Pacific Journal of Mathematics

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SHAMIM ISMAIL ANSARI

Volume 169 No. 2 June 1995

PACIFIC JOURNAL OF MATHEMATICS
Vol. 169, No. 2, 1995

ON BANACH SPACES Y FOR WHICH $B(C(\Omega), Y) = K(C(\Omega), Y)$

S.I. Ansari

Let Ω be a compact Hausdorff space. In this paper we give some necessary conditions and some sufficient conditions on a Banach space Y in order that all continuous linear operators from $C(\Omega)$ into Y are compact. We prove that for a nonscattered compact Hausdorff space Ω, for Y belonging to a large class of Banach spaces all operators from $C(\Omega)$ into Y are compact if and only if all operators from l^2 into Y are compact.

Introduction. In this paper by the word "operator" we will mean a "continuous linear operator." E. Dubinsky, A. Pelczynski, and H.P. Rosenthal [**8**] have given a characterization of all Banach spaces Y for which all operators from $\mathcal{L}_\infty$ into Y are absolutely 2-summing. Here, our aim is to characterize all Banach spaces Y for which all operators from a $C(\Omega)$-space into Y are compact. We noticed that such a characterization depends on whether the compact Hausdorff space Ω is scattered (dispersed) or nonscattered (nondispersed). So we consider two cases separately.

Case 1: Ω is an infinite scattered compact Hausdorff space. In this case, from some known results we deduce that *all operators from $C(\Omega)$ into a Banach space Y are compact if and only if all operators from a closed subspace of c_0 into Y are compact if and only if Y does not contain a copy of c_0.*

Case 2: Ω is a nonscattered compact Hausdorff space. In this case, we present a necessary condition on a Banach space Y for all operators from $C(\Omega)$ into Y to be compact. Specifically, *if each operator from $C(\Omega)$ into Y is compact, then each operator from l^2 into Y is compact.* Consequently, *for a Banach space Y for which each operator from $C(\Omega)$ into Y is absolutely 2-summing, each operator from $C(\Omega)$ into Y is compact if and only if each operator from l^2 into Y is compact.* Another necessary condition is given by a theorem of T. Terzioglu. Namely, *if each operator from $C(\Omega)$ into Y is compact, then each operator from $C(\Omega)$ into Y factors through a closed subspace of c_0*. Next, we see that the above two necessary conditions together are also sufficient. Putting together: *Each operator from $C(\Omega)$ into Y is compact if and only if each operator from l^2 into Y is compact and each operator from $C(\Omega)$ into Y factors through a closed subspace of c_0.*

In order to prove that another related condition is also sufficient we first generalize a theorem of N.J. Kalton. Then, employing this generalization, and a result of L. Drewnowski we prove: *Each operator from $C(\Omega)$ into Y is compact if and only if each operator from l^2 into Y is compact and each operator form $C(\Omega)$ into Y has a weak unconditional compact netted expansion* (Definition 3.5). Consequently, for a Banach space Y with an unconditional basis consisting of finite dimensional subspaces *all operators from $C(\Omega)$ into Y are compact if and only if all operators from l^2 into Y are compact.* The conclusion is that the class of all Banach spaces Y for which all operators from $C(\Omega)$ into Y are compact if and only if all operators from l^2 into Y are compact is big (see Conclusion 3.12).

In the way we present a necessary and sufficient condition on a Banach space Y for all operators from l^p into Y to be compact for each $p \in [1, \infty)$. We conclude this paper with some results that relate the space of all compact operators on $C(\Omega)$ with the space $\Phi_{c_0}(C(\Omega))$ for all operators factoring through c_0.

1. Notations. Suppose X and Y are Banach spaces. We will denote the space of all bounded linear operators, compact operators, and absolutely 2-summing operators from X into Y by $B(X,Y)$, $K(X,Y)$, and $\Pi_2(X,Y)$, respectively. By "$X \hookrightarrow Y$" we will mean "Y contains a copy of X."

1.1. Scattered-Compact Spaces. Recall that a topological space S is said to be **scattered** or **dispersed** if every nonempty closed subset of S has an isolated point in its induced topology (see [**22**]). In this section we will assume that S is a scattered compact Hausdorff space.

PROPOSITION 1.1. *Suppose X is an infinite dimensional closed subspace of c_0 and Y is a Banach space. Then,* $B(X,Y) = K(X,Y)$

if and only if Y does not contain any copy of c_0.

