Pacific Journal of Mathematics

FINE STRUCTURE OF THE MACKEY MACHINE FOR ACTIONS OF ABELIAN GROUPS WITH CONSTANT MACKEY OBSTRUCTION

SIEGFRIED ECHTERHOFF AND JONATHAN ROSENBERG

Volume 170 No. 1 September 1995

PACIFIC JOURNAL OF MATHEMATICS
Vol. 170 No. 1 1995

FINE STRUCTURE OF THE MACKEY MACHINE FOR ACTIONS OF ABELIAN GROUPS WITH CONSTANT MACKEY OBSTRUCTION

SIEGFRIED ECHTERHOFF AND JONATHAN ROSENBERG

Let G be a locally compact group, $\omega \in Z^2(G,\mathbb{T})$ a (measurable) multiplier on G, and denote by $C^*(G,\omega)$ the twisted group C^*-algebra of G defined by ω. We are only interested in multipliers up to equivalence, so we always tacitly assume that one is free to vary a multiplier within its cohomology class in $H^2(G,\mathbb{T})$. In this paper we are basically concerned with the following two problems: the first is to determine the structure of $C^*(G,\omega)$, where ω is a type I multiplier on the locally compact abelian group G, and the second is to describe the crossed product $A \rtimes_\alpha G$ of a continuous-trace C^*-algebra A by an action of an abelian group G, such that the corresponding action of G on $\widehat{A}$ has constant stabilizer N, and the Mackey obstruction to extending an irreducible representation ρ of A to a covariant representation (ρ, U) of (A, N, α_N) is equal to a constant multiplier $\omega \in H^2(N,\mathbb{T})$ for all $\rho \in \widehat{A}$. The second of these problems is the obvious starting point for the study of the "fine structure of the Mackey machine", for actions of abelian groups on continuous-trace algebras with "continuously varying" stabilizers and Mackey obstructions.

In case where all Mackey obstructions are trivial, i.e. if α is pointwise unitary on the stabilizer subgroup N, these systems have been investigated extensively in the literature (see for instance [**18**; **22**; **24**; **27**; **28**]), and there are also some results for the case of continuously varying stabilizers [**6**; **26**]. But in recent years there has been almost no progress in the investigation of crossed products with non-trivial Mackey obstructions. However, it turns out that many techniques for the investigation of crossed products by pointwise unitary actions can be used also for the case of non-trivial Mackey obstructions.

Let us explain our results in more detail. In Section 1 we start with investigation of the twisted group algebras $C^*(G,\omega)$ of an abelian group G and a multiplier $\omega \in Z^2(G,\mathbb{T})$. Recall that $C^*(G,\omega)$ can be realized in different ways. It is possible to write $C^*(G,\omega)$ as a completion of $L^1(G,\omega)$, the L^1-algebra of G with convolution and involution twisted by ω, or we can write $C^*(G,\omega)$ as the Green twisted product $\mathbb{C} \rtimes_{\tau_\omega} G(\omega)$, where $G(\omega)$

is the central extension of G by $\mathbb{T}$ defined by ω, and $\tau_\omega : \mathbb{T} \to \mathbb{T} = \mathcal{U}(\mathbb{C})$ is the identity (see [**14**] and [**20**, Appendix 1] for more details). A multiplier ω of G is said to be type I if $C^*(G,\omega)$ is a type I C^*-algebra. It is a classical result by Baggett and Kleppner [**1**] that if G is abelian, then ω is a type I multiplier if and only if the homomorphism $h_\omega : G \to \widehat{G}$ defined by $h_\omega(x)(y) = \omega(x,y)\omega(y,x)^{-1}$ has closed range and is open as a map onto its image. Now let $S_\omega := \ker h_\omega$ denote the **symmetry group** of ω. Then it is well known that $\mathrm{Prim}(C^*(G,\omega))$ is always homeomorphic to $\widehat{S}_\omega$. Our main result in Section 1 is now the fact that in the case where ω is type I, $C^*(G,\omega)$ is always Morita equivalent to $C_0(\widehat{S}_\omega)$. In fact, if G is second-countable and we exclude the case where S_ω is of finite index, $C^*(G,\omega)$ is isomorphic to $C_0(\widehat{S}_\omega) \otimes \mathcal{K}$. As a consequence we will also see that for any type I $[FC]^-$-group G the group C^*-algebra $C^*(G)$ is Morita equivalent to $C_0(\widehat{G})$, which is interesting since these groups are the only known locally compact groups with Hausdorff dual space, and are all such groups if G is connected or discrete [**2**].

In Section 2 we investigate crossed products $A \rtimes_\alpha G$ such that A has continuous trace, the abelian group G acts trivially on $\widehat{A}$, and all Mackey obstructions are similar to a constant multiplier $\omega \in Z^2(G,\mathbb{T})$. It was shown in [**15**] that for these systems $\mathrm{Prim}(A \rtimes_\alpha G)$ is always homeomorphic to $(A \rtimes_{\alpha_S} S)\widehat{\ }$, and we will see that in the case where ω is type I, $A \rtimes_\alpha G$ also has continuous trace. The results in Section 1 may lead to the guess that in this case $A \rtimes_\alpha G$ is Morita equivalent to $A \rtimes_{\alpha_S} S$, but we will see that this is not true in general, by computing explicitly the Dixmier-Douady class of $A \rtimes_\alpha G$ in terms of α and the Dixmier-Douady class of A, at least if G/S is compactly generated.

Crossed products with constant stabilizer N and constant Mackey obstruction ω are investigated in Section 3. If the resulting action of G/N on $\widehat{A}$ is proper, i.e. if the map $G/N \times \widehat{A} \to \widehat{A} \times \widehat{A} : (\dot{x},\pi) \mapsto (\pi \circ \alpha_{x^{-1}}, \pi)$ is proper in the sense that the inverse images of compact sets are compact, then, just as in the case where α_N is pointwise unitary [**18**], we will see that $\mathrm{Prim}(A \rtimes_\alpha G)$ is a proper $\widehat{S}$-space, where S denotes the symmetry group of ω. In fact this result is also true under the more general assumption that all Mackey obstructions of the system have a common symmetry group. If in addition $\widehat{A}$ is a principal G/N-bundle (which is automatic if G/N is a Lie group) and the action α_S of S on A is locally unitary (which is always the case if S is compactly generated), then we will see that $\mathrm{Prim}(A \rtimes_\alpha G)$ is a principal $\widehat{S}$-bundle over $\widehat{A}/G$, and we will also describe how to construct a representative for this bundle in $Z^1(\widehat{A}/G, \widehat{\mathcal{S}})$ (Čech cohomology), where $\widehat{\mathcal{S}}$ denotes the sheaf of germs of continuous $\widehat{S}$-valued functions on $\widehat{A}/G$.

