \beta_j \text{ is minimal} \}.$$

There exists j_0 such that $|B^{12\dots j_0}| = 1$. Take β_0 to be the unique element of $B^{12\dots j_0}$. Applying $L^{3\beta_0}$ to both sides of 3.28, we get that

$$(3.29) \quad (L^{\beta_0} D)^3 (1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k \zeta'_p}(f, \bar{f}, g) \\ + \sum_{\substack{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 3\beta_0 \\ (\gamma_1, \gamma_2, \gamma_3) \neq (\beta_0, \beta_0, \beta_0)}} (L^{\gamma_1} D)(L^{\gamma_2} D)(L^{\gamma_3} D)L^{\gamma_4} \\ \cdot \left((1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k \zeta'_p}(f, \bar{f}, g) \right) = L^{3\beta_0} \tilde{a}_{kp}.$$

By the choice of β_0 , a term of the form

$$(L^{\gamma_1} D)(L^{\gamma_2} D)(L^{\gamma_3} D) \left((1 + \bar{g}^{m'-1} \bar{S}(\bar{f}, f, \bar{g}))^{m'} S_{\zeta'_k \zeta'_p}(f, \bar{f}, g) \right),$$

with $|\gamma_1| = |\gamma_2| = |\gamma_3| = |\beta_0|$ cannot occur unless $\gamma_1 = \gamma_2 = \gamma_3 = \beta_0$. That means that

$$\frac{(L^{\gamma_1} D)(L^{\gamma_2} D)(L^{\gamma_3} D)}{(L^{\beta_0} D)^3}(0) = 0, \quad \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 3\beta_0,$$

by 3.4 and by minimality of β_0 . We divide both sides of 3.29 by $(L^{\beta_0} D)^3$ and we put

$$u = \left\{ \frac{(L^{\gamma_1} D)(L^{\gamma_2} D)(L^{\gamma_3} D)}{(L^{\beta_0} D)^3}, \bar{f}, \bar{g}, L^\alpha \bar{f}, L^\beta \bar{g}, \right. \\ \left. \frac{L^{3\beta_0} \tilde{a}_{kp}}{(L^{\beta_0} D)^3}, |\alpha|, |\beta| \leq |3\beta_0| \right\}.$$

We can apply the same process as for $|\alpha| = 1$, in order to get the desired conclusion for the case $|\alpha| = 2$. For the general case, i.e. for a multi-index α of any length, we observe that

$$g^{m'} S_{\zeta'\alpha}(f, \bar{f}, g) = \frac{d_\alpha}{D^{2|\alpha|-1}},$$

with d_α extending down. We apply the same proof as in the case of $|\alpha| = 1, 2$ to get the desired conclusion 3.16. This completes the proof of Proposition 3.15. \square

We need the following lemma:

LEMMA 3.30. *Let M be given by (3.11). Suppose that M is m -essential. Write*

$$R_m(z, \bar{z}) = \sum_{\alpha} b_{\alpha}(z) \bar{z}^{\alpha}.$$

Then for every z_0 sufficiently small, there exists a multi-index α_0 such that $b_{\alpha_0}(z_0) \neq 0$.

The proof is left to the reader.

PROPOSITION 3.31. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle, with M non flat, M' m' -essential at 0, of infinite type, $\det \left(\frac{\partial F_{ij}^*}{\partial z_k}(z) \right) \neq 0$ and $G \neq 0$. Then each $f_j, j = 1, \dots, n$, satisfies a polynomial equation with analytic coefficients depending on functions which extend down.*

Proof. Let M' given in normal coordinates by 3.11. Using Lemma 3.30 and the Nullstellensatz, we can find N and r such that

$$(3.32) \quad z_j^N = \sum_{\alpha=1}^r c_{j\alpha}(z') b_{\alpha}(z'), \quad j = 1, \dots, n,$$

with $c_{j\alpha}(z')$ convergent power series. We also have

$$(3.33) \quad R_{m'\zeta'\alpha}(z', \bar{z}') = \alpha! b_{\alpha}(z') + \sum_{|s| \geq 1} d_{\alpha s}(z') \bar{z}'^s,$$

with $d_{\alpha s}(z')$ convergent power series. As $f_j(0) = 0$, we can substitute f_j in 3.32 and we get