Proof. Suppose Y does not contain any copy of c_0. Let $T \in B(X,Y)$. Let $\{x_n\}$ be any norm bounded sequence in E. We will show that $\{Tx_n\}$ has a norm convergent subsequence. Since c_0 does not contain any copy of l^1, the space E does not contain any copy of l^1. So by the celebrated l^1-theorem of H.P. Rosenthal [**20**], a subsequence of $\{x_n\}$ is weakly Cauchy. By passing to the subsequence we can assume that the $\{x_n\}$ itself is weakly Cauchy. Let $y_{m,n} = x_n - x_m$. Then the net $\{y_{m,n}\}$ is weakly null. So is the net $\{Ty_{m,n}\}$. We claim that $\|Ty_{m,n}\| \longrightarrow 0$. To arrive at a contradiction suppose this is not the case. Then there exists an $\epsilon > 0$ and sequences $\{m_k\}$ and $n_k\}$ of natural numbers such that $m_k > m_{k-1} \geq k-1$, $n_k > n_{k-1} \geq k-1$, and $\|Ty_{m_k,n_k}\| > \epsilon$. Now by a theorem of C. Bessaga and A. Pelczynski [**4**] a subsequence of Ty_{m_k,n_k} itself is a basic sequence. Since y_{m_k,n_k} is a weakly null sequence in c_0 such that $\inf \|y_{m_k,n_k}\| > 0$, a subsequence of this sequence is a basic sequence and a subsequence of the basic sequence is equivalent to a block basis of the standard basis of c_0. Since every normalized block basis of the standard basis is equivalent to the standard basis, it follows that a subsequence of $\{y_{m_k,n_k}\}$ is equivalent to the standard basis. By passing to the subsequence we can assume that $\{y_{m_k,n_k}\}$ itself is such a sequence. That is, $\{y_{m_k,n_k}\}$ is equivalent to the standard basis of c_0. Now it is easy to verify that $\sum a_k y_{m_k,n_k}$ converges if and only if $\sum a_k T y_{m_k,n_k}$ does. So, the subspace $[Ty_{m_k,n_k}]$ of Y is isomorphic to c_0. This contradicts the hypothesis. The converse is obvious. □

The next result is a corollary of some known results and Proposition 1.1.

COROLLARY 1.2. *For a Banach space Y the following are equivalent*

(a) *For all infinite scattered compact Hausdorff spaces S, we have $B(C(S),Y) = K(C(S),Y)$.*

(b) *For some infinite scattered compact Hausdorff space S, we have $B(C(S),Y) = K(C(S),Y)$.*

(c) *Y does not contain a copy of c_0.*

(d) *For all infinite dimensional subspaces X of c_0, we have*

$B(X, Y) = K(X, Y)$.

(e) *For some infinite dimensional subspace X of c_0, we have* $B(X, Y) = K(X, Y)$.

Proof. (a) $\Rightarrow$ (b) This is obvious.

(b) $\Rightarrow$ (c) By way of contradiction, suppose that Y contains a copy of c_0. Since S is an infinite scattered space, there exists a complemented subspace M of $C(S)$ isomorphic to c_0 see [**19**, p. 201]). Let P be the projection of $C(S)$ onto M and T be an isomorphism of M onto an isomorphic copy of c_0 in Y. Then $TP \in B(C(S), Y)$ is a noncompact operator. This contradiction proves *(c)*.

(c) $\Rightarrow$ (a) Let S be an arbitrary infinite scattered compact Hausdorff space. Let $T \in B(C(S), Y)$ be arbitrary. Since Y does not contain any copy of c_0, by a result of A. Pelczynski [**17**], the operator T is weakly compact. So, its adjoint $T^* : Y^* \longrightarrow C(S)^*$ is weakly compact. By a well known theorem of W. Rudin [**21**], or (see [**22**, Corollary 19.7.7]), we have $C(S)^* \cong l^1(S)$. By a theorem of Schur (see [**22**, p. 338]), the space $l^1(S)$ has the Schur property. So, T^* is compact. Hence, T is compact.

(c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) This is Proposition 1.1. □

COROLLARY 1.3 (Pitt). *For $1 \le p < \infty$, we have $B(c_0, l^p) = K(c_0, l^p)$.*

Proof. We know that $c_0 \cong C(S)$ for the infinite scattered compact Hausdorff space $S = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. We also know that l^p does not contain any copy of c_0. So, by Corollary 1.2, we have $B(c_0, l^p) = K(c_0, l^p)$. □

1.2. l^p_w-Sequences. This section gives a complete characterization of all Banach spaces Y (in terms of l^q_w-sequences) for which $B(X, Y) = K(X, Y)$ for $X = c_0$ or l^p $(1 \le p < \infty)$. The results for $X = c_0$ and l^2 are already known. We fill in the gap by giving the characterization in the case $X = l^p$ for $1 \le p < \infty$. This ties the results for c_0, l^2, and l^p $(p \neq 2)$ together.

Recall that a sequence $\{y_n\}$ of elements in a Banach space Y is said to be a **weak l^p-sequence**, or in short an **l^p_w-sequence** in Y, where $p \in [1, \infty)$, if for every $f \in Y^*$ we have $\sum_{n=1}^{\infty} |f(y_n)|^p < \infty$. The set of all l^p_w-sequences of a Banach space Y is denoted by $l^p_w(Y)$

(see [**6**]). For any real number $p > 1$, we denote the number $p/(p-1)$ by q. Note that $1/p + 1/q = 1$.

REMARK. (a) If $\{y_n\} \in l^p_w(Y)$, $p \geq 1$, then $\{y_n\} \in l^r_w(Y)$ for any $r \geq p$.

(b) If $\{e_n\}$ is the standard unit vector basis of l^p, $1 < p < \infty$, then $\{e_n\} \in l^q_w(l^p)$.

(c) If $\{e_n\}$ is the standard unit vector basis of c_0, then $\{e_n\} \in l^1_w(c_0)$.

The next proposition is motivated by [**3**] and [**4**].