Having done this, we go back to the investigation of crossed products with continuous trace. Extending a similar result of Green [13] to covariant systems (A, G, α), where A has continuous trace and the abelian group G acts freely on $\widehat{A}$, Olesen and Raeburn have shown [18, Theorem 3.1] that $A \rtimes_\alpha G$ has continuous trace if and only if G acts properly on $\widehat{A}$. As a first step we will generalize this result to arbitrary A by showing that if an abelian group G acts on A so that G acts freely on $\mathrm{Prim}(A)$, then the crossed product $A \rtimes_\alpha G$ has continuous trace if and only if A has continuous trace and G acts properly on $\widehat{A}$. Using this and the Packer-Raeburn stabilization trick [19] (which will also be used extensively in Section 2), we will then see that if A has continuous trace, a crossed product $A \rtimes_\alpha G$ with constant stabilizer N and constant Mackey obstruction ω has continuous trace if and only if ω is type I and G/N acts properly on $(A \rtimes_{\alpha_N} N)\widehat{\ }$. The last statement is always true if the action of G/N on $\widehat{A}$ is proper, but it turns out that the converse is not true in general. In fact we will construct a covariant system (A, G, α) with constant stabilizer N such that α_N is pointwise unitary, A and $A \rtimes_\alpha G$ have continuous trace, but the action of G on $\widehat{A}$ is even not smooth. This example shows that Williams's description of transformation group algebras with continuous trace [33] cannot be extended to actions of abelian groups on continuous-trace algebras, even if all Mackey obstructions vanish. Finally, we will discover that it is also possible to construct a covariant system (A, G, α) with G abelian, such that A and $A \rtimes_\alpha G$ have continuous trace, but where the stability groups do not vary continuously, thus showing also that the other necessary condition for transformation groups having continuous-trace transformation group algebra does not apply for actions on continuous-trace algebras.

For any C^*-algebra A we will as usual denote by $\mathcal{M}(A)$ the multiplier algebra of A and by $\mathcal{U}(A)$ the group of unitaries in $\mathcal{M}(A)$. If X is a locally closed subset of $\widehat{A}$ (i.e. X is open in its closure), then there exist two ideals I_X and J_X of A such that $X \cong \widehat{I_X/J_X}$. For simplicity we will denote the quotient I_X/J_X always by $A|_X$. If α is an action of the locally group G on A and X is G-invariant, then I_X and J_X are G-invariant, too. Thus α defines canonically an action of G on $A|_X$, which will also be denoted by α. Finally, if $\mathcal{H}$ is a Hilbert space, then we denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators on $\mathcal{H}$.

This work was begun while the first author was visiting the Department of Mathematics of the University of Maryland at College Park. He wants to take this opportunity to thank the members of the Department for their warm hospitality and the stimulating working atmosphere.

1. Decompositions of multipliers and the structure of $C^*(G,\omega)$.

We start our investigations with a study of decompositions of multipliers on G. To fix notations: if ω is a multiplier on G and H a closed subgroup of G, then we denote by ω_H the restriction of ω to H. Furthermore, if $G = G_1 \times G_2$, and if ω_1 and ω_2 are multipliers of G_1 and G_2, respectively, then we denote by $\omega_1 \otimes \omega_2$ the multiplier on G defined by

$$\omega_1 \otimes \omega_2((x_1,x_2),(y_1,y_2)) = \omega_1(x_1,y_1)\omega_2(x_2,y_2),$$

$x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$. Let us also recall that a multiplier ω on the abelian group G is called **totally skew** if (and only if) the symmetry group S_ω is trivial. If ω is totally skew, then it is type I if and only if h_ω maps G isomorphically onto $\widehat{G}$ [**1**].

Lemma 1. *Let ω be a multiplier on a locally compact abelian group G. Suppose G has a closed subgroup H such that ω_H is totally skew and type I, and such that the group extension*

$$1 \to H \to G \to G/H \to 1$$

splits. Then there is a complement L to H in G (i.e., a subgroup with $G = H \times L$) such that (after perhaps replacing ω by a similar multiplier) ω splits as $\omega = \omega_H \otimes \omega_L$.

Proof. Choose any splitting $\varphi : G/H \to G$ for the group extension. Let us denote by $h_\omega : G \to \widehat{G}$ the canonical homomorphism defined by $h_\omega(x)(y) = \omega(x,y)\omega(y,x)^{-1}$. Since ω_H is totally skew and type I by assumption, it follows from [**1**] that $h_{\omega_H} : H \to \widehat{H}$ is a homeomorphism. Thus we can define $\psi : G/H \to G$ by

$$\psi(x) = \varphi(x)h_{\omega_H}^{-1}[(h_\omega(\varphi(x))|_H)^{-1}].$$

This is again a splitting for the group extension, and we let L denote its image, so that $G = H \times L$. Furthermore, for each $h \in H$ and $x \in G/H$, we have

$$\begin{aligned} h_\omega(\psi(x))(h) &= h_\omega(\varphi(x))(h)h_\omega \circ h_{\omega_H}^{-1}[(h_\omega(\varphi(x))|_H)^{-1}](h) \\ &= h_\omega(\varphi(x))(h)[h_\omega(\varphi(x))(h)]^{-1} = 1. \end{aligned}$$

This implies that there exists a similar multiplier ω' such that $\omega'(x,h) = \omega'(h,x) = 1$ for each $x \in L$ and $h \in H$. It follows easily from the cocycle identities that ω' has the decomposition given in the lemma. $\square$

The next lemma shows that the answer to the question of whether a totally skew multiplier ω on a compactly generated abelian Lie group G is type I only depends on the structure of G. Recall first the structure theory of compactly generated abelian Lie groups: any such group can be written in the form $G = V \times Z \times T \times F$, where V is a vector group, Z is finitely generated free abelian, T is a torus, and F is finite abelian. The dimensions of V and of T, the rank of Z, and the isomorphism class of F are a complete set of isomorphism invariants for G.

