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GENERALIZED GENERALIZED SPIN MODELS (FOUR-WEIGHT SPIN MODELS)

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The concept of spin model was introduced by V. F. R. Jones. Kawagoe, Munemasa and Watatani generalized it by dropping the symmetric condition, and defined a generalized spin model. In this paper, by further generalizing the concept using four functions, we define a generalized generalized spin model (four-weight spin model). Namely, (X, w_1, w_2, w_3, w_4) is a generalized generalized spin model (four-weight spin model), if X is a finite set and w_i $(i = 1, 2, 3, 4)$ are complex valued functions on $X \times X$ satisfying the following conditions:

$$(1) \quad w_1(\alpha, \beta)w_3(\beta, \alpha) = 1, \; w_2(\alpha, \beta)w_4(\beta, \alpha) = 1$$

for any α, β in X,

$$(2) \quad \sum_{x \in X} w_1(\alpha, x)w_3(x, \beta) = n\delta_{\alpha,\beta}, \; \sum_{x \in X} w_2(\alpha, x)w_4(x, \beta) = n\delta_{\alpha,\beta}$$

for any α and β in X,

$$(3a) \quad \sum_{x \in X} w_1(\alpha, x)w_1(x, \beta)w_4(\gamma, x) = Dw_1(\alpha, \beta)w_4(\gamma, \alpha)w_4(\gamma, \beta)$$

and

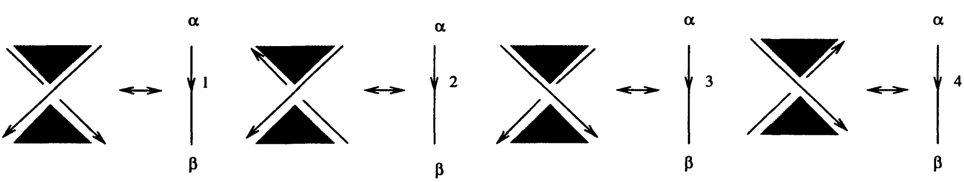
$$(3b) \quad \sum_{x \in X} w_1(x, \alpha)w_1(\beta, x)w_4(x, \gamma) = Dw_1(\beta, \alpha)w_4(\alpha, \gamma)w_4(\beta, \gamma)$$

for any α, β, and γ in X, where $D^2 = n = |X|$.
We call as generalized spin models (two-weight spin models), the special cases of generalized generalized spin models (four-weight spin models), where there are only two functions w_+ and w_- from $X \times X$ to C with two of w_1, w_2, w_3, w_4 being in $\{w_+, {}^t w_+\}$ and the remaining two of w_1, w_2, w_3, w_4 being in $\{w_-, {}^t w_-\}$. We see that we have three types of generalized spin models (two-weight spin models), namely Jones type, pseudo-Jones type, and Hadamard type. We also see that Kawagoe-Munemasa-Watatani's generalized spin model is one special case of Jones type, and Jones' original spin model is a further special case of it. Here we emphasize that there are actually interesting spin models which are considerably different from the original concept of spin model defined by Jones.

1. Introduction.

The concept of spin model was defined by Jones [**6**] (see Definition 7 below). Kawagoe, Munemasa and Watatani [**7**] generalized it by dropping the symmetric condition, and defined a generalized spin model (i.e., the generalized spin model (two-weight spin model) of Jones type in Definition 7 (ii)). In §1 of the present paper, we further generalize the concept by using four functions w_i $(i = 1, 2, 3, 4)$, and define generalized generalized spin models (four-weight spin models) (see Definition 2). The purpose of §1 is to discuss the background of this new definition. In the subsequent sections, we study the special cases where there are only two functions w_+ and w_- from $X \times X$ to $\mathbf{C}$ with two of w_1, w_2, w_3, w_4 being in $\{w_+, {}^t w_+\}$ and the remaining two of w_1, w_2, w_3, w_4 being in $\{w_-, {}^t w_-\}$. We call these models generalized spin models (two-weight spin models), and they are divided into three types (though these types are not exclusive of each other): Jones type, pseudo-Jones type and Hadamard type. They are discussed in §2, §3, and §4 respectively. We also see that Kawagoe-Munemasa-Watatani's generalized spin model is the generalized spin model (two-weight spin model) of Jones type (in Definition 7 (ii)) and that Jones' original spin model is a further special case of it. Here we emphasize that there are actually interesting spin models which are considerably different from the original concept of spin model defined by Jones [**6**].

For any diagram L of an oriented link, we color the regions (a region is a connected component of the complement of L in the plane of L) in black and white so that the unbounded region is white and adjacent regions have different colors as in a chess board. Then we get exactly four kinds of crossings. We construct a numbered oriented graph whose vertices are the black regions and edges are the crossings. For each edge (crossing) assign a number and an orientation in the following manner.



For any edge $\alpha \to \beta, c(\alpha \to \beta)$ denotes the number attached to the edge according to the definition given above.

For a diagram L of a link, $v(L)$ denotes the number of black regions (number of the vertices of the corresponding graph).

Let X be a finite set with $|X| = n$ and let $D \in \mathbf{C}$ be such that $D^2 = n$. Let w_1, w_2, w_3, and w_4 be complex valued functions defined on $X \times X$.

Now we define the partition function Z_L of L by

$$Z_L = D^{-v(L)} \sum_{\substack{\sigma \\ \text{states}}} \prod_{\substack{\alpha\to\beta \\ \text{edges}}} w_{c(\alpha\to\beta)}(\sigma(\alpha), \sigma(\beta)),$$

where a state σ is a map from the set of vertices of the graph of L to X.