PROPOSITION 2.1. *If $\{y_n\}$ is a sequence in a Banach space Y and $1 < p < \infty$, then the following three conditions are equivalent.*

(a) *The sequence $\{y_n\} \in l^p_w(Y)$.*

(b) *The series $\sum_{n=1}^{\infty} a_n y_n$ converges unconditionally for all $\{a_n\} \in l^q$.*

(c) *There exists an operator $T \in B(l^q, Y)$ such that $Te_n = y_n$, where $\{e_n\}$ is the standard unit vector basis of l^q.*

Proof. (*a*) $\Rightarrow$ (*b*) We suppose that $\{y_n\} \in l^p_w(Y)$, that is, $\{f(y_n)\} \in l^p$ for each $f \in Y^*$. First define a linear operator $S : Y^* \longrightarrow l^p$ by $Sf = \{f(y_n)\}$ for $f \in Y^*$. We will use the closed graph theorem to prove continuity of S. So suppose $\{f_n \oplus Sf_n\}$ is a Cauchy sequence in the product space $Y^* \oplus l^p$. Then both $\{f_n\}$ and $\{Sf_n\}$ are Cauchy sequences in Y^* and l^p, respectively. Let $f_n \longrightarrow f \in Y^*$. We will show that $Sf_n \longrightarrow Sf$. For every $\epsilon > 0$ there exists a natural number n_0 such that $\|Sf_i - Sf_j\|_p < \epsilon$ for all $i, j > n_0$. That is, $\sum_{n=1}^{\infty} |f_i(y_n) - f_j(y_n)|^p < \epsilon^p$ for all $i, j > n_0$. In particular, $\sum_{n=1}^{N} |f_i(y_n) - f_j(y_n)|^p < \epsilon^p$, for all natural numbers N and all natural numbers $i, j > n_0$. By letting $j \longrightarrow \infty$ we get $\sum_{n=1}^{N} |f_i(y_n) - f(y_n)|^p \leq \epsilon^p$. Since this holds for all natural numbers N we get

$$\|Sf_i - Sf\|_p^p = \sum_{n=1}^{\infty} |f_i(y_n) - f(y_n)|^p \leq \epsilon^p$$

for all $i > n_0$. So, $Sf_n \longrightarrow Sf$ in norm. Hence, S is continuous.

Now let $\{a_n\} \in l^q$ be arbitrary, $f \in Y^*$ be such that $\|f\| = 1$, and

i, j be any natural numbers. Then

$$\left\|f\left(\sum_{n=1}^{j} a_n y_n\right)\right\| = \left|\sum_{n=1}^{j} a_n f(y_n)\right|$$
$$= |\{0, \ldots, 0, a_i, \ldots, a_j, 0, 0, \ldots\} S(f)|$$
$$\leq \left(\sum_{n=1}^{j} |a_n|^q\right)^{\frac{1}{q}} \|S\|,$$

where $(0, \ldots, 0, a_i, \ldots, a_j, 0, 0, \ldots)$ is treated as an element of $(l^p)^*$. So,

$$\sup_{\|f\| \leq 1} \left|f\left(\sum_{n=1}^{j} a_n y_n\right)\right| \leq \left(\sum_{n=i}^{j} |a_n|^q\right)^{\frac{1}{q}} \|S\|.$$

Since

$$\sup_{\|f\| \leq 1} \left|f\left(\sum_{n=i}^{j} a_n y_n\right)\right| = \left\|\sum_{n=1}^{j} a_n y_n\right\|,$$

we obtain

$$\left\|\sum_{n=i}^{j} a_n y_n\right\| \leq \left(\sum_{n=i}^{j} |a_n|^q\right)^{\frac{1}{q}} \|S\|, \tag{1}$$

for all natural numbers i, j. Since $\{a_n\} \in l^q$, $\left(\sum_{n=i}^{j} |a_n|^q\right)^{\frac{1}{q}} \longrightarrow 0$ as $n \longrightarrow \infty$. So, $\left\|\sum_{n=i}^{j} a_n y_n\right\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, the series $\sum_{n=1}^{\infty} a_n y_n$ converges. Since $\{a_n\} \in l^q$ implies $\{\epsilon_n a_n\} \in l^q$, for any sequence $\{\epsilon_n\}$ of numbers $+1$ and -1, we certainly have that the series $\sum_{n=1}^{\infty} \epsilon_n a_n y_n$ converges. That is, the series $\sum_{n=1}^{\infty} a_n y_n$ converges unconditionally in Y.

(b) $\Rightarrow$ (c) Define the operator $T : l^q \longrightarrow Y$ by $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n y_n$. Clearly, T is linear and $T(e_n) = y_n$. We will prove that T is bounded. Let S be the bounded linear operator defined above. By letting $i = 1$ and $j \to \infty$ in (1), we obtain $\|\sum_{n=1}^{\infty} a_n y_n\| \leq \|\{a_n\}\| \|S\|$. So, $\|T\| \leq \|S\|$.

(c) $\Rightarrow$ (a) Suppose $T \in B(l^q, Y)$ and $T(e_n) = y_n$, for $n = 1, 2, \ldots$. We need to prove that $\{y_n\} \in l_w^p(Y)$. Let $f \in Y^*$ be arbitrary. Then $\sum_{n=1}^{\infty} |f(y_n)|^p = \sum_{n=1}^{\infty} |f \circ T(e_n)|^p < \infty$, because $f \circ T \in (l^q)^*$ and $\{e_n\} \in l_w^p(l^q)$. □

REMARK. On replacing "l_w^p" by "l_w^1" and "l^q" by "c_0" in the statement of Proposition 2.1, we obtain a result of C. Bessaga and A. Pelczynski [4], whereas on replacing "l_w^p" by "l_w^2" and l^q by l^2 we get a result given in the paper of R. Anantharaman and J. Diestel [3].

The next proposition is motivated by a paper of L. Drewnowski [7]. Part (c) of the proposition is well known and is included here for the sake of completeness.