Lemma 2. *Let ω be a totally skew multiplier on a compactly generated abelian Lie group G. Let n be the dimension of the maximal torus T of G, and let l be the rank of a maximal free abelian quotient of G (this may be computed as $\dim_{\mathbb{Q}}(G/G_0) \otimes_{\mathbb{Z}} \mathbb{Q}$, where G_0 is the connected component of the identity in G). Then $l \geq n$, and $l = n$ if and only if ω is of type I. Furthermore, if ω is type I and V is a maximal vector subgroup of G, then ω_V is also totally skew and type I.*

Proof. The assumption that ω is totally skew means that h_ω is an injection with dense range, from the abelian Lie group G to the abelian group $\widehat{G}$. Let G_0 be the connected component of the identity in G, let K be the maximal compact subgroup of G, and let V be a maximal vector subgroup of G, so that $G_0 = T \times V$. The connected component of the identity in $\widehat{G}$ is $K^\perp$. Since G has dimension $n + \dim V$ and $\widehat{G}$ has dimension $l + \dim V$, the fact that h_ω is an injection with dense range forces $l \geq n$, with equality if and only if h_ω induces an isomorphism of Lie algebras. If this isn't the case, h_ω is clearly not an isomorphism, while if it is, h_ω has closed range and thus ω is type I. Furthermore, in this case, h_ω gives a non-degenerate pairing of V with itself, since if $h_\omega(v)$ annihilates V for some $v \in V$, then since $h_\omega(v)$ lies in the connected component of the identity in $\widehat{G}$, it also annihilates K and thus lies in $(V \cdot K)^\perp$, which is compact, hence (since h_ω is an isomorphism) $h_\omega(v) = 1$. Thus ω_V is also skew and type I. □

The following lemma is crucial for the proofs of several of our main results. It shows that for any type I totally skew multiplier ω on a compactly generated abelian group G there exists a decomposition $G = H \times L$ such that ω_H and ω_L are trivial.

Lemma 3. *Let ω be a totally skew type I multiplier on the compactly generated locally compact abelian group G. Then there exists a splitting $G = V_1 \times V_2 \times Z \times T \times F_1 \times F_2$ such that V_1 and V_2 are isomorphic vector groups, Z is a free abelian group of rank n, T is a torus of the same dimension n, and F_1 and F_2 are isomorphic finite abelian groups. We can choose the splitting such that*

(1) *if* $H = V_1 \times T \times F_1$ *and* $L = V_2 \times Z \times F_2$, *then* ω_H *and* ω_L *are trivial;*
(2) *if* $V = V_1 \times V_2, M = Z \times T$, *and* $F = F_1 \times F_2$, *then* ω *is similar to* $\omega_V \otimes \omega_M \otimes \omega_F$ *and all these multipliers are totally skew and type I.*

Proof. Since ω is totally skew and type I, it follows that h_ω is an isomorphism between G and $\widehat{G}$. Thus $\widehat{G}$ is also compactly generated, which means G is a Lie group. Let V be a maximal vector subgroup of G. By Lemma 2, ω_V is also totally skew and type I, and furthermore, by Lemma 1, after replacing ω by a similar multiplier, we may write $G = V \times G'$ and $\omega = \omega_V \otimes \omega_{G'}$. (A special case of this was pointed out in [**1**, p. 316].) Since ω_V is totally skew, we may assume that ω_V is a "Heisenberg multiplier" of the form e^{iJ}, where J is a symplectic form on the vector space V. By existence of polarizations for symplectic vector spaces, there exists a splitting $V = V_1 \times V_2$ such that ω_{V_1} and ω_{V_2} are trivial.

Thus for rest of the proof, we may and do replace G by G' and assume that the maximal torus T of G is also the connected component of the identity in G. By Lemma 2, the dimension n of T coincides with the rank of a maximal free abelian subgroup of G/T. Let K be the maximal compact subgroup of G, so that $K \cong T \times F$ for some finite abelian group F. Note that h_ω must map T isomorphically onto the maximal torus in $\widehat{G}$, which is $K^\perp$, and must map K isomorphically onto the maximal compact subgroup of $\widehat{G}$, which is $T^\perp$. Let Z be any complement to K in G, so $Z \cong \mathbb{Z}^n$. Let $F = K \cap h_\omega^{-1}(Z^\perp)$, so $h_\omega(F) = T^\perp \cap Z^\perp$. Then F is a complement to T in K, and for $x \in F$, $h_\omega(x)$ must be non-trivial on F since it annihilates T and Z. Thus ω_F is totally skew (and of course type I). By Lemma 1, we may assume $\omega = \omega_F \otimes \omega_{TZ}$. It is thus enough to prove the lemma for two remaining cases: when G is a product of a torus and a free abelian group, and when G is finite.

First consider the case where $G \cong \mathbb{T}^n \times \mathbb{Z}^n$. We prove the existence of the splitting by induction on the dimension n of the maximal torus T. If $n = 0$, there is nothing to prove. So suppose $n > 0$. Choose an infinite cyclic discrete subgroup Z_1 of G whose image in $G/T \cong \mathbb{Z}^n$ is a direct summand. Let $G_1 = h_\omega^{-1}(Z_1^\perp)$ and $\omega_1 = \omega_{G_1}$. Then $G = G_1 \times T_1$ where T_1 is a one-dimensional torus. By [**1**, Lemma 3.3] we know that the symmetry group S_{ω_1} of ω_1 is equal to Z_1 and we can write $G_1 = Z_1 \times G_2$, where $G_2 \cong \mathbb{T}^{n-1} \times \mathbb{Z}^{n-1}$. It is clear that $\omega_2 = \omega_{G_2}$ is totally skew and it is one consequence of Lemma 2 that ω_2 is also type I. Hence by induction there exists a splitting $G_2 = Z' \times T'$ with the properties as in the Lemma. If we now define $Z = Z' \times Z_1$ then the splitting $G = Z \times T$ also has the desired properties.