$$(3.34) \quad f_j^N = \sum_{\alpha=1}^r c_{j\alpha}(f) b_{\alpha}(f).$$

Using Proposition 3.15, 3.33 and 3.34, we obtain

$$(3.35) \quad f_j^N + H_j(f, u) = 0,$$

with H_j holomorphic at $(0, u(0))$, $H_j(Z, u(0)) \equiv 0$, $Z \in \mathbb{C}^n$, and u a set of functions which extend down. Using 3.35, the Weierstrass Preparation Theorem and the classical Newton's Theorem for symmetric functions, we claim that f_j , $1 \leq j \leq n$, satisfies a polynomial equation with holomorphic coefficients depending on the set of functions u which extend down. The proof of the claim follows by inspecting the very end of Lemma 6.1 in [3]. Hence, we get the desired conclusion. \square

We have the following proposition:

PROPOSITION 3.36. *Let (M, M', \mathcal{H}) satisfy the hypothesis of the reflection principle. If either*

- (1) \mathcal{H} is not totally degenerate at 0, or
- (2) M is not flat, M' is of infinite type at 0 given by (3.11), $G \neq 0$ and $\det \left(\frac{\partial F_{ij}^*}{\partial z_k}(z) \right) \neq 0$,

then there exists $r > 0$ such that for every $z_0 \in \mathbb{C}^n$ fixed, $|z_0| < r$ and every multi-index α , there exist functions $a(s + it)$ and $b(s + it)$ holomorphic in the domain

$$R = \{s + it \text{ such that } |s| < r, -r < t < 0\},$$

smooth in $R \cup (-r, +r)$, such that

$$Q_{\zeta^\alpha}(f, \bar{f}, g)(z_0, \bar{z}_0, s) = \frac{a(s)}{b(s)}, \quad |s| < r.$$

The proof is similar to that of Lemma 5.3 in [3] and is left to the reader.

Proof of Theorem 1 and Theorem 3. By Proposition 2.11, (1) implies (2) in Theorem 1. Therefore we only have to prove Theorem 1 for condition (2) or (3). As 1.7 implies 1.6, we only have to prove Theorem 1 for condition (3) and Theorem 3 for condition (2). Using Proposition 3.31 and Lemma 7.1 in [3], we conclude that for each

α , $Q_{\zeta^\alpha}(f, \bar{f}, g)$ satisfies a polynomial relation with coefficients which are analytic functions depending on functions which extend down. Using Proposition 3.36, Lemma 7.1 in [3] and Lemma 8.15 in [1], we conclude that $Q_{\zeta^\alpha}(f, \bar{f}, g)$ extends down for every α , and that

$$|Q_{\zeta^\alpha}(f, \bar{f}, g)(z, \bar{z}, s + it)| \leq C^\alpha \alpha!$$

Therefore, following the proof of Theorem 1 in [3], we can conclude that

$$Q(f, \lambda, g)(z, \bar{z}, s) = \sum_{\alpha=0}^{\infty} \frac{(\lambda - \bar{f})^\alpha}{\alpha!} Q_{\zeta^\alpha}(f, \bar{f}, g)(z, \bar{z}, s)$$

extends up and down, uniformly in λ . Taking $\lambda = 0$, we get that g extends down, as we work in normal coordinates. Consider

$$\frac{Q(f, \lambda, g) - g}{g^{m'}} = S(f, \lambda, g).$$

By Lemma 3.14, $S(f, \lambda, g)$ extends up and down. Again, inspecting the proof of Theorem 1 in [3], using the Weierstrass Preparation Theorem, we obtain that f extends down. We are able to complete the proof of (3) of Theorem 1 and the proof of (2) of Theorem 3 by using the following Criterion proved in [2]: \mathcal{H} extends holomorphically through 0 in \mathbb{C}^{n+1} if and only if the function $s \rightarrow H(z, \bar{z}, s)$ extends holomorphically through 0 in \mathbb{C} , uniformly in z .