PROPOSITION 2.2. *For a Banach space Y and an arbitrary $1 < p < \infty$, the following statements are true.*

(a) *The equality $B(l^p, Y) = K(l^p, Y)$ holds if and only if every l_w^q-sequence in Y is a norm null sequence.*

(b) *The equality $B(c_0, Y) = K(c_0, Y)$ holds if and only if every l_w^1-sequence in Y is a norm null sequence.*

(c) *The equality $B(l^1, Y) = K(l^1, Y)$ holds if and only if Y is of finite dimension.*

Proof. (a) Suppose $B(l^p, Y) = K(l^p, Y)$. Let $\{y_n\}$ be an arbitrary l_w^q-sequence in Y. By Proposition 2.1, there is an operator $T \in B(l^p, Y)$ such that $T(e_n) = y_n$ for all $n = 1, 2, \ldots$, where $\{e_n\}$ is the standard unit vector basis of l^p. By way of contradiction, suppose that $\{y_n\}$ is not norm null. So, there exists a subsequence, say $\{y_{nk}\}$, such that $\|y_{nk}\| > \epsilon$ for some $\epsilon > 0$ and for all $k = 1, 2, \ldots$. Since $\{e_{nk}\}$ is a norm bounded sequence, and T is a compact operator, the sequence $\{Te_{nk}\}$, (i.e., $\{y_{nk}\}$) has a norm convergent subsequence, say $\{y_{nkl}\}$. Suppose $y_{nkl} \xrightarrow{\|\ \|} y \in Y$. Then $y_{nkl} \xrightarrow{w} y \; inY$. Since $\{y_n\}$ is an l_w^q-sequence, it is a weakly null sequence. So, $y_{nkl} \xrightarrow{w} 0$. Thus, $y = 0$. Hence, $\|y_{nkl}\| \xrightarrow{\|\ \|} 0$, a contradiction.

For the converse, suppose that every l_w^q-sequence of Y is a norm null sequence and take an arbitrary $T \in B(l^p, Y)$. Let $\{x_n\}$ be any norm bounded sequence in l^p. We will show that $\{T(x_n)\}$ has a norm convergent subsequence. Since l^p is reflexive, the sequence $\{x_n\}$ has a weakly convergent subsequence. Without loss of generality we can assume that $\{x_n\}$ itself is weakly convergent. Suppose $x_n \xrightarrow{w} x \in l^p$. If $\liminf \|x_n - x\| = 0$, then $\{x_n\}$ has a norm convergent subsequence, and consequently, $\{T(x_n)\}$ has a norm convergent subsequence. So suppose that $\lim \|x_n - x\| > 0$. By the Bessaga-

Pelczynski theorem (see [**6**]), there exists a subsequence of $\{x_n - x\}$ which is a basic sequence. Since $\{x_n - x\}$ is a basic sequence in l^p and $\liminf \|x_n - x\| > 0$, by a theorem of A. Pelczynski [**16**, p. 7], there is a subsequence of $\{x_n - x\}$, which is equivalent to a block basis of the standard basis of l^p. Again by passing to a subsequence, we can assume that $\{x_n - x\}$ itself is equivalent to a block basis of the standard basis. Since every block basis of the standard basis of l^p is equivalent to the standard basis (see [**16**]), $\{x_n - x\}$ is equivalent to the standard basis. Since the standard basis is an l^q_w-sequence, $\{x_n - x\}$ is an l^q_w-sequence. And so $\{T(x_n - x)\}$ is an l^q_w-sequence. Consequently, by the hypothesis, $\{T(x_n - x)\}$ is a norm null sequence. That is, $Tx_n \longrightarrow Tx$ in norm. In other words, for every norm bounded sequence $\{x_n\}$ the sequence $\{Tx_n\}$ has a norm convergent subsequence.

(b) Suppose $B(c_0, Y) = K(c_0, Y)$. Let $\{y_n\} \in l^1_w(Y)$ be arbitrary. By Proposition 2.1 there is an operator $T \in B(c_0, Y)$ such that $T(e_n) = y_n$. Note that $\{y_n\}$ converges weakly to zero. So, every subsequence of it converges weakly to zero. Since T is compact, every subsequence of $\{Te_n\}$ (i.e., of $\{y_n\}$) has a subsequence which converges to zero in norm. So, $\{y_n\}$ itself converges to zero in norm.

For the converse, suppose that every l^1_w-sequence of Y converges in norm to zero. Notice that the standard unit vector basis $\{e_n\}$ of c_0 is an l^1_w-sequence, which does not converge to zero in norm. So, Y does not contain any copy of c_0. Since $c_0 \cong C(S)$, for some infinite scattered compact Hausdorff space S, Corollary 1.2 implies that all operators from c_0 into Y are compact.

(c) This follows from the well known fact that every separable Banach space is a quotient of l^1. □

NOTE 2.3. For the comparison we mention now the following result that follows from Corollary 3.11. If a Banach space Y has an unconditional basis of finite dimensional subspaces (or more generally, a weak unconditional compact netted expansion of identity), then $B(l_\infty, Y) = K(l_\infty, Y)$ if and only if every l^2_w-sequence in Y is a norm null sequence.

COROLLARY 2.4. *Suppose Y is a Banach space and suppose $p \in [1, \infty)$. If $B(l^p, Y) = K(l^p, Y)$, then*
(a) $B(l^r, Y) = K(l^r, Y)$ *for all* $r \in [p, \infty)$ *and*

(b) $B(c_0, Y) = K(c_0, Y)$.