Finally let us consider the case of finite groups. This is certainly well-known, but for completeness we prove it anyway. So let ω be a totally skew multiplier on the finite abelian group F, and let F' be a cyclic subgroup of maximal order in F. Then we can find a splitting $F = F' \times F''$, and since $\omega_{F'}$ is trivial (since F' is cyclic) it follows that $h_\omega^{F'}$ induces a surjective homomorphism from F'' onto $\widehat{F'}$. This implies that we can split $F'' = F_1 \times F_2$ such that F_2 is the kernel of this homomorphism. It follows now easily for $N = F_1 \times F'$ that ω_N is totally skew and we can assume by Lemma 1 that $\omega = \omega_N \otimes \omega_{F_2}$. By induction on the order of F we get a splitting for F_2 as in the lemma and we are done. □

We are now ready to prove the main result of this section.

Theorem 1. *Let G be a locally compact abelian group, ω a type* I *multiplier on G with symmetry group S. Assume G/S is compactly generated. Then $C^*(G,\omega)$ is strongly Morita equivalent to $C^*(S) \cong C_0(\widehat{S})$. In fact, unless G/S is finite, $C^*(G,\omega)$ is isomorphic (non-canonically) to $C_0(\widehat{S}) \otimes \mathcal{K}$.*

Proof. Since G/S carries a totally skew type I multiplier, it is self-dual, hence is a compactly generated abelian Lie group. First, let's simplify the S. The group G/S must be of the form $\mathbb{R}^{2n} \times \mathbb{T}^m \times \mathbb{Z}^m \times F$, for some finite group F. Thus there exists a group $G' \cong \mathbb{R}^{2n+m} \times \mathbb{Z}^k$ (for some $k \geq m$) such that G/S is isomorphic to G'/D for some finitely generated torsion-free abelian and discrete subgroup D of G'. Now if $q : G \to G/S$ and $q' : G' \to G'/D \cong G/S$ denote the quotient maps, we define

$$\widetilde{G} = \{(x,y) \in G \times G' : q(x) = q'(y)\}.$$

Then $\widetilde{G}$ is a locally compact abelian group with $\widetilde{G}/D \cong G$, where we identify D with its canonical image in $\widetilde{G}$. Thus we may lift the cocycle ω to a cocycle $\widetilde{\omega}$ of $\widetilde{G}$ with symmetry group $\widetilde{S} = S \times D$. However, G' may be identified with the quotient of $\widetilde{G}$ by the canonical image of S in $\widetilde{G}$, via the projection on the second factor. Since every quotient of the form $\mathbb{R}^{2n+m} \times \mathbb{Z}^k$ of an abelian locally compact group splits, we may write

$$\widetilde{G} = S \times (\mathbb{R}^{2n+m} \times \mathbb{Z}^k), \; S \subseteq \widetilde{S}.$$

Note that, since $D \subseteq \widetilde{S}$, $C^*(G,\omega)$ is a quotient algebra of $C^*(\widetilde{G},\widetilde{\omega})$. So if the latter is Morita equivalent to an abelian algebra, or a tensor product of an abelian algebra with $\mathcal{K}$, so is $C^*(G,\omega)$. By [1], $C^*(G,\omega)$ has Hausdorff spectrum $\widehat{S}$, so this will prove the theorem. But since S lies in the symmetry group for $\widetilde{\omega}$ and also splits as a direct factor in $\widetilde{G}$,

$$\begin{aligned} C^*(\widetilde{G},\widetilde{\omega}) &\cong C^*(S) \otimes C^*(\mathbb{R}^{2n+m} \times \mathbb{Z}^k, \text{ res of } \widetilde{\omega}) \\ &\cong C_0(\widehat{S}) \otimes C^*(\mathbb{R}^{2n+m} \times \mathbb{Z}^k, \text{ res of } \widetilde{\omega}). \end{aligned}$$

The abelian factor is clearly harmless, so this shows that after replacing G by $\mathbb{R}^{2n+m} \times \mathbb{Z}^k$ and S by D, we may assume that G is a torsion-free compactly generated abelian Lie group, and S is a free abelian group sitting discretely in G. We assume this hereafter.

In making the above reduction, we haven't changed the fact that G/S is a self-dual compactly generated abelian Lie group, and we may suppose ω is lifted from a totally skew bicharacter on G/S. By Lemma 3, there is an open subgroup G_1 of G, of finite index, such that G_1/S is isomorphic to $\mathbb{R}^{2n} \times \mathbb{T}^m \times \mathbb{Z}^m$ and such that the restriction ω_1 of ω to G_1 still has symmetry group S. (Fix any splitting of G/S as $G' \times F$ with F finite, $G' \cong \mathbb{R}^{2n} \times \mathbb{T}^m \times \mathbb{Z}^m$, and such that after replacing ω by a similar multiplier, ω splits as $\omega_{G'} \otimes \omega_F$. Then let G_1 be the inverse image of G' in G.) G_1 is an open subgroup of G of finite index, so that $C^*(G_1, \omega_1)$ is a hereditary subalgebra of $C^*(G, \omega)$ of finite index, and they have homeomorphic spectra (since ω and ω_1 have the same symmetry group S). In fact, any irreducible representation of $C^*(G, \omega)$ restricts to a multiple of an irreducible representation of $C^*(G_1, \omega_1)$, since in both cases there is one and only one irreducible representation inducing any given character of S. We will first show that $C^*(G_1, \omega_1)$ is strongly Morita equivalent to an abelian algebra, then deduce the same fact for $C^*(G, \omega)$.

By Lemma 3, we may choose a decomposition $G_1/S = H' \times L'$ such that the image of ω_1 is trivial when restricted to either H' or L' and such that $H' \cong \mathbb{T}^m \times \mathbb{R}^n$, $L' \cong \mathbb{Z}^m \times \mathbb{R}^n$. Let H be the inverse image of $H' \subseteq G_1/S$ in G_1. Since any locally compact abelian extension of $\mathbb{Z}^m \times \mathbb{R}^n$ by some other locally compact abelian group must split, there is a subgroup L of G_1 such that L maps isomorphically onto L' in G_1/S and such that ω_1 is trivial when restricted to either H or L.