It is easy to see that there are the following eight kinds of Reidemeister moves of type II and sixteen kinds of type III, as follows

type II_1

type II_2

type II_3

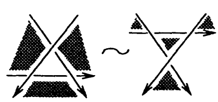
type II_4

type II_5

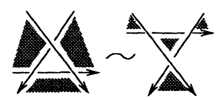
type II_6

type II_7

type II_8

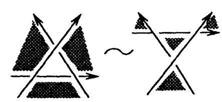


type III_1

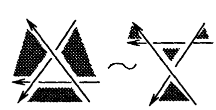


type III_2

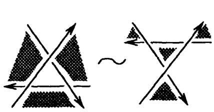
type III_3



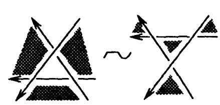
type III_4



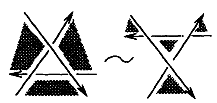
type III_5



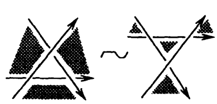
type III_6



type III_7

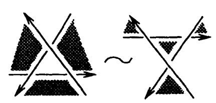


type III_8

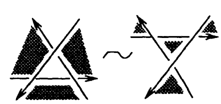


type III_9

type III_{10}

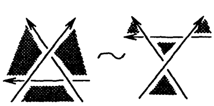


type III_{11}

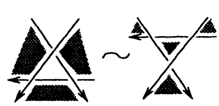


type III_{12}

type III_{13}



type III_{14}



type III_{15}

type III_{16}

The partition function is invariant under Reidemeister moves of type II_1, $\cdots$, type II_8 and type III_1, $\cdots$, type III_{16} if the following conditions $\mathrm{II}_1, \cdots, \mathrm{II}_8$ and $\mathrm{III}_1, \cdots, \mathrm{III}_{16}$ hold respectively.

II_1. $w_2(\alpha, \beta) w_4(\beta, \alpha) = 1$,

II_2. $w_1(\beta,\alpha)w_3(\alpha,\beta) = 1$,

II_3. $\sum_x w_1(\alpha,x)w_3(x,\beta) = n\delta_{\alpha,\beta}$,

II_4. $\sum_x w_4(\beta,x)w_2(x,\alpha) = n\delta_{\alpha,\beta}$,

II_5. $w_2(\beta,\alpha)w_4(\alpha,\beta) = 1$,

II_6. $w_1(\alpha,\beta)w_3(\beta,\alpha) = 1$,

II_7. $\sum_x w_3(\beta,x)w_1(x,\alpha) = n\delta_{\alpha,\beta}$,

II_8. $\sum_x w_2(\alpha,x)w_4(x,\beta) = n\delta_{\alpha,\beta}$, each of these is meant for any $\alpha,\beta \in X$.

III_1. $\sum_x w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta)$,

III_2. $\sum_x w_1(\alpha,x)w_3(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_2(\alpha,\gamma)w_4(\gamma,\beta)$,

III_3. $\sum_x w_2(x,\alpha)w_3(x,\beta)w_3(\gamma,x) = Dw_2(\beta,\alpha)w_2(\gamma,\alpha)w_3(\gamma,\beta)$,

III_4. $\sum_x w_1(\alpha,x)w_2(x,\beta)w_3(x,\gamma) = Dw_2(\alpha,\beta)w_3(\alpha,\gamma)w_4(\beta,\gamma)$,

III_5. $\sum_x w_1(x,\alpha)w_3(\beta,x)w_4(\gamma,x) = Dw_1(\beta,\alpha)w_2(\alpha,\gamma)w_4(\gamma,\beta)$,

III_6. $\sum_x w_2(\alpha,x)w_2(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_3(\alpha,\gamma)w_3(\gamma,\beta)$,

III_7. $\sum_x w_1(x,\alpha)w_2(x,\beta)w_3(\gamma,x) = Dw_2(\alpha,\beta)w_3(\gamma,\alpha)w_4(\beta,\gamma)$,

III_8. $\sum_x w_2(\alpha,x)w_4(x,\beta)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_1(\alpha,\gamma)w_3(\gamma,\beta)$,

III_9. $\sum_x w_1(\alpha,x)w_2(\beta,x)w_3(x,\gamma) = Dw_2(\beta,\alpha)w_3(\alpha,\gamma)w_4(\gamma,\beta)$,

III_{10}. $\sum_x w_1(\alpha,x)w_3(x,\beta)w_4(x,\gamma) = Dw_1(\alpha,\beta)w_2(\gamma,\alpha)w_4(\beta,\gamma)$,

III_{11}. $\sum_x w_2(x,\alpha)w_4(\beta,x)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_1(\gamma,\alpha)w_3(\beta,\gamma)$,

III_{12}. $\sum_x w_2(x,\alpha)w_2(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_3(\gamma,\alpha)w_3(\beta,\gamma)$,

III_{13}. $\sum_x w_2(\alpha,x)w_3(x,\beta)w_3(\gamma,x) = Dw_2(\alpha,\beta)w_2(\alpha,\gamma)w_3(\gamma,\beta)$,

III_{14}. $\sum_x w_1(x,\alpha)w_1(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\alpha,\gamma)w_4(\beta,\gamma)$,

III_{15}. $\sum_x w_1(x,\alpha)w_2(\beta,x)w_3(\gamma,x) = Dw_2(\beta,\alpha)w_3(\gamma,\alpha)w_4(\gamma,\beta)$,

III_{16}. $\sum_x w_1(x,\alpha)w_3(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_2(\gamma,\alpha)w_4(\beta,\gamma)$, each of these is meant for any $\alpha,\beta,\gamma \in X$.

Let $W_i = (w_i(\alpha,\beta))_{\alpha\in X,\beta\in X}$ for $i = 1,2,3,4$. Let I be the identity matrix and J be the matrix whose entries are all 1. Let $Y^{i,j}_{\alpha,\beta}$ be an n-dimensional column vector whose x-entry is given by $Y^{i,j}_{\alpha,\beta}(x) = w_i(\alpha,x)w_j(x,\beta)$ for any $i,j \in \{1,2,3,4\}$ and $\alpha,\beta \in X$.

The matrix expressions of II_6 and $\mathrm{II}_2,\mathrm{II}_5$ and $\mathrm{II}_1,\mathrm{II}_7$ and $\mathrm{II}_3,\mathrm{II}_8$ and II_4 are ${}^tW_1 \circ W_3 = J$, ${}^tW_2 \circ W_4 = J$, $W_1W_3 = nI$ and $W_2W_4 = nI$ respectively.

The following $\mathrm{III}_1{}', \mathrm{III}_2{}', \cdots, \mathrm{III}_{16}{}'$ are the matrix expressions of $\mathrm{III}_1, \mathrm{III}_2, \cdots, \mathrm{III}_{16}$ respectively.