4. Proof of Theorem 2 and Theorem 4.

Proof of Theorem 2. The case $m = 0$ has been considered in [3] and [4]. Assume $m > 0$. Using Propositions 2.2 and 2.11, Theorem 2 is proved for condition (1). We have to prove Theorem 2 under condition (2). Inspecting the proof of Proposition 2.11, and using Proposition 2.2, we conclude that

$$h(z, \bar{z})\varphi(z, \bar{z}, 0) \equiv \psi(F(z, 0), \bar{F}(\bar{z}, 0), 0),$$

with $h(z, \bar{z})$ a formal power series such that $h(0) \neq 0$. Inspecting the proof of Theorem 2 in [3], which uses tools of commutative algebra, we conclude that M is m -essential. Using condition (1), we get the desired conclusion.

Proof of Theorem 4. By 3.11, M' can be parametrized by

$$w' - \bar{w}' = \sum_{j=1}^{\infty} P_j(z', \bar{z}') \bar{w}'^j,$$

with $P_j(z', 0) \equiv P_j(0, \bar{z}') \equiv 0$. M can be parametrized by

$$w = \bar{w}(1 + \bar{w}^{m-1} R(z, \bar{z}, \bar{w})) = s\lambda,$$

where $s = \bar{w}$ and $\lambda = 1 + \bar{w}^{m-1} R(z, \bar{z}, \bar{w})$, with

$$(4.1) \quad R(0, \bar{z}, \bar{w}) \equiv R(z, 0, \bar{w}) \equiv 0.$$

As $\mathcal{H}(M) \subset M'$, we have

$$(4.2) \quad G(z, s\lambda) - \bar{G}(\bar{z}, s) \equiv \sum_{j=1}^{\infty} P_j(F(z, s\lambda), \bar{F}(\bar{z}, s)) (\bar{G}(\bar{z}, s))^j.$$

Putting $\bar{z} = 0$ in 4.2, we get

$$(4.3) \quad G(z, s) \equiv \bar{G}(0, s) + \sum_{j=1}^{\infty} P_j(F(z, s), \bar{F}(0, s)) \bar{G}(0, s)^j.$$

Taking the complex conjugate of 4.3, we get (s taken to be real)

$$\bar{G}(z, s) \equiv G(0, s) + \sum_{j=1}^{\infty} \bar{P}_j(\bar{F}(\bar{z}, s), F(0, s)) G(0, s)^j.$$

Substituting for $\bar{G}(\bar{z}, s)$ in the right hand side of 4.2, we get

$$(4.4) \quad \begin{aligned} G(z, s\lambda) - \bar{G}(\bar{z}, s) & \\ & \equiv \sum_{j=1}^{\infty} P_j(F(z, s\lambda), \bar{F}(\bar{z}, s)) \left(G(0, s) \right. \\ & \quad \left. + \sum_{q=1}^{\infty} \bar{P}_q(\bar{F}(\bar{z}, s), F(0, s)) G(0, s)^q \right)^j. \end{aligned}$$

From 4.4, we get

$$(4.5) \quad \begin{aligned} (s\lambda)^{k_0} G^{k_0}(z, s\lambda) - s^{k_0} \bar{G}^{k_0}(\bar{z}, s) & \\ & \equiv \sum_{j=1}^{\infty} P_j(F(z, s\lambda), \bar{F}(\bar{z}, s)) \left(s^{k_0} G^{k_0}(0, s) \right. \\ & \quad \left. + \sum_{q=1}^{\infty} \bar{P}_q(\bar{F}(\bar{z}, s), F(0, s)) s^{k_0 q} G^{k_0}(0, s)^q \right)^j. \end{aligned}$$

By the binomial formula,

$$(4.6) \quad (s + s^m R)^{k_0} G^{k_0}(z, s\lambda) \\ \equiv s^{k_0} G^{k_0}(z, s\lambda) + C s^{m+k_0-1} R(G^{k_0}(z, s\lambda) + \alpha(z, \bar{z}, s)),$$

with C a constant different from 0, and $\alpha(z, \bar{z}, s)$ another formal power series such that $\alpha(0, \bar{z}, s) \equiv \alpha(z, 0, s) \equiv 0$. Hence, we get from 4.5

$$(4.7) \quad s^{k_0} G^{k_0}(z, s\lambda) - s^{k_0} \bar{G}^{k_0}(\bar{z}, s) \\ + C s^{m+k_0-1} R(G^{k_0}(z, s\lambda) + \alpha(z, \bar{z}, s)) \\ \equiv \sum_{j=1}^{\infty} P_j(F(z, s\lambda), \bar{F}(\bar{z}, s)) \left(s^{k_0} G^{k_0}(0, s) \right. \\ \left. + \sum_{q=1}^{\infty} \bar{P}_q(\bar{F}(\bar{z}, s), F(0, s)) s^{k_0 q} G^{k_0}(0, s)^q \right)^j.$$