Consider the central extension $G_1(\omega_1)$ of G_1 by $\mathbb{T}$ associated to ω_1. Since ω_1 is trivial on both H and L, the normal subgroup $H(\omega_1|_H)$ is just $\mathbb{T} \times H$, and we have a nilpotent extension

$$1 \to (\mathbb{T} \times H) \to G_1(\omega_1) \to L \to 1$$

which must split **both algebraically and topologically**, so

$$G_1(\omega_1) = (\mathbb{T} \times H) \rtimes L$$

(with the copy of $\mathbb{T}$ central and the whole group nilpotent). By definition, $C^*(G_1, \omega_1)$ is the quotient of $C^*(G_1(\omega_1))$ corresponding to the identity character of $\mathbb{T}$. So $C^*(G_1, \omega_1)$ is a C^*-crossed product $C^*(H) \rtimes L$ and thus isomorphic to a transformation group C^*-algebra

$$C_0(\widehat{H}) \rtimes L.$$

Since L has trivial intersection with S, the action here of L on $\widehat{H}$ is free, with Hausdorff quotient isomorphic to $\widehat{S}$. By Green's Theorem [**13**], it follows that $C^*(G_1, \omega_1)$ is strongly Morita equivalent to $C_0(\widehat{S})$.

It remains to show that $C^*(G, \omega)$ is also strongly Morita equivalent to $C_0(\widehat{S})$, and in fact isomorphic to $C_0(\widehat{S}) \otimes \mathcal{K}$ unless G/S is finite. Since $C^*(G, \omega)$ is already known to have continuous trace ([**16**], but see the corrected proof in [**9**, Lemma 6]) with spectrum $\widehat{S}$, which is a torus, we only have to show that its Dixmier-Douady class vanishes. (For the result on isomorphism, we are using [**4**, Théorème 14], which says that an $\aleph_0$-homogeneous continuous-trace algebra over a finite-dimensional compact metrizable space is necessarily locally trivial.) We will use the fact that the integral cohomology and K-theory of a torus are torsion-free, together with the transfer on K_0. Let $\iota : C^*(G_1, \omega_1) \hookrightarrow C^*(G, \omega)$ be the inclusion. It induces a map

$$\iota_* : K_0(C^*(G_1, \omega_1)) \to K_0(C^*(G, \omega)),$$

and since $C^*(G, \omega)$ is a finitely generated free (left) $C^*(G_1, \omega_1)$-module, there is also a **transfer map**

$$\iota^* : K_0(C^*(G, \omega)) \to K_0(C^*(G_1, \omega_1)),$$

such that $\iota^* \circ \iota_*$ is multiplication by $[G : G_1]$. Since the Dixmier-Douady class vanishes for $C^*(G_1, \omega_1), K_0(C^*(G_1, \omega_1)) \cong K^0(\widehat{S})$, which is free abelian of rank $2^{\operatorname{rank} S - 1}$. Hence ι_* is injective and $K_0(C^*(G, \omega))$ has rank $\geq 2^{\operatorname{rank} S - 1}$. But by [**29**, Theorem 6.5], if the Dixmier-Douady class of $C^*(G, \omega)$ were non-zero, the d_2 differential in the Atiyah-Hirzebruch spectral sequence for its K-theory would be non-zero, and so its K-theory would have rank $< 2^{\operatorname{rank} S - 1}$ (the rank of the E_2 term), a contradiction. This completes the proof. □

Remarks. If G/S is finite, say of order n^2, then $C^*(G, \omega)$ is homogeneous C^*-algebra *locally* isomorphic to $C_0(\widehat{S}) \otimes M_n(\mathbb{C})$, but it is known that $C^*(G, \omega)$ need not be isomorphic to $C_0(\widehat{S}) \otimes M_n(\mathbb{C})$ in general, even if G is free abelian (the case of a "non-commutative torus" or "rational rotation algebra" – see [**3**] for further information about this case). So our result is best possible.

An important special case of our results is thus that any type I non-commutative torus is strongly Morita-equivalent to the continuous functions on an ordinary torus. This fact was known before (see for instance [**3**] for the two-dimensional case), but the argument above is new. If one is willing to assume the result in this case, then one can avoid the K-theoretic part of the above argument, as follows. By the reductions which have been made before, we may split both G **and** the multiplier ω to get $G = H \times Z$ and

$\omega = \omega_H \otimes \omega_Z$, where $Z \cong \mathbb{Z}^n$, and where the quotient of Z by the symmetry group of ω_Z is just the finite part of G/S_ω. Then the result follows for G, if it follows for H and Z. Now the arguments used for G_1 in the original proof give the result for H, while the result for Z follows by the fact that non-commutative rational tori are Morita equivalent to $C(\mathbb{T}^n)$.

Finally, one can give an independent proof (not using K-theory) of the fact that non-commutative rational tori are Morita equivalent to $C(\mathbb{T}^n)$ via our Theorem 3 below. See §2 below for further details.

Recall that a locally compact group G is called an $[FC]^-$-group if for all $x \in G$ the conjugacy class of x is relatively compact. It is well known [**16**;**9**] that the group C^*-algebra $C^*(G)$ of an $[FC]^-$-group has continuous trace if and only if $C^*(G)$ is type I, i.e. G is a type I group. It is also known [**2**] that any connected or discrete group with continuous-trace group C^*-algebra must be an $[FC]^-$-group. We will see that the group C^*-algebras of these groups are always Morita equivalent to commutative C^*-algebras. At the same time we will show that the conclusion of Theorem 1 also holds for arbitrary locally compact abelian groups.

Theorem 2. *Let G be any type I $[FC]^-$-group. Then $C^*(G)$ is strongly Morita equivalent to $C_0(\widehat{G})$. Similarly, if ω is a type I multiplier on a locally compact abelian group H, then $C^*(H,\omega)$ is strongly Morita equivalent to $C_0(\widehat{S})$, where S is the symmetry group for ω. Furthermore, if H is second-countable and H/S is not finite, then $C^*(H,\omega) \cong C_0(\widehat{S}) \otimes \mathcal{K}$.*