$\mathrm{III}_1{}'$. $W_1 Y^{4,1}_{\alpha,\beta} = Dw_4(\alpha,\beta) Y^{4,1}_{\alpha\beta}$,

$\mathrm{III}_2{}'$. $W_4 Y^{1,3}_{\alpha,\beta} = Dw_1(\alpha,\beta) Y^{2,4}_{\alpha,\beta}$,

$\mathrm{III}_3{}'$. ${}^tW_3 Y^{3,2}_{\alpha,\beta} = Dw_2(\alpha,\beta) Y^{3,2}_{\alpha,\beta}$,

$\mathrm{III}_4{}'$. ${}^tW_2 Y^{1,3}_{\alpha,\beta} = Dw_3(\alpha,\beta) Y^{2,4}_{\alpha,\beta}$,

$\mathrm{III}_5{}'$. $W_3 Y^{4,1}_{\alpha,\beta} = Dw_2(\beta,\alpha) Y^{4,1}_{\alpha,\beta}$,

$\mathrm{III}_6{}'$. $W_2 Y^{2,4}_{\alpha,\beta} = Dw_3(\beta,\alpha) Y^{1,3}_{\alpha,\beta}$,

$\mathrm{III}_7{}'$. ${}^tW_1 Y^{3,2}_{\alpha,\beta} = Dw_4(\beta,\alpha) Y^{3,2}_{\alpha\beta}$,

$\mathrm{III}_8{}'$. ${}^tW_4 Y^{2,4}_{\alpha,\beta} = Dw_1(\beta,\alpha) Y^{1,3}_{\alpha,\beta}$,

$\mathrm{III}_9{}'$. $W_1 Y^{2,3}_{\alpha,\beta} = Dw_4(\beta,\alpha) Y^{2,3}_{\alpha,\beta}$,

$\mathrm{III}_{10}{}'$. ${}^tW_3 Y^{1,4}_{\alpha,\beta} = Dw_2(\beta,\alpha) Y^{1,4}_{\alpha,\beta}$,

$\mathrm{III}_{11}{}'$. $W_4 Y^{42}_{\alpha\beta} = Dw_1(\beta,\alpha) Y^{3,1}_{\alpha,\beta}$,

$\mathrm{III}_{12}{}'$. ${}^tW_2 Y^{4,2}_{\alpha,\beta} = Dw_3(\beta,\alpha) Y^{3,1}_{\alpha,\beta}$,

$\mathrm{III}_{13}{}'$. $W_3 Y^{2,3}_{\alpha,\beta} = Dw_2(\alpha,\beta) Y^{2,3}_{\alpha,\beta}$,

$\mathrm{III}_{14}{}'$. ${}^tW_1 Y^{1,4}_{\alpha,\beta} = Dw_4(\alpha,\beta) Y^{1,4}_{\alpha,\beta}$,

$\mathrm{III}_{15}{}'$. $W_2 Y^{3,1}_{\alpha,\beta} = Dw_3(\alpha,\beta) Y^{4,2}_{\alpha,\beta}$,

$\mathrm{III}_{16}{}'$. ${}^tW_4 Y^{3,1}_{\alpha,\beta} = Dw_1(\alpha,\beta) Y^{4,2}_{\alpha,\beta}$, each of these is meant for any $\alpha, \beta \in X$.

We have the following theorem.

Theorem 1. *Let X be finite set with $\mid X \mid = n = D^2$. Let w_1, w_2, w_3, w_4 be complex valued functions on $X \times X$ which satisfy the following conditions:*

(1) $w_1(\alpha,\beta)w_3(\beta,\alpha) = 1$, $w_2(\alpha,\beta)w_4(\beta,\alpha) = 1$ *for any* $\alpha, \beta \in X$,

(2) $\sum_{x \in X} w_1(\alpha,x)w_3(x,\beta) = n\delta_{\alpha,\beta}$, $\sum_{x \in X} w_2(\alpha,x)w_4(x,\beta) = n\delta_{\alpha,\beta}$ *for any* $\alpha, \beta \in X$. *Then the conditions* III_1 *to* III_8 *are equivalent to each other, as well as* III_9 *to* III_{16}. *(Note that the condition* $\mathrm{II}_1 + \mathrm{II}_2, \mathrm{II}_5 + \mathrm{II}_6$ *and* (1) *are equivalent to each other, as well as* $\mathrm{II}_3 + \mathrm{II}_4, \mathrm{II}_7 + \mathrm{II}_8$ *and* (2).)

Proof. The matrix expressions of the conditions (1), (2) and III_1' through III_{16}' show that III_1, III_2, III_3, III_4, III_9, III_{10}, III_{11}, III_{12} are equivalent to III_5, III_6, III_7, III_8, III_{13}, III_{14}, III_{15}, III_{16} respectively. □

By III_1 we have

$$\sum_{\gamma}\left\{\sum_{x} w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x)\right\} w_2(y,\gamma)w_3(\beta,\alpha)$$
$$=\sum_{\gamma}(Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta))w_2(y,\gamma)w_3(\beta,\alpha)$$

for any $\alpha,\beta,y \in X$. Since $W_2W_4 = nI$ and ${}^tW_1 \circ W_3 = J$, we have III_8. Similarly from $\mathrm{III}_8, \mathrm{III}_7, \mathrm{III}_2$, by summing over β, γ, and β respectively, we have $\mathrm{III}_7, \mathrm{III}_2$, and III_1 respectively. Therefore $\mathrm{III}_1, \mathrm{III}_8, \mathrm{III}_7, \mathrm{III}_2$ are equivalent to each other. A similar method on $\mathrm{III}_{15}, \mathrm{III}_{13}, \mathrm{III}_{12}, \mathrm{III}_{10}$, summing over α, α, β, and α respectively, gives $\mathrm{III}_{13}, \mathrm{III}_{12}, \mathrm{III}_{10}, \mathrm{III}_{15}$. Therefore $\mathrm{III}_{15}, \mathrm{III}_{13}, \mathrm{III}_{12}, \mathrm{III}_{10}$ are equivalent to each other.

Theorem 1 tells us that the following definition of generalized generalized spin model (four-weight spin model) is meaningful.

