

*Pacific
Journal of
Mathematics*

A DIOPHANTINE EQUATION CONCERNING FINITE GROUPS

MAOHUA LE

A DIOPHANTINE EQUATION CONCERNING FINITE GROUPS

MAOHUA LE

In this paper we prove that all solutions (y, m, n) of the equation $3^m - 2y^n = \pm 1$, $y, m, n \in \mathbb{N}$, $y > 1, m > 1, n > 1$, satisfy $y < 10^{6 \cdot 10^9}$, $m < 1,4 \cdot 10^{15}$ and $n < 1,2 \cdot 10^5$.

1. Introduction. Let $\mathbb{Z}, \mathbb{N}, \mathbb{P}, \mathbb{Q}$ be the sets of integers, positive integers, odd primes and rational numbers respectively. In [2], Creszeno considered the solutions (p, q, m, n, δ) of the equation

$$(1) p^m - 2q^n = \delta, p, q \in \mathbb{P}, m, n \in \mathbb{N}, m > 1, n > 1, \delta \in \{-1, 1\},$$

which is concerned with finite groups. He claimed that if $(p, q, m, n, \delta) \neq (239, 13, 2, 4, -1)$, then $(m, n, \delta) = (2, 2, -1)$. However, we notice that (1) has another solution $(p, q, m, n, \delta) = (3, 11, 5, 2, 1)$ with $(m, n, \delta) \neq (2, 2, -1)$. Thus it can be seen that the above result is not correct. If we follow the proof of Creszeno, we can argue as follows. The above result is deduced from the following lemma:

LEMMA A ([2, Lemma 1]). *Suppose that $q \in \mathbb{P}$ and $x, m, n \in \mathbb{N}$. If*

$$(2) x^m - 2q^n = \pm 1, x > 1, m > 1, n > 1,$$

then m is a power of 2. Furthermore, the sign of the term ± 1 must be negative.

Notice that if $2 \nmid xm$, then from (2) we get

$$2q^k = x \mp 1$$

for some $k \in \mathbb{N}$ with $k < n$. Now there are two cases:

$$(3) x = \begin{cases} 3, & \text{if } k = 0, \\ 2q^k \pm 1, & \text{if } k > 0. \end{cases}$$

Hence, Lemma A is false since the first case of (3) was not considered in [2]. The lemma must be replaced by:

LEMMA A'. *Suppose that $q \in \mathbb{P}$ and $x, m, n, \in \mathbb{N}$. If (2) hold, then either $x = 3$ or $x > 3$ and m is a power of 2. Furthermore, in the last case, the sign of the term ± 1 of (2) must be negative.*

Thus, the correct main statement of [2] should be that the solutions (p, q, m, n, δ) of (1) satisfy either $p = 3$ or $p > 3$ and $(m, n, \delta) = (2, 2, -1)$ except when $(p, q, m, n, \delta) = (239, 13, 2, 4, -1)$. In this paper, we deal with the solutions of (1) with $p = 3$. We shall prove a general result as follows:

THEOREM. *The equation*

$$(4) \quad 3^m - 2y^n = \delta, \quad y, m, n, \in \mathbb{N}, \quad y > 1, m > 1, n > 1, \delta \in \{-1, 1\},$$

has only finitely many solutions (y, m, n, δ) . Moreover, all solutions of (4) satisfy $y < 10^{6 \cdot 10^9}$, $m < 1.4 \cdot 10^{15}$ and $n < 1.2 \cdot 10^5$.

2. Lemmas.

LEMMA 1. *Let $k \in \mathbb{N}$ with $\gcd(6, k) = 1$. If $k > 1$ and (X, Y, Z) is solution of the equation*

$$X^2 - 3Y^2 = k^Z, \quad X, Y, Z, \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0,$$

then there exist $X_1, Y_1 \in \mathbb{N}$ such that

$$X_1^2 - 3Y_1^2 = k, \quad \gcd(X_1, Y_1) = 1,$$

$$1 < \frac{X_1 + Y_1\sqrt{3}}{X_1 - Y_1\sqrt{3}} < (2 + \sqrt{3})^2,$$

$$X + Y\sqrt{3} = (X_1 + \lambda Y_1\sqrt{3})^Z (u + v\sqrt{3}), \quad \lambda \in \{-1, 1\},$$

where (u, v) is a solution of the equation

$$(5) \quad u^2 - 3v^2 = 1, \quad u, v \in \mathbb{Z}.$$

Proof. Since the class number of the binary quadratic forms with discriminant 12 is equal to 1 and $2 + \sqrt{3}$ is a fundamental solution of (5), the lemma follows immediately from [6, Lemma 7]. \square