Proof. (a) For $p = 1$ the result follows from Proposition 2.2(c). Suppose now that $1 < p \leq r < \infty$ and $B(l^p, Y) = K(l^p, Y)$. Then by Proposition 2.2(a) every l^q_w-sequence of elements in Y converges to zero in norm. Since $p \leq r$ implies that the conjugate number r' satisfies $r' \leq q$, we see that every $l^{r'}_w$-sequence of elements in Y is an l^q_w-sequence. So, every $l^{r'}_w$-sequence of elements in Y converges to zero in norm. By Proposition 2.2(a), we get $B(l^r, Y) = K(l^r, Y)$.

(b) Since $B(l^p, Y) = K(l^p, Y)$ for some $1 \leq p < \infty$, the space Y does not contain any copy of c_0. Since $c_0 \cong C(s)$, for some infinite compact scattered Hausdorff space, by Corollary 1.2 we get $B(c_0, Y) = K(c_0, Y)$. □

We conclude this section with the following remark.

REMARK 2.5. For a Banach space Y the following are equivalent.

(a) For all infinite dimensional Hilbert spaces H we have $B(H, Y) = K(H, Y)$.

(b) For some infinite dimensional Hilbert space H we have $B(H, Y) = K(H, Y)$.

(c) We have $B(l^2, Y) = K(l^2, Y)$.

(d) Every l^2_w-sequence in Y is a norm null sequence.

3. Nonscattered-Compact Spaces. Recall that a topological space Ω is said to be **nonscattered** or **nondispersed** if Ω contains a nonempty closed set which has no isolated point in its induced topology. In this section we assume that Ω is a nonscattered compact Hausdorff space. We begin with a note whose proof is left to the readers.

NOTE 3.1. If Y is a Banach space with the Schur property, then $B(C(\Omega), Y) = K(C(\Omega), Y)$.

THEOREM 3.2. *Let Ω be a nonscattered compact Hausdorff space, Y be a Banach space. If $B(C(\Omega), Y) = K(C(\Omega), Y)$, then $B(l^2, Y) = K(l^2, Y)$. Furthermore, if $B(C(\Omega), Y) = K(C(\Omega), Y)$, then $B(l^p, Y) = K(l^p, Y)$ for $p \geq 2$.*

Proof. By Corollary 2.4 only the case $p = 2$ needs a proof. We proceed by contradiction and assume that $B(l^2, Y) \neq K(l^2, Y)$. Then there is a noncompact operator T in $B(l^2, Y)$. From the proof of Proposition 2.2 it follows that there is a basic sequence $\{u_n\}$ in l^2 equivalent to a block basis of the standard basis of l^2 such that $\{Tu_n\}$ is an l^2_w-sequence with no norm convergent subsequence.

Now we will define a bounded linear operator $\Psi(T) : C(\Omega) \to Y$ which is not compact. Since Ω is a nonscattered compact Hausdorff space, by a theorem of A. Pelczynski, W. Rudin, and Z. Semedeni (see [**22**, Theorem 19.7.6]) there exists a purely nonatomic Borel probability measure μ on Ω. Let $\{r_n\}$ be a sequence of Rademacher like functions in $L^2(\mu)$. Then the sequence $\{r_n\}$ is a basic sequence of orthonormal functions. Observe that since μ is a regular Borel measure, for each function r_n and for each natural number k there exists an $f_{nk} \in C(\Omega)$ such that $||f_{nk}|| = \sup\{|f_{nk}(\omega)| : w \in \Omega\} = 1$ and $||f_{nk} - r_n||_2 < \frac{1}{k}$. Let M be the closed subspace of $L^2(\mu)$ spanned by the sequence $\{r_n\}$ and the sequences $\{f_{nk}\}$ for $n = 1, 2, \ldots$. Let M_1 be the closed subspace of M spanned by the sequence $\{r_n\}$ and M_0 be the orthogonal complement of M_1 in M. Then M is the internal direct sum of M_1 and M_0 (i.e., $M = \{x_1 + x_2 : x_1 \in M_1, x_2 \in M_2\}$ and $||x_1 + x_2|| = (||x_1||^2 + ||x_2||^2)^{\frac{1}{2}}$). Let N be the closed linear subspace spanned by $\{u_n\}$. We have

$$C(\Omega) \xrightarrow{\Lambda} L^2(\mu) \xrightarrow{P} M \xrightarrow{I} M_1 \oplus M_0 \xrightarrow{J} N \xrightarrow{T|_N} Y,$$

where $\Lambda(f) = f =$ the equivalence class of f in $L^2(\mu)$; the operator P is the orthogonal projection from $L^2(\mu)$ onto M; I is the identity map from M onto $M_1 \oplus M_0$; and $J : M_1 \oplus M_0 \to N$ is the operator defined by $J(r_n) = u_n$ for $n = 1, 2, \ldots$ and $J(x) = 0$ for each $x \in M_0$. (Since $\{u_n\}$ is a basic sequence in l^2, J is an isomorphism from M_1 onto N.) Let $\Psi(T) = T|_N JIP\Lambda$. Clearly, $\Psi(T)$ maps $C(\Omega)$ into Y. We claim that $\Psi(T)$ is not compact. For this it is enough to show that $\{Tu_n\} \subseteq \overline{\{\Psi(T)(f) : f \in C(\Omega) \text{ and } ||f|| = 1\}}$. To this end, note that

$$\begin{aligned}
||Tu_n - \Psi(T)f_{nk}|| &= ||TJPr_n - TJIP\Lambda f_{nk}|| \\
&\leq ||T|| ||JPr_n - JPf_{nk}|| \\
&\leq ||T|| ||J|| ||P|| \frac{1}{k} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty.
\end{aligned}$$