Definition 2. Let X be a finite set, and let w_i $(i = 1,2,3,4)$ be functions on $X \times X$ to $\mathbf{C}$. Then $(X, w_1, w_2, w_3, w_4; D)$ is a generalized generalized spin model (four-weight spin model) of loop variable D if the following conditions are satisfied for any α, β and $\gamma \in X$:

(1) $w_1(\alpha,\beta)w_3(\beta,\alpha) = 1,\ w_2(\alpha,\beta)w_4(\beta,\alpha) = 1,$

(2) $\sum_{x\in X} w_1(\alpha,x)w_3(x,\beta) = n\delta_{\alpha,\beta}, \sum_{x\in X} w_2(\alpha,x)w_4(x,\beta) = n\delta_{\alpha,\beta},$

(3a) $\sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta),$

(3b) $\sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\alpha,\gamma)w_4(\beta,\gamma).$

Note. (1) is $\mathrm{II}_2 + \mathrm{II}_1$ and (2) is $\mathrm{II}_3 + \mathrm{II}_8$, (3a) and (3b) are III_1 and III_{14} respectively.

Proposition 3. *The following are equivalent:*

(i) *$(X, w_1, w_2, w_3, w_4; D)$ is a four-weight spin model,*
(ii) *$(X, -w_1, w_2, -w_3, w_4; -D)$ is a four-weight spin model,*
(iii) *$(X, w_1, -w_2, w_3, -w_4; -D)$ is a four-weight spin model,*
(iv) *$(X, -w_1, -w_2, -w_3, -w_4; D)$ is a four-weight spin model.*

Note. In what follows we simply write (X, w_1, w_2, w_3, w_4) to represent $(X, w_1, w_2, w_3, w_4; D)$ whenever there is no confusion.

Note. If (X, w_1, w_2, w_3, w_4) is a four-weight spin model, then the partition function Z_L of an oriented link diagram L is invariant under the Reidemeister moves of type II and III.

We have the following matrix expressions of (1), (2), (3a) and (3b).

(1)′ ${}^tW_1 \circ W_3 = J,\ {}^tW_2 \circ W_4 = J,$

(2)′ $W_1W_3 = nI,\ W_2W_4 = nI,$

(3a)′ $W_1Y^{4,1}_{\gamma,\beta} = Dw_4(\gamma,\beta)Y^{4,1}_{\gamma,\beta}$ for any $\gamma,\beta \in X$,

(3b)′ ${}^tW_1Y^{1,4}_{\beta,\gamma} = Dw_4(\beta,\gamma)Y^{1,4}_{\beta,\gamma}$ for any $\gamma,\beta \in X$.

Proposition 4. *Let (X, w_1, w_2, w_3, w_4) be a four-weight spin model. Then we have*

(4) $\sum_{x\in X} w_2(\alpha,x) = \sum_{x\in X} w_2(x,\alpha) = Dw_3(\alpha,\alpha) = Da^{-1},$

(5) $\sum_{x\in X} w_4(\alpha,x) = \sum_{x\in X} w_4(x,\alpha) = Dw_1(\alpha,\alpha) = Da$

for any $\alpha \in X$ with some nonzero $a \in \mathbf{C}$. We call this number a the modulus of (X, w_1, w_2, w_3, w_4).

Proof. In $\mathrm{III}_{15}, \mathrm{III}_{12}, \mathrm{III}_2$ and III_{16}, put $\alpha = \gamma, \beta = \gamma, \alpha = \beta$, and $\alpha = \beta$ respectively. Then, by (1) and (2), we have the proposition. □

The following Proposition 5 is the matrix expression of Proposition 4.

Proposition 5. *Let (X, w_1, w_2, w_3, w_4) be a four-weight spin model of modulus a. Then we have the following relations.*

(4)′ $W_2J = {}^tW_2J = Da^{-1}J,\ W_3 \circ I = a^{-1}I,$

(5)′ $W_4J = {}^tW_4J = DaJ,\ W_1 \circ I = aI.$

Corollary 6. *Let (X, w_1, w_2, w_3, w_4) be a four-weight spin model of loop variable D. Then $Dw_4(\alpha,\beta)$ and $Dw_2(\alpha,\beta)$ are eigenvalues of W_1 and W_3 respectively.*

Proof. Obvious from $\mathrm{III}_1{}'$ and $\mathrm{III}_5{}'$. □

(Note that $Dw_1(\alpha,\beta)$ is not necessarily an eigenvalue of W_4, and that $Dw_3(\alpha,\beta)$ is not necessarily an eigenvalue of W_2.)

2. Generalized spin models of Jones type.

In this section we consider the special case of four-weight spin models, where there are only two functions w_+ and w_- on $X \times X$ to $\mathbf{C}$ with $w_1, w_2 \in$

$\{w_\epsilon, {}^t w_\epsilon\}$ and $w_3, w_4 \in \{w_{\epsilon'}, {}^t w_{\epsilon'}\}$, where $\{\epsilon, \epsilon'\} = \{+, -\}$, with ${}^t w_\epsilon(\alpha, \beta) = w_\epsilon(\beta, \alpha)$ for any $\alpha, \beta \in X$ and $\epsilon \in \{+, -\}$. First we define the following conditions on the ordered triple (X, w_+, w_-) with $\mid X \mid = n = D^2$.

(0) $w_+(\alpha, \beta) = w_+(\beta, \alpha)$, $w_-(\alpha, \beta) = w_-(\beta, \alpha)$ for any α and β in X,

(1J) $w_+(\alpha, \beta) w_-(\beta, \alpha) = 1$ for any α and β in X,

(1JT) $w_+(\alpha, \beta) w_-(\alpha, \beta) = 1$ for any α and β in X,

(2J) $\sum_{x \in X} w_+(\alpha, x) w_-(x, \beta) = n \delta_{\alpha,\beta}$ for any α and β in X,

(2JT) $\sum_{x \in X} w_+(\alpha, x) w_-(\beta, x) = n \delta_{\alpha,\beta}$ for any α and β in X,

(3J) $\sum_{x \in X} w_+(\alpha, x) w_+(x, \beta) w_-(x, \gamma) = D w_+(\alpha, \beta) w_-(\alpha, \gamma) w(\beta, \gamma)$ for any α, β and γ in X,

(3JT) $\sum_{x \in X} w_+(\alpha, x) w_+(x, \beta) w_-(\gamma, x) = D w_+(\alpha, \beta) w_-(\gamma, \alpha) w_-(\gamma, \beta)$ for any α, β and γ in X.