Proof. First of all, if G is a type I $[FC]^-$-group, $C^*(G)$ has continuous trace, and in particular has Hausdorff spectrum, by [**9**, Lemma 6]. Furthermore, we claim G has a compact normal subgroup K such that G/K is locally compact abelian. The argument for this is sketched in [**16**]: indeed, by [**17**, Theorem 2.2], there exists a compact normal subgroup K' of G such that $G/K' \cong V \times D$ for a vector group V and a discrete $[FC]$ group D. Since G is type I, the same is true for D, hence by Thoma's Theorem we know that D contains an abelian subgroup of finite index, thus (since D is also $[FC]$) it follows that the commutator subgroup of D is finite. Taking the pull-back of this finite subgroup in G we obtain a compact subgroup K of G such that G/K is abelian. (For the central extension $G = H(\omega)$ of a locally compact abelian H by $\mathbb{T}$ associated to a multiplier, it is already clear that one has such a structure.) Without loss of generality, we can enlarge K if necessary and assume that G/K is an abelian Lie group, though possibly with a very large group of components. Since $\widehat{K}$ is discrete, the orbits of G on $\widehat{K}$ are both open and closed, and it is easy to see that $C^*(G)$ decomposes as a C^* (c_0) direct sum of subalgebras associated to the various orbits. By the Mackey machine (e.g., [**14**, §§4-5]), these subalgebras are strongly Morita equivalent to the $C^*(G_\rho/K,\ \omega^{-1})$, where we choose one $\rho \in \widehat{K}$ from each orbit, G_ρ is the stabilizer of ρ in G, and ω is the associated Mackey obstruction. Furthermore, if G is type I, this means all these multipliers ω are type I. Let us note also that the central extension $G = H(\omega)$ of a locally compact abelian H by $\mathbb{T}$ associated to a type I multiplier ω is not necessarily a type I group (see counterexample below), but at least the direct summand of $C^*(G)$ corresponding to the identity character of $\mathbb{T}$ is type I, and the argument to be given below will apply locally to this portion of the algebra. It is therefore not necessary to give a separate argument for this case.

We may therefore assume we are looking at a type I multiplier ω on G_ρ, where G_ρ is the stabilizer in G of some $\rho \in \widehat{K}$, and where G/K is an abelian Lie group. So G_ρ/K is also an abelian Lie group. The quotient of this group by the symmetry group S for ω is now an abelian Lie group admitting a type I totally skew multiplier, so it is compactly generated. The statement of the theorem now follows from Theorem 1. (The fact that $C^*(H,\omega)$ is stable when H is second-countable and S is not of finite index again follows from [**4**, Théorème 14].) $\square$

Remark. It might be of interest to see an example of a type I multiplier on a locally compact abelian group H such that the corresponding central extension G of H by $\mathbb{T}$ is not type I. Since the portion of $C^*(G)$ living over the nth power of the identity character of $\mathbb{T}$ can be identified with $C^*(G,\omega^n)$, it's enough to give an example where ω is type I but ω^n is not. Now, to see whether it is or isn't, one can assume ω is totally skew, so h_ω is an isomorphism $H \to \widehat{H}$. But $h_{\omega^n} = h_\omega \circ \beta_n$, where $\beta_n : x \mapsto x^n$, by an easy calculation. So for h_{ω^n} to have closed range (which is the condition for type I-ness, at least in the separable case), it's necessary and sufficient for β_n to have closed range. There are groups where it doesn't; for example, take $n > 1$ and let H' be the restricted direct product of countably many copies of $\mathbb{Z}/(n^2)$ with respect to the open subgroups of order $n, n\mathbb{Z}/(n^2) \cong \mathbb{Z}/(n)$. For this group, β_n kills the compact subgroup $\prod_{i=1}^{\infty} \mathbb{Z}/(n)$ and so the range of β_n can be seen to be a countable dense subgroup of this compact subgroup. This group H' doesn't have a type I totally skew multiplier, but $H = H' \times \widehat{H}'$ does, and the nth power of its canonical totally skew multiplier isn't type I.

We take this opportunity to point out a correction: in §2 of [**10**], it was asserted that if ω is type I, so is ω^n for any n. This is true when H/S is an abelian Lie group, which was the case needed there, though as we have seen it is false in general.

2. The structure of crossed products in the case of trivial action on the spectrum.

We are now going to investigate the structure of crossed products $A \rtimes_\alpha G$ of separable covariant systems (A, G, α), where A has continuous trace, G is abelian, the action of G on $\widehat{A}$ is trivial, and the Mackey obstruction at each point $\rho \in \widehat{A}$ is similar to a constant type I multiplier $\omega \in Z^2(G, \mathbb{T})$. These systems play a crucial role in the investigation of the Mackey machine for certain group extensions (see for instance the discussions in [**31**] and [**18**, Section 3]). If S denotes the symmetry group of ω, then it is well known that $(A \rtimes_\alpha G)\widehat{\ }$ is homeomorphic to $(A \rtimes_{\alpha_S} S)\widehat{\ }$[**15**], which is a proper $\widehat{S}$-space [**18**], and we will see that $A \rtimes_\alpha G$ is always a continuous-trace algebra (here α_S denotes the restriction of α to S). The preceding results about $C^*(G, \omega)$ may lead to the guess that $A \rtimes_\alpha G$ is always Morita equivalent to $A \rtimes_{\alpha_S} S$, but by computing explicitly the Dixmier-Douady class of $A \rtimes_\alpha G$ in terms of the Dixmier-Douady class $\delta(A)$ of A and the action of G, we will see that this is not true in general. However, it **is** true for the very simplest such systems, the projective unitary actions.

Let (A, G, α) be a separable covariant system such that A is of type I and G acts trivially on $\widehat{A}$. If ω is the Mackey obstruction for the action α at $\rho \in \widehat{A}$, then there is an ω-representation $U : G \to \mathcal{U}(\mathcal{H}_\rho)$ such that U implements α at ρ, which means that $\rho(\alpha_x(a)) = U_x \rho(a) U_x^*$ for all $a \in A$ and $x \in G$. The action α of G on A is called **pointwise unitary** if U can be chosen as a homomorphism, or equivalently if all Mackey obstructions are trivial. We say α is **unitary** if there exists a strictly continuous homomorphism $u : G \to \mathcal{U}(A)$, the group of unitaries in the multiplier algebra $\mathcal{M}(A)$ of A, such that α is implemented by u, which means that $\alpha_x(a) = u_x a u_x^*$ for all $x \in G$ and $a \in A$. Similarly, α is called **locally unitary** if, for each $\rho \in \widehat{A}$, there exists an open neighborhood U of $\rho \in \widehat{A}$ and a strictly continuous homomorphism $u : G \to \mathcal{U}(A|_U)$ such that u implements α in a neighborhood of ρ, which simply means that $\pi(\alpha_x(a)) = \pi(u_x a u_x^*)$ for all $\pi \in U$, $x \in G$ and $a \in A$. This condition is equivalent to saying that for each $\rho \in \widehat{A}$ there exists a closed two-sided ideal $I \subseteq A$ such that $I \not\subseteq \ker \rho$ and the restriction of α to I is unitary. Locally unitary actions have been studied extensively in the literature [**22;24**] and there are also some remarkable results concerning pointwise unitary actions [**18**]. If (A, G, α) is separable and A has continuous trace, then it is known for large class of groups G that pointwise unitary actions α of G on A are automatically locally unitary. This is for instance true for all compactly generated abelian and compact groups, but also for many other groups (see [**30**, Corollary 2.2]).