□

COROLLARY 3.3. *If Y is a Banach space such that $B(C(\Omega), Y) = \Pi_2(C(\Omega), Y)$, then $B(C(\Omega), Y) = K(C(\Omega), Y)$ if and only if $B(l^2, Y) = K(l^2, Y)$.*

Proof. In view of Theorem 3.2 we need only to prove that if $B(l^2, Y) = K(l^2, Y)$, then $B(C(\Omega), Y) = K(C(\Omega), Y)$. This follows from Remark 2.5 and the factorization theorem of A. Pietsch [**18**], which states that every absolutely 2-summing operator factors through a Hilbert space. □

COROLLARY 3.4. *For any compact nonscattered Hausdorff space Ω and any Banach space Y, the following are equivalent.*

(a) $B(C(\Omega), Y) = K(C(\Omega), Y)$.

(b) *$B(l^2, Y) = K(l^2, Y)$ and each $T \in B(C(\Omega), Y)$ factors through a closed subspace of c_0.*

Proof. (a)$\Longrightarrow$ (b) This follows from a theorem of T. Terzioglu [**24**] (or see [**1**, Theorem 16.5]) and Theorem 3.2.

(b)$\Longrightarrow$(a) Since $B(l^2, Y) = K(l^2, Y)$, Y does not contain any copy of c_0. So, every operator from c_0 into Y is compact. Now (a) is clear. □

To present Theorem 3.9 we need some discussion on the spaces of compact operators. Recall [**11**] that an operator $T \in B(X, Y)$ is said to have an **unconditional compact expansion** if there is a sequence $\{T_n\}$ of compact operators from X into Y such that for each $x \in X$ we have $Tx = \sum_{n=1}^{\infty} T_n x$, where the series converges unconditionally in Y. Recall also that T is said to have a **finite dimensional expansion** if the operators T_n are of finite rank. We shall now formulate the following definitions.

DEFINITION 3.5. An operator $T \in B(X, Y)$ is said to have a **weak unconditional compact netted expansion** if there is a net $\{T_\mu\}$ of compact operators from X into Y such that for each $x \in X$

$$Tx = \sum_{\mu} T_\mu x,$$

where the series converges weakly unconditionally in Y

DEFINITION 3.6. A Banach space B is said to have a **weak unconditional compact netted expansion of identity** if the identity operator I_B on B has a weak unconditional compact netted expansion.

Recall that if I_B in the above definition has an unconditional finite dimensional expansion, then B is said to have an **unconditional finite dimensional expansion of identity**.

REMARKS. Suppose T in $B(X, Y)$ factors through a Banach space E.

(a) If E has a weak unconditional compact netted expansion of identity, then T has a weak unconditional compact netted expansion.

(b) If E has an unconditional finite dimensional expansion of identity, then T has an unconditional finite dimensional expansion.

The part (a) of the next proposition is motivated by a result of N.J. Kalton [**13**] and is slightly more general than other known generalizations of the same result.

PROPOSITION 3.7. *Suppose c_0 does not embed in $K(X, Y)$ and $T \in B(X, Y)$.*

(a) *If T has a weak unconditional compact netted expansion, then T is compact.*

(b) *If T has a weak unconditional compact netted expansion, then T factors through a closed subspace of c_0.*

Proof. (a) Let $\{T_\mu\}$ be a weak unconditional compact netted expansion of T. We claim that $\{T_\mu\}$ is an unconditional compact netted expansion of T. By way of contradiction suppose that for some $x \in B$ the series $\sum_\mu T_\mu x$ does not converge unconditionally. Then there exists an $\epsilon > 0$ and sequences (F_n), (F'_n) of finite subsets of the index set such that for all m and n the sets F_n and F'_m are disjoint and

$$\left\| \sum_{\eta \in F_n} \epsilon_\eta T_\eta x - \sum_{\eta \in F'_n} \epsilon_\eta T_\eta x \right\| > \epsilon.$$

for some choices of signs ϵ_η. Set $y_n = \sum_{\eta \in F_n} \epsilon_\eta T_\eta x - \sum_{\eta \in F'_n} \epsilon_\eta T_\eta x$. Then, the series $\sum_n y_n$ converges weakly unconditionally Cauchy in Y and $\inf \|y_n\| \geq \epsilon$. So, by a theorem of Bessaga and Pelczynski [**4**] the space Y contains a copy of c_0. This contradicts the hypothesis.

Since the series $\sum_\mu T_\mu x$ converges unconditionally for every $x \in B$, by the uniform boundedness principle

$$\sup \left\| \sum_{\mu \in F} T_\mu \right\| < \infty,$$

where the supremum is taken over all finite subsets F of the index set M. Equivalently, the series $\sum_\mu T_\mu$ is weakly unconditionally Cauchy in $K(X,Y)$. Since $K(X,Y)$ does not contain any copy of c_0 by a theorem of Bessaga and Pelczynski [**4**], the series converges in norm. Clearly, it converges to T.

(b) This is immediate from (a) and a theorem of T. Terzioglu [**24**]. □

This completes the necessary discussion on the spaces of compact operators. The following theorem due to L. Drewnowski [**7**] will also be useful in the proof of Theorem 3.9. Here, the Banach space of all countably additive vector measures from the σ-algebra Σ into the Banach space Y is denoted by $ca(\Sigma, Y)$.