Proposition 7. *Let* (X, w_+, w_-) *satisfy the conditions* (1J) *and* (2J). *Then each of the conditions* III_1 *to* III_{16} *for* $(X, w_+, {}^t w_+, w_-, {}^t w_-)$ *is equivalent to* (3J). *In particular* (3JT) *is equivalent to* (3J).

Proof. Clearly $(X, w_+, {}^t w_+, w_-, {}^t w_-)$ satisfies the conditions (1) and (2) of Theorem 1. The condition III_7 and III_{14} are both equivalent to (3JT) for (X, w_+, w_-) and the condition III_1 is equivalent to (3J) for (X, w_+, w_-). Hence by Theorem 1 we have the proposition. □

Definition 8. (i) (The original spin model due to Jones [**6**].) (X, w_+, w_-) is a symmetric spin model of Jones type if the conditions (0), (1J), (2J) and (3J) are satisfied.

(ii) (X, w_+, w_-) is a generalized spin model (two-weight spin model) of Jones type if the conditions (1J), (2J) and (3J) are satisfied.

(iii) (X, w_+, w_-) is a generalized spin model (two-weight spin model) of transposed Jones type if the conditions (1JT), (2JT) and (3J) are satisfied.

Note. The symmetric spin models of Jones type (i) are special cases of (ii) and (iii) of Definition 8.

Theorem 9. *The following are equivalent.*

(i) (X, w_+, w_-) *is a two-weight spin model of Jones type,*

(ii) (X, w_-, w_+) *is a two-weight spin model of Jones type,*

(iii) $(X, {}^t w_+, {}^t w_-)$ *is a two-weight spin model of Jones type,*

(iv) $(X, w_+, {}^t w_-)$ *is a two-weight spin model of transposed Jones type,*

(v) $(X, {}^tw_+, w_-)$ *is a two-weight spin model of transposed Jones type.*

Proof. The condition (1J) for (X, w_+, w_-), (X, w_-, w_+), and $(X, {}^tw_+, {}^tw_-)$, and the condition (1JT) for $(X, {}^tw_+, {}^tw_-)$ and $(X, {}^tw_+, w_-)$ are equivalent to each other. The conditions (3J) or $(X, w_+, {}^tw_-)$ and (3J) for $(X, {}^tw_+, w_-)$ are equivalent to the conditions (3JT) and (3J) for (X, w_+, w_-) respectively. Hence, by Proposition 7, (i), (iv) and (v) are equivalent to each other. The condition III_{10} for $(X, w_+, {}^tw_+, w_-, {}^tw_-)$ is the condition (3JT) for (X, w_-, w_+). Hence by Proposition 7, (ii) is equivalent to (i). Since the condition (3J) for $(X, {}^tw_+, {}^tw_-)$ is exactly the condition (3JT) for (X, w_+, w_-), (iii) is equivalent to (i). □

Corollary 10. *(X, w_+, w_-) is a two-weight spin model of transposed Jones type if and only if (X, w_-, w_+) is.*

Proof. Immediate from Theorem 9. □

Theorem 11. *Let (X, w_1, w_2, w_3, w_4) be a four weight spin model. If $w_1, w_2 \in \{w_\epsilon, {}^t w_\epsilon\}$ and $w_3, w_4 \in \{w_{\epsilon'}, {}^t w_{\epsilon'}\}$ where $\{\epsilon, \epsilon'\} = \{+, -\}$, then the conditions* (3a) *and* (3b) *in Definition* 2 *are equivalent and (X, w_+, w_-) is either a two-weight spin model of Jones type or that of transposed Jones type.* (*Note that it is possible to have $w_1 = w_2 \neq {}^tw_1$ or $w_3 = w_4 \neq {}^tw_3$.*)

Proof. Case (i). $w_1 = w_+, w_2 = w_+, w_3 = w_-, w_4 = w_-$.

The conditions (1) and (2) in Definition 2 show that conditions (1J) and (2J) are satisfied. Both conditions III_1 and III_{15} in §1 give (3JT). Since III_1 and III_{15} are equivalent to (3a) and (3b) respectively, the conditions (3a) and (3b) are equivalent. Since (3JT) is equivalent to (3J) under the conditions (1J) and (2J), (X, w_+, w_-) is a two-weight spin model of Jones type.

Case (ii). $w_1 = w_+, w_2 = w_+, w_3 = w_-, w_4 = {}^tw_-$.

By (1) of Definition 2, we have $w_+(\alpha, \beta)w_-(\beta, \alpha) = w_+(\alpha, \beta)w_-(\alpha, \beta) = 1$. Therefore ${}^tw_+ = w_+$ and ${}^tw_- = w_-$. Hence the conditions (3a) and (3b) both give condition (3J), and (X, w_+, w_-) is a symmetric spin model of Jones type.

Case (iii). $w_1 = w_+, w_2 = {}^tw_+, w_3 = w_-, w_4 = w_-$.

A similar argument as in (ii) proves that (X, w_+, w_-) is a symmetric spin model of Jones type.

Case (iv). $w_1 = w_+, w_2 = {}^tw_+, w_3 = w_-, w_4 = {}^tw_-$.

The conditions III_1 and III_9 in §1 both give (3J). Since III_1 and III_9 are equivalent to (3a) and (3b) respectively, (3a) and (3b) are equivalent. Therefore (X, w_+, w_-) is a two-weight spin model of Jones type.

Case (v). $w_1 = w_+, w_3 = {}^tw_-$.

Cases (i) and (iv) show that $(X, w_+, {}^t w_-)$ is a two-weight spin model of Jones type and cases (ii) and (iii) show that $(X, w_+, {}^t w_-)$ is a symmetric spin model of Jones type. Therefore (X, w_+, w_-) is a two-weight spin model of transposed Jones type or a symmetric spin model of Jones type.

Case (vi). $w_1 = {}^t w_+$.

Cases (i), (ii), (iii), (iv), (v) and Theorem 9 show that (X, w_+, w_-) is a two-weight spin model of Jones type, transposed Jones type, or a symmetric spin model of Jones type.

Case (vii). $w_1 \in \{w_-, {}^t w_-\}$.

Cases (i), (ii), $\cdots$, (vi) show that (X, w_-, w_+) is a two-weight spin model of Jones type, transposed Jones type or a symmetric spin model of Jones type. Therefore by Theorem 9, the proof is completed. $\square$

Remark. Combining Theorem 9 and Corollary 10, we can conclude that in order to study two-weight spin models of transposed Jones type, we essentially have to consider the two-weight spin models of Jones type.