In the sequel it will be necessary to decompose a given crossed product $A \rtimes_\alpha G$ with respect to a normal subgroup N of G, at least modulo Morita equivalence. Hence, before we start with the investigation of crossed products with constant Mackey obstructions let us recall basic ingredients of the stabilization trick of Packer and Raeburn [**19**]. (For another approach, see also [**8**].) For this let us start with a separable covariant system (A, G, α) and suppose that N is a closed normal subgroup of G. Then there is a canonical action γ of G on $A \rtimes_{\alpha_N} N$ which is given on the dense subalgebra $C_c(N, A)$ by the formula

$$(\gamma_x(f))(n) = \Delta_{G,N}(x)\alpha_x(f(x^{-1}nx))$$

$x \in G,\ n \in N$, where $\Delta_{G,N} : G \to \mathbb{R}^+$ is defined by the formula

$$\int_N g(n)\, dn = \Delta_{G,N}(x) \int_N g(x^{-1}nx)\, dn$$

for all $g \in C_c(N)$. Now suppose that $c : G/N \to G$ is a Borel cross-section with $c(\dot{e}) = e$. If we define $u : G/N \times G/N \to \mathcal{U}(A \rtimes_{\alpha_N} N)$ by

$$u(\dot{x}, \dot{y}) = i_N(c(\dot{x})c(\dot{y})c(\dot{x}\dot{y})^{-1}),$$

where i_N denotes the canonical embedding of N into $\mathcal{U}(A \rtimes_{\alpha_N} N)$, then $(\gamma \circ c, u)$ is a twisted action (in the sense of Packer and Raeburn) of G/N on $B := A \rtimes_{\alpha_N} N$ such that $A \rtimes_\alpha G \cong B \rtimes_{\gamma\circ c, u} G/N$ [**19**, Theorem 4.2]. Furthermore, if (π, U) is a covariant representation of (A, G, α), then the corresponding representation of the twisted covariant system $(B, G/N, \gamma \circ c, u)$ is given by $(\pi \times U|_N,\ U \circ c)$.

For the next step let us denote $\mathcal{K}(L^2(G/N))$ simply by $\mathcal{K}$. It was shown in [**19**, Theorem 3.4] that there exists an ordinary action β of G/N on $B \otimes \mathcal{K}$ such that β is exterior equivalent to the twisted action $((\gamma \circ c) \otimes \mathrm{id}, u \otimes 1)$. This means that there exists a Borel map $v : G/N \to \mathcal{U}(B \otimes \mathcal{K})$ such that

(1) $\beta_{\dot{x}} = \mathrm{Ad}\, v_{\dot{x}} \circ (\gamma_{c(\dot{x})} \otimes \mathrm{id})$, and
(2) $v_{\dot{x}\dot{y}} = v_{\dot{x}}(\gamma_{c(\dot{x})} \otimes \mathrm{id})(v_{\dot{y}})u(\dot{x}, \dot{y})$,

for all $\dot{x}, \dot{y} \in G/N$. As a consequence [**19**, Lemma 3.3] we know that $(B\otimes\mathcal{K}) \rtimes_\beta G/N$ is isomorphic to $(B\otimes\mathcal{K}) \rtimes_{(\gamma\circ c\otimes\mathrm{id}, u\otimes 1)} G/N$, which in fact is isomorphic to $(A\otimes\mathcal{K}) \rtimes_{\alpha\otimes\mathrm{id}} G$. The covariant representation (π, U) of (A, G, α) now corresponds first to the representation $(\pi\otimes\mathrm{id}, U\otimes 1)$ of $(A\otimes\mathrm{id}, G, \alpha\otimes\mathrm{id})$, which by the isomorphism above corresponds to the covariant representation $(\pi \times U|_N \otimes \mathrm{id}, V)$ of $(B \otimes \mathcal{K}, G/N, \beta)$, where

$$V_{\dot{x}} = \pi \times U|_N \otimes \mathrm{id}(v_{\dot{x}})U_{c(\dot{x})} \otimes 1.$$

If A is stable, we may identify $A \otimes \mathcal{K}$ with $A, \alpha \otimes \mathrm{id}$ with α (at least modulo exterior equivalence by [**24**, Lemma 1.14]) and $(\pi\otimes\mathrm{id}, U\otimes 1)$ with (π, U). In the same way we may suppose that β is an action of G/N on B rather than on $B \otimes \mathcal{K}$. Hence in this case we have an isomorphism between $A \rtimes_\alpha G$ and $B \rtimes_\beta G/N$, and if (π, U) is a covariant representation of (A, G, α), then the corresponding representation of $(B, G/N, \beta)$ is given by $(\pi \times U|_N, V)$ such that

$$V_{\dot{x}} = \pi \times U|_N(v_{\dot{x}})U_{c(\dot{x})},$$

where $v : G/N \to \mathcal{U}(B)$ implements the exterior equivalence between $(\gamma \circ c, u)$ and β. We need the following Proposition:

Proposition 1. *Let (A, G, α) be a separable covariant system such that A is stable. Furthermore, let N be a closed normal subgroup of G and let β be the action of G/N on $B := A \rtimes_{\alpha_N} N$ as constructed as above. Then the following is true:*

(1) If H is any closed subgroup of G containing N, then $A \rtimes_{\alpha_H} H \cong B \rtimes_{\beta_{H/N}} H/N$, and the restriction of any representation (π, U) of (A, H, α_H) to $B = A \rtimes_{\alpha_N} N$ is equal to the restriction of the corresponding representation of $(B, H/N, \beta_{H/N})$ to B.

(2) If $\pi \times V \in \widehat{B}$, and if γ denotes the canonical action of G on B, then $(\pi \times V) \circ \gamma_x$ is equivalent to $(\pi \times V) \circ \beta_{\dot{x}}$ for all $x \in G$.