LEMMA 2 ([3]). Let $a, b, k, n \in \mathbb{Z} \setminus \{0\}$ with $n \geq 3$. All solutions (X, Y) of the equation

$$aX^n - bY^n = k, \quad X, Y \in \mathbb{Z},$$

satisfy $\max(|X|, |Y|) \leq 2n^{(n-1)/2-1/n} H^{n-3/n} |k|^{1/n}$, where $H = \max(|a|, |b|)$.

LEMMA 3 ([7]). The equation

$$1 + X^2 = 2Y^n, \quad X, Y, n \in \mathbb{N}, \quad X > 1, Y > 1, n > 2,$$

has no solution (X, Y, n) .

LEMMA 4 ([9]). The equation

$$(6) \quad \frac{X^m - 1}{X - 1} = Y^n, \quad X, Y, m, n \in \mathbb{N}, \quad X > 1, Y > 1, m > 2, n > 1,$$

has only solution $(X, Y, m, n) = (7, 20, 4, 2)$ with $4|m$.

LEMMA 5 ([10]). The equation (6) has only solutions $(X, Y, m, n) = (3, 11, 5, 2), (7, 20, 4, 2)$ with $2|n$.

Let α be an algebraic number with the minimal polynomial

$$a_0 z^d + \cdots + a_{d-1} z + a_d = a_0 \prod_{i=1}^d (z - \sigma_i \alpha), \quad a_0 \in \mathbb{N},$$

where $\sigma_1 \alpha, \dots, \sigma_d \alpha$ are all conjugates of α . Then

$$h(\alpha) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log \max(1, |\sigma_i \alpha|) \right)$$

is called the logarithmic absolute height of α .

LEMMA 6. Let α_1, α_2 be real algebraic numbers with $\alpha_1 \geq 1$, $\alpha_2 \geq 1$, and let D denote the degree of $\mathbb{Q}(\alpha_1, \alpha_2)$. Let $b_1, b_2 \in \mathbb{N}$, and let $b = b_1/Dh(\alpha_2) + b_2/Dh(\alpha_1)$. For any T with $T > 1$, if $0.52 + \log b \geq T$ and $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2 \neq 0$, then

$$\log |\Lambda| \geq -70 \left(1 + \frac{0.1137}{T} \right)^2 D^4 h(\alpha_1) h(\alpha_2) (0.52 + \log b)^2.$$

Proof. Let $B = \log(5c_4/c_1) + \log b$, $K = [c_1 D^3 B h(\alpha_1) h(\alpha_2)]$, $L = [c_2 D B]$, $R_1 = [c_3 D^{3/2} B^{1/2} h(\alpha_2)] + 1$, $S_1 = [c_3 D^{3/2} B^{1/2} h(\alpha_1)] + 1$, $R_2 = [c_4 D^2 B h(\alpha_2)]$, $S_2 = [c_4 D^2 B h(\alpha_1)]$, $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$, where c_1, c_2, c_3, c_4 are positive numbers. Notice that $(u - 1/T)v < [uv] \leq uv$ for any real numbers u, v with $u \geq 0$ and $v \geq T$. By the proof of [4, Theorem 1 and 3], if $B \geq T$,

$$(7) \quad \sqrt{c_1} = \frac{\rho + 1}{(\log \rho)^{3/2}} + \sqrt{\frac{(\rho + 1)^2}{(\log \rho)^3} + \frac{\rho + 1}{T \log \rho}}, c_2 > \frac{2}{\log \rho},$$

$$c_3 = \max(\sqrt{c_1}, \sqrt{c_2}), c_4 = \sqrt{2c_1 c_2} + 1/T,$$

for any ρ with $\rho > 1$, then

$$(8) \quad \log |\Lambda| \geq -(c_1 c_2 \log \rho + 1) D^4 h(\alpha_1) h(\alpha_2) B^2.$$