Note. For a given two-weight spin model of Jones type (X, w_+, w_-), there are several ways to construct partition functions for oriented links which are possibly different from each other. For example (X, w_+, w_+, w_-, w_-) and $(X, w_+, {}^t w_+, w_-, {}^t w_-)$ are four-weight spin models. We can construct partition functions according to the definition given in §1.

3. Generalized spin models of pseudo-Jones type.

In this section we consider the four-weight spin model with $w_1, w_4 \in \{w_\epsilon, {}^t w_\epsilon\}$ and $w_2, w_3 \in \{w_{\epsilon'}, {}^t w_{\epsilon'}\}$ where $\{\epsilon, \epsilon'\} = \{+, -\}$.

First we define a condition for (X, w_+, w_-) with $|X| = n = D^2$ in addition to (0), $\cdots$, (3JT) given in §2.

$$\text{(3P)} \qquad \sum_{x \in X} w_+(\alpha, x) w_+(x, \beta) w_+(x, \gamma) = D w_+(\alpha, \beta) w_+(\alpha, \gamma) w_+(\beta, \gamma)$$

for any α, β and γ in X.

Proposition 12. *Let* (X, w_+, w_-) *satisfy* (0), (1J) *and* (2J). *The conditions* III_1 *to* III_{16} *for* (X, w_+, w_-, w_-, w_+) *are all equivalent to* (3P).

Proof. Clearly (X, w_+, w_-, w_-, w_+) satisfies the conditions (1) and (2) of Definition 2. Clearly the conditions III_1 and III_{14} are both equivalent to (3P). Therefore by Theorem 1 we have Proposition 12. $\square$

Definition 13. (X, w_+, w_-) is a generalized spin model (two-weight spin model) of pseudo-Jones type if the conditions (0), (1J), (2J) and (3P) are satisfied.

Theorem 14. *(X, w_+, w_-) is a two-weight spin model of pseudo-Jones type if and only if (X, w_-, w_+) is.*

Proof. Since the condition III_{13} for (X, w_+, w_-, w_-, w_+) is the condition (3P) for (X, w_-, w_+), by Proposition 12 we have Theorem 14. □

Theorem 15. *Let (X, w_1, w_2, w_3, w_4) be a four-weight spin model. If $w_1, w_4 \in \{w_\epsilon, {}^t w_\epsilon\}$ and $w_2, w_3 \in \{w_{\epsilon'}, {}^t w_{\epsilon'}\}$, where $\{\epsilon, \epsilon'\} = \{+, -\}$, with some w_+ and w_-, then (X, w_+, w_-) is a two-weight spin model of pseudo-Jones type. (In the assumption it is possible to have $w_1 = w_4 \neq {}^t w_1$ and $w_2 = w_3 \neq {}^t w_2$.)*

Proof. First we will show that w_+ and w_- are symmetric.

Case (i). $w_1 = w_+, w_4 = w_+, w_2 = w_-, w_3 = w_-$.

By the assumptions we have $Y^{3,1}_{\alpha,\beta} = Y^{2,4}_{\alpha,\beta}$ and $Y^{1,3}_{\alpha,\beta} = Y^{4,2}_{\alpha,\beta}$. Then by ${\mathrm{III}_{12}}'$ and ${\mathrm{III}_4}'$ we have $w_-(\beta, \alpha) = w_-(\alpha, \beta)$. Therefore w_+ and w_- are symmetric.

Case (ii). $w_1 = w_+, w_4 = w_+, w_2 = w_-, w_3 = {}^t w_-$.

By (1) of Definition 2, clearly w_+ and w_- are symmetric.

Case (iii). $w_1, w_4 \in \{w_-, {}^t w_-\}$ and $w_2, w_3 \in \{w_+, {}^t w_+\}$. The similar arguments as for case (i) and case (ii) show that w_+ and w_- are symmetric.

Thus we see that w_+ and w_- are symmetric. Therefore w_1, w_2, w_3, and w_4 are symmetric and the conditions (3a) and (3b) in Definition 2 are equivalent. For the case $w_1 = w_4 = w_+$, the condition (3a) gives (3P). Therefore (X, w_+, w_-) is a two-weight spin model of pseudo-Jones type. For the case $w_1 = w_4 = w_-$, we have $w_2 = w_3 = w_+$, and III_{13} gives the condition (3P). Therefore, in both cases, (X, w_+, w_-) is a two-weight spin model of pseudo-Jones type. □

Note. For a given two-weight spin model of psudo-Jones type (X, w_+, w_-), (X, w_+, w_-, w_-, w_+) and (X, w_-, w_+, w_+, w_-) are four-weight spin models. We can obtain partition functions from these two spin models according to the definition given in §1.

4. Generalized spin models of Hadamard type.

In this section we consider the cases where $w_1, w_3 \in \{w_\epsilon, {}^t w_\epsilon\}$ and $w_2, w_4 \in \{w_{\epsilon'}, {}^t w_{\epsilon'}\}$, where $\{\epsilon, \epsilon'\} = \{+, -\}$. In these cases, W_+ or W_- is an Hadamard matrix. We call these spin models Hadamard type.

First we define additional conditions for (X, w_+, w_-) with $\mid X \mid = n = D^2$.

(0_ϵ) $w_\epsilon(\alpha,\beta) = w_\epsilon(\beta,\alpha)$ for any α and β in X,

$(1\mathrm{H}_\epsilon)$ $w_\epsilon(\alpha,\beta)w_\epsilon(\alpha,\beta) = 1,\ w_{\epsilon'}(\alpha,\beta)w_{\epsilon'}(\beta,\alpha) = 1$ for any α and β in X,

$(2\mathrm{H}_\epsilon)$ $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(\beta,x) = n\delta_{\alpha,\beta},\ \sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta) = n\delta_{\alpha,\beta}$ for any α and β in X,

$(3\mathrm{a}_\epsilon)$

$$\sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta)w_\epsilon(x,\gamma) = Dw_{\epsilon'}(\alpha,\beta)w_\epsilon(\alpha,\gamma)w_\epsilon(\beta,\gamma)$$

for any α,β and γ in X,

$(3\mathrm{b}_\epsilon)$ $\sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta)w_\epsilon(\gamma,x) = Dw_{\epsilon'}(\alpha,\beta)w_\epsilon(\gamma,\alpha)w_\epsilon(\gamma,\beta)$ for any α,β and γ in X.