THEOREM 3.8 (Drewnowski). *If a σ-algebra Σ admits an atomless probability measure, then for any Banach space Y the following statements are equivalent.*

(a) $l_\infty \hookrightarrow ca(\Sigma, Y)$.

(b) $c_0 \hookrightarrow ca(\Sigma, Y)$.

(c) $B(l^2, Y) \neq K(l^2, Y)$.

The following theorem gives another necessary and sufficient condition on a Banach space Y for all operators from $C(\Omega)$ into Y to be compact.

THEOREM 3.9. *For any compact nonscattered Hausdorff space Ω and any Banach space Y the following are equivalent.*

(a) $B(C(\Omega), Y) = K(C(\Omega), Y)$.

(b) $B(l^2, Y) = K(l^2, Y)$ *and each* $T \in B(C(\Omega), Y)$ *has a weak unconditional compact netted expansion.*

Proof. (a)$\Longrightarrow$(b) We get the equality $B(l^2, Y) = K(l^2, Y)$ from Theorem 3.2 and that each $T \in B(C(\Omega), Y)$ admits a weak unconditional compact netted expansion is obvious.

(b)$\Longrightarrow$(a) Since Ω is nonscattered, by a theorem of A. Pelczynski, W. Rudin, and Z. Semadeni (see [**22**, p. 338]), it admits an atomless regular Borel probability measure. Since $B(l^2, Y) = K(l^2, Y)$, by Theorem 3.8, it follows that $c_0 \not\hookrightarrow ca(\Sigma, Y)$, where Σ denotes the σ-algebra of all Borel subsets of Ω. Since $K(C(\Omega), Y)$ is isometrically embeddable in $ca(\Sigma, Y)$ (see [**5**, pp. 152–154]), $c_0 \not\hookrightarrow K(C(\Omega), Y)$. Now the conclusion follows from Proposition 3.7. □

COROLLARY 3.10. *If for some p with $1 \leq p \leq 2$, $B(l^p, Y) = K(l^p, Y)$ and each operator in $B(C(\Omega), Y)$ has a weak unconditional compact netted expansion, then $B(C(\Omega), Y) = K(C(\Omega), Y)$.*

Proof. This follows from Corollary 2.4 and Theorem 3.9. □

Recall that a Banach space is said to be **separably universal** if it contains an isometric copy of every separable Banach space. Recall also that for a compact Hausdorff space Ω the space $C(\Omega)$ is separably universal if and only if Ω is nonscattered (see [**14**]). Note that if μ is a regular Borel measure whose support is an infinite compact Hausdorff space, then there exists a nonscattered compact Hausdorff space Ω' such that $L^\infty(\mu) \cong C(\Omega')$. In particular, $l^\infty \cong C(\Omega')$ for some nonscattered compact Hausdorff space Ω'.

COROLLARY 3.11. *For any nonscattered compact Hausdorff space Ω, any Banach space Y with a weak unconditional compact netted expansion of identity, and any regular Borel measure μ on a compact Hausdorff space the following statements hold.*

(a) *$B(C(\Omega), Y) = K(C(\Omega), Y)$ if and only if $B(l^2, Y) = K(l^2, Y)$.*
(b) *For any nonscattered compact Hausdorff space Ω' we have $B(C(\Omega), Y) = K(C(\Omega), Y)$ if and only if $B(C(\Omega'), Y) = K(C(\Omega'), Y)$.*
(c) *$B(C(\Omega), l^p) = K(C(\Omega), l^p)$ for $1 \leq p < 2$.*
(d) *$B(C(\Omega), l^p) \neq K(C(\Omega), l^p)$ for $2 \leq p < \infty$.*
(e) *$B(L^\infty(\mu), l^p) = K(L^\infty(\mu), l^p)$ for $1 \leq p < 2$.*
(f) *$B(L^\infty(\mu), l^p) \neq K(L^\infty(\mu), l^p)$ for $2 \leq p < \infty$.*

Proof. (a) This follows from Theorem 3.2 and Theorem 3.9.

(b) This follows from *(a)*.

(c) Since $1 \leq p < 2$, by a result of H.R. Pitt [**16**], we have $B(l^2, l^p) = K(l^2, l^p)$. We know that l^p has a weak unconditional compact netted expansion of identity, so by *(a)* we get $B(C(\Omega), l^p) = K(C(\Omega), l^p)$.

(d) Since $2 \leq p < \infty$, we obviously have $B(l^2, l^p) \neq K(l^2, l^p)$. Now the conclusion follows from Theorem 3.2.

(e) follows from (c) and (f) follows from *(d)*. □

Parts (e) and (f) of Corollary 3.11 follow also from [**19**, Remark 2].

The following conclusion is clear from what we have proved so far.

CONCLUSION 3.12. *Let $\Sigma(\Omega)$ denote the class of all Banach spaces Y for which all operators from $C(\Omega)$ into Y are compact iff all operators from l^2 into Y are compact. Then, for a Banach space Y the following statements hold.*

(a) *If Y has an unconditional basis, then $Y \in \Sigma(\Omega)$.*

(b) *If Y has an unconditional basis consisting of finite dimensional subspaces, then $Y \in \Sigma(\Omega)$.*

(c) *If Y has a weak conditional compact netted expansion of identity, then $Y \in \Sigma(\Omega)$.*

(d) *If each operator from $C(\Omega)$ into Y admits a weak unconditional compact netted expansion, then $Y \in \Sigma(\Omega)$.*

(e) *If each operator from $C(\Omega)$ into Y factors through a closed subspace of c_0, then $Y \in \Sigma(\Omega)$.*

(f) *If each operator from $C(\Omega)$ into Y is absolutely 2-summing, then $Y \in \Sigma(\Omega)$.*

(g) *If Y has the Schur property, then $Y \in \Sigma(\Omega)$.*

We conclude this section with a remark, whose proof is left to the reader.