Setting $\rho = 5.803$. We may choose c_1, c_2, c_3, c_4 which make (7) hold and

$$(9) \quad c_1 c_2 \log \rho + 1 < 70 \left(1 + \frac{0.1137}{T}\right)^2, B < 0.52 + \log b.$$

Substituting (9) into (8), the lemma is proved. □

3. Proof of Theorem. Let (y, m, n, δ) be a solution of (4). Since

$$(10) \quad \text{ord}_2(3^m + 1) = \begin{cases} 1, & \text{if } 2|m, \\ 2, & \text{if } 2 \nmid m, \end{cases}$$

for any $m \in \mathbb{N}$, if $\delta = -1$, then m must be even. Further, by Lemma 3, it is impossible. By Lemma 4, (4) has no solution with $\delta = 1$ and $4|m$, and by Lemma 5, (4) has only solutions $(y, m, n, \delta) = (2, 2, 2, 1)$ and $(11, 5, 2, 1)$ with $\delta = 1$ and $2|n$. If $2|m$ and $2 \nmid n$, then from (4) we get

$$(11) \quad 3^{m/2} + 1 = y_1^n, 3^{m/2} - 1 = 2y_2^n, y_1 y_2 = y, y_1, y_2 \in \mathbb{N}, 2|y_1.$$

Since $n > 2$, (11) is impossible by (10). Therefore, if $(y, m, n, \delta) \neq (2, 2, 2, 1)$ or $(11, 5, 3, 1)$, then $\delta = 1$ and $2 \nmid mn$. It is a well known

fact that $(3^m - 1)/(3 - 1)$ has a prime factor l with $l \equiv 1 \pmod{m}$ (see [1]). So by (4) we have

$$(12) \quad y \geq 2m + 1 > 2n + 1.$$

If $2 \nmid m$, then from (4) we get

$$(13) \quad A^2 - 3B^2 = y^n, \quad A, B \in \mathbb{N},$$

where

$$(14) \quad A = \frac{3^{(m+1)/2} - 1}{2}, \quad B = \frac{3^{(m-1)/2} - 1}{2}.$$

Since $\gcd(6, y) = \gcd(A, B) = 1$ by Lemma 1, we see from (13) that

$$(15) \quad A + B\sqrt{3} = (X_1 + \lambda Y_1\sqrt{3})^n(u + v\sqrt{3}), \quad \lambda \in \{-1, 1\},$$

where (u, v) is a solution of (5), $X_1, Y_1 \in \mathbb{N}$ such that

$$(16) \quad X_1^2 - 3Y_1^2 = y, \quad \gcd(X_1, Y_1) = 1,$$

$$(17) \quad 1 < \frac{X_1 + Y_1\sqrt{3}}{X_1 - Y_1\sqrt{3}} < (2 + \sqrt{3})^2.$$

Let

$$(18) \quad \rho = 2 + \sqrt{3}, \quad \bar{\rho} = 2 - \sqrt{3}, \quad \varepsilon = X_1 + Y_1\sqrt{3}, \quad \bar{\varepsilon} = X_1 - Y_1\sqrt{3}.$$

Since $A = 3B + 1$ by (14), we have

$$1 < \frac{A + B\sqrt{3}}{A - B\sqrt{3}} = \frac{(\sqrt{3} + 1)(\sqrt{3^m} - 1)}{(\sqrt{3} - 1)(\sqrt{3^m} + 1)} < 2.$$