Definition 16. (i) (X, w_+, w_-) is a generalized spin model (two-weight spin model) of symmetric Hadamard type (SH_ϵ) if the conditions (0), $(1\mathrm{H}_\epsilon)$, $(2\mathrm{H}_\epsilon)$ and $(3\mathrm{a}_\epsilon)$ are satisfied.

(ii) (X, w_+, w_-) is a generalized spin model (two-weight spin model) of Hadamard type (H_ϵ) if the condition (0_ϵ), $(1\mathrm{H}_\epsilon)$, $(2\mathrm{H}_\epsilon)$ and $(3\mathrm{a}_\epsilon)$ are satisfied.

(iii) (X, w_+, w_-) is a generalized spin model (two-weight spin model) of Hadamard type (HA_ϵ) if the conditions $(1\mathrm{H}_\epsilon), (2\mathrm{H}_\epsilon), (3\mathrm{a}_\epsilon)$ and $(3\mathrm{b}_\epsilon)$ are satisfied.

(iv) (X, w_+, w_-) is a generalized spin model (two-weight spin model) of Hadamard type (HB_ϵ) if the conditions $(1\mathrm{H}_\epsilon), (2\mathrm{H}_\epsilon), (3\mathrm{a}_{\epsilon'})$ and $(3\mathrm{b}_{\epsilon'})$ are satisfied.

(v) (X, w_+, w_-) is a generalized spin model (two-weight spin model) of Hadamard type (HC_ϵ) if the conditions $(0_{\epsilon'}), (1\mathrm{H}_\epsilon), (2\mathrm{H}_\epsilon), (3\mathrm{a}_\epsilon)$ and $(3\mathrm{b}_\epsilon)$ are satisfied.

Note. In Definition 16 the matrix W_ϵ is a Hadamard matrix but $W_{\epsilon'}$ need not be.

Note. Spin models (i) are a special case of (ii), (iii), (iv) and (v).

Theorem 17. *If (X, w_+, w_-) is a two-weight spin model of type $(H_\epsilon), (HA_\epsilon), (HB_\epsilon), (HC_\epsilon)$, then $(X, w_+, {}^t w_-)$ and $(X, {}^t w_+, w_-)$ are also two-weight spin models of type (H_ϵ), (HA_ϵ), (HB_ϵ), or (HC_ϵ) respectively and (X, w_-, w_+) is a two-weight spin model of type $(\mathrm{H}_{\epsilon'}), (\mathrm{HA}_{\epsilon'}), (\mathrm{HB}_{\epsilon'})$ or $(\mathrm{HC}_{\epsilon'})$ respectively.*

Proof. Immediate from Definition 16. □

Theorem 18. *Let (X, w_1, w_2, w_3, w_4) be a four-weight spin model. Assume that $w_1, w_3 \in \{w_\epsilon, {}^t w_\epsilon\}$ and $w_2, w_4 \in \{w_{\epsilon'}, {}^t w_{\epsilon'}\}$ for some functions w_+ and w_- on $X \times X$. Then (X, w_+, w_-) is one of the two-weight spin models of type (H_ϵ), (HA_ϵ), (HB_ϵ), and (HC_ϵ), where $\{\epsilon, \epsilon'\} = \{+, -\}$. (Note that it is possible to have $w_1 = w_3 \neq {}^t w_1$ or $w_2 = w_4 \neq {}^t w_2$.)*

Proof. Case (i). $w_1 = w_+, w_2 = w_-, w_3 = w_+, w_4 = w_-$. Since $Y^{4,1}_{\alpha,\beta} = Y^{2,3}_{\alpha,\beta}$ by $\mathrm{III}_1{}'$ and $\mathrm{III}_9{}'$ of §2, we have $w_4(\alpha, \beta) = w_4(\beta, \alpha)$. Hence (0_-) is satisfied. By (1) and (2) of Definition 2, we have (1H) and (2H). Since $w_4 = w_-$ is symmetric, (3a) and (3b) of Definition 2 are both equivalent to $(3\mathrm{a}_-)$. Therefore (X, w_+, w_-) is a two-weight spin model of type (H_-).

Case (ii). $w_1 = w_+, w_2 = w_-, w_3 = w_+, w_4 = {}^t w_-$. By (1) and (2) of Definition 2, we have $(1\mathrm{H}_-)$ and $(2\mathrm{H}_-)$. (3a) and (3b) of Definition 2 give $(3\mathrm{a}_-)$ and $(3\mathrm{b}_-)$ respectively. Therefore (X, w_+, w_-) is a two-weight spin model of type (HA_-).

Case (iii). $w_1 = w_+, w_2 = w_-, w_3 = {}^t w_+, w_4 = w_-$. By (1) and (2) of Definition 2, we have $(1\mathrm{H}_+)$ and $(2\mathrm{H}_+)$. (3a) and (3b) of Definiton 2 give $(3\mathrm{a}_-)$ and $(3\mathrm{b}_-)$ respectively. Therefore (X, w_+, w_-) is a two-weight spin model of type (HB_+).

Case (iv). $w_1 = w_+, w_2 = w_-, w_3 = {}^t w_+, w_4 = {}^t w_-$. Since $w_2 = {}^t w_4$, by III_5' and III_9' we get ${}^t w_+ = w_+$, i.e., (0_+). Therefore by (1) and (2) of Definition 2, we have $(1\mathrm{H}_-)$, $(2\mathrm{H}_-)$. (3a) and (3b) of Definition 2 give $(3\mathrm{a}_-)$ and $(3\mathrm{b}_-)$ respectively. Therefore (X, w_+, w_-) is of type (HC_-).

Case (v). $w_1 = w_+, w_2 = {}^t w_-$.

By (i) to (iv), $(X, w_+, {}^t w_-)$ is of type (H_-), type (HA_-), type (HB_+), or type (HC_-) respectively. Hence by Theorem 17 (X, w_+, w_-) is one of those types.

Case (vi). $w_1 = {}^t w_+$.