REMARK. In Theorem 3.8 the space l^2 can not be replaced by an l^p-space with $p \neq 2$.

4. Factorization. In this section Ω is any (scattered or nonscattered) compact Hausdorff space. Now we will use some of our earlier theorems to prove some results regarding the space $\Phi_{c_0}(C(\Omega))$ of all operators on $C(\Omega)$ factoring through c_0.

PROPOSITION 4.1. *For an infinite compact Hausdorff space Ω, and for a closed subspace X of c_0 the following inclusions hold.*
(a) $\Phi_X(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega))$.
(b) $K(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega))$, *but* $K(C(\Omega)) \neq \Phi_{c_0}(C(\Omega))$.

Proof. (a) Let $T \in \Phi_X(C(\Omega))$ be arbitrary and $T = T_2T_1$ be a factorization of T through X. Since X is a closed subspace of c_0, by a theorem of J. Lindenstrauss and A. Pelczynski [**15**, Theorem 3.1], T_2 extends to a bounded linear operator $\widehat{T}_2$ from c_0 into $C(\Omega)$. Clearly, $T = \widehat{T}_2T_1 \in \Phi_{c_0}(C(\Omega))$.

(b) Let $T \in K(C(\Omega))$ be arbitrary. Then by the theorem of T. Terzioglu [**24**], T factors through a closed subspace of c_0. Hence, by (a) $T \in \Phi_{c_0}(C(\Omega))$, (i.e., $K(C(\Omega)) \subset \Phi_{c_0}(C(\Omega))$). To prove that $K(C(\Omega)) \neq \Phi_{c_0}(C(\Omega))$ let us first suppose Ω is scattered. Since Ω is an infinite set, the space $C(\Omega)$ contains a complemented subspace M isomorphic to c_0 (see [**19**, p. 201]). Let $P : C(\Omega) \to M$ be a continuous projection onto M, let $M \to C(\Omega)$ be the inclusion map. Clearly, JP factors through c_0 and is noncompact. Now suppose Ω is nonscattered. First note that there is a noncompact operator T in $B(C(\Omega))$. (For, otherwise our Theorem 3.2 would imply that $B(l^2, c_0) = K(l^2, c_0)$. On the other hand, the formal identity map from l^2 to c_0 is not compact.) Now note that since Ω is nonscattered there exists an isometry J in $B(c_0, C(\Omega))$. Clearly, $JT \in \Phi_{c_0}(C(\Omega))$ and JT is noncompact. □

THEOREM 4.2. *For a compact Hausdorff space Ω and for a separable Banach space X the following are equivalent.*
(a) $\Phi_X(C(\Omega)) \subseteq K(C(\Omega))$.
(b) $\Phi_X(C(\Omega)) \subseteq \Phi_{c_0}(C(\Omega))$, *but* $\Phi_X(C(\Omega)) \neq \Phi_{c_0}(C(\Omega))$.

Proof. *(a)* $\Longrightarrow$ *(b)* This is immediate from Proposition 4.1.

(b)$\Longrightarrow$(a) First observe that $c_0 \not\hookrightarrow X$. For, otherwise since X is separable, a theorem of Sobczyk [**23**], would imply that an isomorphic copy of c_0 is complemented in X. So, we would get $\Phi_{c_0}(C(\Omega)) \subseteq \Phi_X(C(\Omega))$, contrary to our assumption. To prove that $\Phi_X(C(\Omega)) \subseteq K(C(\Omega))$, it suffices to prove that $B(C(\Omega), X) = K(C(\Omega), X)$. If Ω is scattered, then $B(C(\Omega), X) = K(C(\Omega), X)$ by Corollary 1.2. If Ω is nonscattered, then $C(\Omega)$ is separably universal. So, there is an isometry $J : X \to C(\Omega)$. If $T \in B(C(\Omega), X)$, then by our hypothesis $JT \in \Phi_{c_0}(C(\Omega))$. So, suppose $JT = T_2T_1$ is a factorization through c_0. Note that $T_2 \in B(c_0, J(X))$ and $c_0 \cong C(S)$ for some scattered compact Hausdorff space S. Since $c_0 \not\hookrightarrow J(X)$, by Corollary 1.2 the operator T_2 is compact. □

Acknowledgment. This is part of the author's thesis written under the supervision of Professors Y.A. Abramovich and C.D. Aliprantis. It is my pleasure to thank them for giving helpful comments and suggestions and for helping in the exposition of this material. Without their encouragement and interest this work would not have been possible. The author would also like to thank the referee for his careful reading and for valuable suggestions that have been helpful in presenting the material in a relatively compact form.

Added in proof. After this paper was accepted for publication we learned that Corollary 1.2 (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c) was already known. See Proposition 2 of the following paper.

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Received December 10, 1992.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,
Berkeley, CA 94720, A NON-PROFIT CORPORATION
This publication was typeset using AMS-LATEX,
the American Mathematical Society's TEX macro system.

PACIFIC JOURNAL OF MATHEMATICS

Volume 169 No. 2 June 1995