On the other hand, by Taylor's expansion, we have

$$(4.8) \quad F(z, s\lambda) \equiv F(z, s) + s^m R(z, \bar{z}, s) T(z, \bar{z}, s),$$

with T another formal power series. Therefore, 4.7 becomes

$$(4.9) \quad s^{k_0} G^{k_0}(z, s\lambda) - s^{k_0} \bar{G}^{k_0}(\bar{z}, s) \\ + C s^{m+k_0-1} R(G^{k_0}(z, s\lambda) + \alpha(z, \bar{z}, s)) \\ \equiv \sum_{j=1}^{\infty} P_j(F(z, s), \bar{F}(\bar{z}, s)) \left(s^{k_0} G^{k_0}(0, s) \right. \\ \left. + \sum_{q=1}^{\infty} \bar{P}_q(\bar{F}(\bar{z}, s), F(0, s)) s^{k_0 q} G^{k_0}(0, s)^q \right)^j \\ + \sum_{j=1}^{\infty} s^m \tilde{T}_j(z, \bar{z}, s) \left(s^{k_0} G^{k_0}(0, s) \right. \\ \left. + \sum_{q=1}^{\infty} \bar{P}_q(\bar{F}(\bar{z}, s), F(0, s)) s^{k_0 q} G^{k_0}(0, s)^q \right)^j,$$

where \tilde{T}_j is another formal power series. We can rewrite 4.9 as

$$\begin{aligned}
 (4.10) \quad & s^{k_0} G^{k_0}(z, s + s^m R) - s^{k_0} \overline{G}^{k_0}(\bar{z}, s) \\
 & + C s^{m+k_0-1} R(G^{k_0}(z, s\lambda) \\
 & + \alpha(z, \bar{z}, s)) + s^{m+k_0} U(z, \bar{z}, s) \\
 & \equiv \sum_{j=1}^{\infty} P_j(F(z, s), \overline{F}(\bar{z}, s)) \left(s^{k_0} G^{k_0}(0, s) \right. \\
 & \left. + \sum_{q=1}^{\infty} \overline{P}_q(\overline{F}(\bar{z}, s), F(0, s)) s^{k_0 q} G^{k_0}(0, s)^q \right)^j,
 \end{aligned}$$

with $U(z, \bar{z}, s)$ another formal power series.

Write $F(z, s) = a_p s^p + \dots + F_l^*(0) s^l + s^l \mathcal{F}^*$, with $\mathcal{F}^*(z, s) = F_l^*(z) - F_l^*(0) + s\gamma$, where γ is another formal power series. Therefore the right hand side of 4.10 can be written as

$$s^{n_1} h_1(s) + s^{n_2} h_2(\mathcal{F}^*, \overline{\mathcal{F}^*}, s) + s^{n_3} h_3(\mathcal{F}^*, \overline{\mathcal{F}^*}, s),$$

where h_1, h_2, h_3 are formal power series and $h_1(0) \neq 0, h_2(x, y, 0) \not\equiv 0, h_2$ contains only pure power of \mathcal{F}^* and $\overline{\mathcal{F}^*}, h_3(x, y, 0) \not\equiv 0$ and h_3 contains no pure power of \mathcal{F}^* and $\overline{\mathcal{F}^*}$.

We claim that $m + k_0 - 1 = n_3$. The proof of the claim is similar to that of (2) of Proposition 2.2 and is left to the reader. Differentiating 4.10 with respect to s $m + k_0 - 1 = n_3$ times and putting $s = 0$, we get

$$h(z, \bar{z}, 0)R(z, \bar{z}, 0) = h_3(F_l^*(z) - F_l^*(0), \overline{F_l^*}(\bar{z}) - \overline{F_l^*}(0), 0),$$

with h a formal power series such that $h(0) \neq 0$, by (1) of Proposition 2.2. The rest of the proof is similar to that of Proposition 2.11 and is left to the reader.

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