Then cases (i), (ii), (iii), (iv), (v) show that $(X, {}^t w_+, w_-)$ is a two-weight spin model of type (H_-), (HA_-), (HB_+), and (HC_+). Therefore (X, w_+, w_-) is also one of those types.

Case (vii). $w_1 \in \{w_-, {}^t w_-\}$.

Then cases (i) to (iv) show that (X, w_-, w_+) is a two-weight spin model of type (H_-), (HA_-), (HB_+) or (HC_+). Therefore (X, w_+, w_-) is of type (H_+), (HA_+), (HB_-) or (HC_+). □

Note. As for the partition function Z_L of an oriented link diagram L attached to the two-weught spin models of Hadamard type, there are several ways to construct it. For example if (X, w_+, w_-) is a spin model of Hadamard type (HA_-), then $(X, w_+, w_-, w_+, {}^t w_-)$, $(X, w_+, {}^t w_-, w_+, w_-)$,

$(X, {}^t w_+, w_-, {}^t w_+, w_-)$, $(X, {}^t w_+, {}^t w_-, {}^t w_+, w_-)$ are four-weight spin models. We can construct partition functions from each of those according to the definition given in §1.

5. Concluding Remarks.

Generalized generalized spin models (four-weight spin models) (X, w_1, w_2, w_3, w_4) seem to exist considerably in abundance when compared with the original (symmetric) spin models due to Jones. The generalized spin models (two-weight spin models) considered in §2, §3, §4 are special cases of four-weight spin models, but they exist also in abundance.

As we have discussed in §2, §3, and §4, we have three types of two-weight spin models: Jones type, pseudo-Jones type and Hadamard type.

1) In order to consider (non-symmetric) Jones type, essentially we only have to consider Definition 8 (ii) (because of Theorem 9). Such two-weight spin models were first considered by Kawagoe, Munemasa and Watatani [**7**]. They gave three explicit examples with $n = 3, 4$ and 5. A family of such examples were constructed on the group association schemes of finite cyclic groups by Bannai and Bannai [**1**]. For symmetric Jones type, there are many examples attached to symmetric association schemes, in particular to strongly regular graphs (*cf.* [**4**], [**5**]). Nomura [**8**] systematically gives examples of symmetric spin models (in the original sense of Jones) attached to an Hadamard graph, i.e., the distance-regular graph of intersection array

$$\left\{ \begin{matrix} * & 1 & m & 2m-1 & 2m \\ 0 & 0 & 0 & 0 & 0 \\ 2m & 2m-1 & m & 1 & * \end{matrix} \right\}$$

which is canonically constructed from each Hadamard matrix (see [**3**, p.19]).

2) In pseudo-Jones type, the matrices W_+ and W_- are always symmetric. The following is an explicit example of pseudo-Jones type with loop variable $D = 2$ and modulus $a = 1$ which is not of Jones type nor of Hadamard type

$$W_+ = \begin{pmatrix} 1 & i & 1 & -i \\ i & 1 & -i & 1 \\ 1 & -i & 1 & i \\ -i & 1 & i & 1 \end{pmatrix}, \quad W_- = \begin{pmatrix} 1 & -i & 1 & i \\ -i & 1 & i & 1 \\ 1 & i & 1 & -i \\ i & 1 & -i & 1 \end{pmatrix}$$

with $i = \sqrt{-1}$. (We can check all the conditions in Definition 13 easily.) It is expected that there are many other two-weight spin models of pseudo-Jones type.

3) In Hadamard type, we only have to consider the following ones: Hadamard type (H_+), (HA_+), (HB_+), and (HC_+) (because of Theorem 17).

i) The following is an example of symmetric Hadamard type with $D = 2$ and $a = 1$, which is not of Jones type, nor of pseudo-Jones type.

$$W_+ = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \quad W_- = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

(We can easily check the conditions in Definition 16 (i).)

ii) The following is an example of non-symmetric Hadamard type (H_-) with $D = 2$ and $a = 1$, which is not of Jones type, nor of pseudo-Jones type.

$$W_+ = \begin{pmatrix} 1 & -i & -1 & -i \\ i & 1 & i & -1 \\ -1 & -i & 1 & -i \\ i & -1 & i & 1 \end{pmatrix}, \quad W_- = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

(We can easily check the conditions in Definition 16(ii).) It is expected that there exist many other two-weight spin models of Hadamard type.

Remark. Let $(X_1, w_1^{(1)}, w_2^{(1)}, w_3^{(1)}, w_4^{(1)})$ and $(X_2, w_1^{(2)}, w_2^{(2)}, w_3^{(2)}, w_4^{(2)})$ be four-weight spin models. Let us set $X = X_1 \times X_2$, $w_i = w_i^{(1)} \otimes w_i^{(2)}$ $(i = 1, 2, 3, 4)$, namely, $W_i = W_i^{(1)} \otimes W_i^{(2)}$, where $W_i^{(j)}$ is the matrix representation of $w_i^{(j)}$. Then (X, w_1, w_2, w_3, w_4) is a four-weight spin model. (We can immediately prove this claim by checking Definition 2.) Also, we can easily see that if $(X_i, w_+^{(i)}, w_-^{(i)})$ $(i = 1, 2)$ are two two-weight spin models of a same type, i.e., symmetric Jones type, Jones type, transposed Jones type, pseudo-Jones type, symmetric Hadamard type, or Hadamard type (H_ϵ), (HA_ϵ), (HB_ϵ), (HC_ϵ), then (X, w_+, w_-) with $X = X_1 \times X_2$, $w_+ = w_+^{(1)} \otimes w_+^{(2)}$, $w_- = w_-^{(1)} \otimes w_-^{(2)}$ is a two-weight spin model of the same type. Therefore, by this tensor product construction, we get many more examples of various spin models. Note that if we take two generalized spin models of different types, then their tensor product is generally not a two-weight spin model, but a four-weight spin model.

Anyway, it seems interesting to notice that in many instances, the existence of spin models is closely connected with the existence of interesting combinatorial objects such as Hadamard matrices, association schemes, etc,. (See [2] and [3] for general information on such combinatorial objects.)

We want to discuss further examples of (various kinds of) spin models and the link invariants attached to them in subsequent papers by looking at more combinatorial objects, and by considering (generalized) generalized spin models, we hope to be able to find missing mechanisms of systematically constructing spin models which Jones [6, p.325] wanted to discover.

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