Hence, by (15) and (17), we have

$$(19) \quad A + B\sqrt{3} = \begin{cases} \varepsilon^n \bar{\rho}^s, \\ \bar{\varepsilon}^n \rho^s, \end{cases} \quad A - B\sqrt{3} = \begin{cases} \bar{\varepsilon}^n \rho^s, & \text{if } \lambda = 1, \\ \varepsilon^n \bar{\rho}^s, & \text{if } \lambda = -1, \end{cases}$$

where $s \in \mathbb{Z}$ with $0 \leq s \leq n$. From (19),

$$\varepsilon^n \bar{\rho}^s(\sqrt{3} - \lambda) = \bar{\varepsilon}^n \rho^s(\sqrt{3} + \lambda) - 2\lambda,$$

whence we obtain

$$(20) \quad \left| (2s + \lambda) \log \rho - n \log \frac{\varepsilon}{\bar{\varepsilon}} \right| = \frac{2}{\sqrt{3^m}} \sum_{j=0}^{\infty} \frac{\rho/h 3^{-mi/2}}{2i + 1} < \frac{4}{\sqrt{3^m}}.$$

Let $\alpha_1 = \rho$ and $\alpha_2 = \varepsilon/\bar{\varepsilon}$. Then $\mathbb{Q}(\alpha_1, \alpha_2) = \mathbb{Q}(\sqrt{3})$. We see from (5), (16) and (18) that

$$(21) \quad h(\alpha_1) = \frac{1}{2} \log \rho, \quad h(\alpha_2) = \log \varepsilon.$$

Furthermore, since $3^m > 2y^n$, by (17) and (21),

$$(22) \quad h(\alpha_2) < \log \rho \sqrt{y}.$$

Let $b = (2n + 1)/2h(\alpha_2) + n/2h(\alpha_1)$. Recall that $0 \leq s \leq n$. By Lemma 6, if $n > 10^5$, then

$$(23) \quad \log \left| (2s + \lambda) \log \rho - n \log \frac{\varepsilon}{\bar{\varepsilon}} \right| > -1145h(\alpha_1)h(\alpha_2)(0.52 + \log b)^2.$$

Since

$$b < \frac{2n + 1}{2 \log \rho \sqrt{y}} + \frac{n}{\log \rho} < \frac{2n + 1}{2 \log 3.732 \sqrt{2n + 1}} + \frac{n}{1.317} < 0.8n,$$

by (12) and (22), the combination of (20) and (23) yields

$$1 + 760(\log \sqrt{y})(0.3 + \log n)^2 > n \log \sqrt{y},$$

whence we deduce that

$$(24) \quad n < 1.2 \cdot 10^5.$$

Let $m = rn + t$, where $r, t \in \mathbb{Z}$ with $r \geq 0$ and $0 \leq t < n$. Then (4) can be written as

$$(25) \quad 3^t(3^r)^n - 2y^n = 1.$$

It implies that $(X, Y) = (3^r, y)$ is a solution of the equation

$$3^t X^n - 2Y^n = 1, \quad X, Y \in \mathbb{Z}.$$

Thus, by Lemma 2, we get from (24) and (25) that $y < 10^{6 \cdot 10^9}$ and $m < 1.4 \cdot 10^{15}$. The theorem is proved.

REMARK. By a better estimates for the lower bound of linear forms in two logarithms by Laurent, Mignotte and Nesterenko [5], the upper bound of solutions of (4) in Theorem can be improved.

Acknowledgment. The author thanks the referee for his valuable suggestions.

REFERENCES

- [1] G.D. Birkhoff and H.S. Vandiver, *On the integral divisor of $a^n - b^n$* , Ann. of Math., (2) **5** (1904), 173-180.
- [2] P. Cresenzo, *A diophantine equation which arises in the theory of finite groups*, Adv. Math., **17** (1975), 25-29.
- [3] N.I. Feldman, *Diophantine equations with a finite number of solutions*, Vestnik Moskov. Univ. Ser. I, Mat. Meh., **26** (1971), 52-58 (Russian).
- [4] M. Laurent, *Linear forms in two logarithms and interpolation determinants*, In: M. Waldschmidt, Linear independence of logarithms of algebraic numbers, The Institute of Mathematical Sciences, IMS Report No 116, Madras, 1992.
- [5] M. Laurent, M. Mignotte and Yu. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, to appear.
- [6] M.-H. Le, *A note on the diophantine equation $x^{2p} - Dy^2 = 1$* , Proc. Amer. Math. Soc., **107** (1989), 27-34.
- [7] W. Ljunggren, *Über die Gleichungen $1 + Dx^2 = 2y^n$ und $1 + Dx^2 = 4y^n$* , Norske Vid. Selsk. Forhandl, **15** No. 30, (1942), 115-118.
- [8] ———, *Noen setninger om ubestemte likninger av formen $(x^n - 1)/(x - 1) = y^q$* , Norsk. Mat. Tidsskr., **25** (1943), 17-20.
- [9] T. Nagell, *Note sur l'équation indéterminée $(x^n - 1)/(x - 1) = y^q$* , Norsk. Mat. Tidsskr., **2** (1920), 75-78.

Received December 7, 1992. Supported by National Natural Science Foundation of China.

ZHANJIANG TEACHERS COLLEGE
P. O. BOX 524048
ZHANJIANG, GUANGDONG
P. R. CHINA

PACIFIC JOURNAL OF MATHEMATICS

Founded by E. F. Beckenbach (1906-1982) and F. Wolf (1904-1989)

EDITORS

Sun-Yung Alice Chang (Managing Editor)

University of California
Los Angeles, CA 90095-1555
pacific@math.ucla.edu

F. Michael Christ
University of California
Los Angeles, CA 90095-1555
christ@math.ucla.edu

Thomas Enright
University of California
San Diego, La Jolla, CA 92093
tenright@ucsd.edu

Nicholas Ercolani
University of Arizona
Tucson, AZ 85721
ercolani@math.arizona.edu

Robert Finn
Stanford University
Stanford, CA 94305
finn@gauss.stanford.edu

Vaughan F. R. Jones
University of California
Berkeley, CA 94720
vfr@math.berkeley.edu

Steven Kerckhoff
Stanford University
Stanford, CA 94305
spk@gauss.stanford.edu

Martin Scharlemann
University of California
Santa Barbara, CA 93106
mgscharl@math.ucsb.edu

Gang Tian
Courant Institute
New York University
New York, NY 10012-1100
tiang@taotao.cims.nyu.edu

V. S. Varadarajan
University of California
Los Angeles, CA 90095-1555
vsv@math.ucla.edu

SUPPORTING INSTITUTIONS

CALIFORNIA INSTITUTE OF TECHNOLOGY
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
STANFORD UNIVERSITY
UNIVERSITY OF ARIZONA
UNIVERSITY OF BRITISH COLUMBIA
UNIVERSITY OF CALIFORNIA
UNIVERSITY OF HAWAII

UNIVERSITY OF MONTANA
UNIVERSITY OF NEVADA, RENO
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
UNIVERSITY OF UTAH
UNIVERSITY OF WASHINGTON
WASHINGTON STATE UNIVERSITY

The supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Manuscripts must be prepared in accordance with the instructions provided on the inside back cover.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$215.00 a year (10 issues). Special rate: \$108.00 a year to individual members of supporting institutions.

Subscriptions, orders for back issues published within the last three years, and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at the University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 6143, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,
Berkeley, CA 94720, A NON-PROFIT CORPORATION

This publication was typeset using AMS-LATEX,
the American Mathematical Society's TEX macro system.
Copyright © 1995 by Pacific Journal of Mathematics

PACIFIC JOURNAL OF MATHEMATICS

Volume 169 No. 2 June 1995

On Banach spaces Y for which $B(C(\Omega), Y) = K(C(\Omega), Y)$	201
SHAMIM ISMAIL ANSARI	
Convergence of infinite exponentials	219
GENNADY BACHMAN	
Cohomologie d'intersection modérée. Un théorème de de Rham	235
BOHUMIL CENKL, GILBERT HECTOR and MARTINTXO SARALEGI-ARANGUREN	
Kleinian groups with an invariant Jordan curve: J -groups	291
RUBEN A. HIDALGO	
Multiplicative functions on free groups and irreducible representations	311
M. GABRIELLA KUHN and TIM STEGER	
A Diophantine equation concerning finite groups	335
MAOHUA LE	
Nilpotent characters	343
GABRIEL NAVARRO	
Smooth extensions and quantized Fréchet algebras	353
XIAOLU WANG	