Proof. Part (1) follows immediately from the construction above and the fact that $\beta_{H/N}$ is just the action of H on B which comes from the same procedure, if we start with the restriction of the Borel cross-section c to H/N.

In order to prove (2) let $v : G/N \to \mathcal{U}(B)$ be the map which implements the exterior equivalence between $(\gamma \circ c, u)$ and β. Then $\beta_{\dot{x}} = \operatorname{Ad} v_{\dot{x}} \circ \gamma_{c(\dot{x})}$. Hence, if $\pi \times V \in \widehat{B}$ and $b \in B$ we get:

$$\begin{aligned}(\pi \times V) \circ \beta_{\dot{x}}(b) &= (\pi \times V) \circ \operatorname{Ad} v_{\dot{x}} \circ \gamma_{c(\dot{x})}(b) \\ &= (\pi \times V)(v_{\dot{x}}\gamma_{c(\dot{x})}(b)v_{\dot{x}}^*) \\ &= (\pi \times V)(v_{\dot{x}})(\pi \times V)(\gamma_{c(\dot{x})}(b))(\pi \times V)(v_{\dot{x}}^*),\end{aligned}$$

which proves the claim, since $(\pi \times V) \circ \gamma_{c(\dot{x})}$ is always equivalent to $(\pi \times V) \circ \gamma_x$. □

Proposition 2. *Let (A, G, α) be a separable covariant system such that G is abelian, A is stable and of type I, G acts trivially on $\widehat{A}$ and all Mackey obstructions of the system are similar to a constant multiplier $\omega \in Z^2(G, \mathbb{T})$. Moreover, let S denote the symmetry group of ω and suppose that ω is lifted from $\tilde{\omega} \in Z^2(G/S, \mathbb{T})$, that is, $\tilde{\omega}(\dot{x}, \dot{y}) = \omega(x, y)$ for all $x, y \in G$. Then, if β is the action of G/S on $B := A \rtimes_{\alpha_S} S$ coming from the Packer-Raeburn* *stabilization trick, all Mackey obstructions of the system* $(B, G/S, \beta)$ *are similar to* $\widetilde{\omega}$.

Proof. Since the action of S on A is pointwise unitary, we can write every irreducible representation of $A \rtimes_{\alpha_S} S$ in the form $\rho \times V$, with $\rho \in \widehat{A}$. Now let $U : G \to \mathcal{U}(\mathcal{H}_\rho)$ be an ω-representation such that U implements α in ρ. Since $U|_S$ is a homomorphism we have $\rho \times U|_S \in \widehat{B}$, and it follows from the general theory that we find a character $\chi \in \widehat{G}$ such that $(U \otimes \chi)|_S = V$. Hence we may as well assume that $U|_S = V$. Now let us define $W : G/S \to \mathcal{U}(\mathcal{H}_\rho)$ by

$$W_{\dot{x}} = \rho \times U|_S(v_{\dot{x}})U_{c(\dot{x})},$$

where C and v are as in the constuction of β above. Then, if γ denotes the canonical action of G on B and $u : G/S \times G/S \to \mathcal{U}(B)$ is given by $u(\dot{x}, \dot{y}) = i_S(c(\dot{x})c(\dot{y})c(\dot{x}\dot{y}^{-1}))$, i_S denoting the canonical embedding of S into $\mathcal{U}(B)$, we get:

$$\begin{aligned}
W_{\dot{x}\dot{y}} &= \rho \times U|_S(v_{\dot{x}\dot{y}})U_{c(\dot{x}\dot{y})} \\
&= \rho \times U|_S(v_{\dot{x}}\gamma_{c(\dot{x})}(v_{\dot{y}})u(\dot{x},\dot{y}))U_{c(\dot{x}\dot{y})} \\
&= \rho \times U|_S(v_{\dot{x}}\gamma_{c(\dot{x})}(v_{\dot{y}}))U_{c(\dot{x})c(\dot{y})c(\dot{x}\dot{y})^{-1}}U_{c(\dot{x}\dot{y})} \\
&= \rho \times U|_S(v_{\dot{x}})\rho \times U|_S(\gamma_{c(\dot{x})}(v_{\dot{y}})) \\
&\cdot U_{c(\dot{x})}U_{c(\dot{y})}U_{c(\dot{x}\dot{y})^{-1}}U_{c(\dot{x}\dot{y})}\omega(c(\dot{x}), c(\dot{y}))\omega(c(\dot{x})c(\dot{y}), c(\dot{x}\dot{y})^{-1}) \\
&= \widetilde{\omega}(\dot{x},\dot{y})\rho \times U|_S(v_{\dot{x}})U_{c(\dot{x})}\rho \times U|_S(v_{\dot{y}})U_{c(\dot{y})} \\
&= \widetilde{\omega}(\dot{x},\dot{y})W_{\dot{x}}W_{\dot{y}}.
\end{aligned}$$

Here we used the relation $\rho \times U|_S(\gamma_{c(\dot{x})}(v_{\dot{y}})) = U_{c(\dot{x})}\rho \times U|_S(v_{\dot{y}})U^*_{c(\dot{x})}$, which follows easily from the fact that U implements α in ρ. Furthermore, we have also used the fact that $\widetilde{\omega}(\dot{x},\dot{y}) = \omega(x, y)$ for all $x, y \in G$, from which in particular follows that $\omega(c(\dot{x})c(\dot{y}), c(\dot{x}\dot{y})^{-1}) = \omega(c(\dot{x}\dot{y}), c(\dot{x}\dot{y})^{-1})$, and the fact that $U^*_x = \omega(x, x^{-1})U_{x^{-1}}$, which implies that

$$\omega(c(\dot{x})c(\dot{y}), c(\dot{x}\dot{y})^{-1})U_{c(\dot{x}\dot{y})^{-1}}U_{c(\dot{x}\dot{y})} = 1.$$

The proposition follows now from the fact the W implements β in $\rho \times U|_S$, which can be shown exactly as in the proof of the correspondence of the covariant representations of (A, G, α) and $(B, G/S, \beta)$ given in [**19**]. □

If G is abelian, H is a closed subgroup of G, and α is pointwise unitary, then there is a well-defined map

$$\mathrm{res}^G_H : (A \rtimes_\alpha G)^\wedge \to (A \rtimes_{\alpha_H} H)^\wedge$$

which is just given by the usual restriction of representations. If H is the trivial subgroup, then we will denote this map simply by res^G.

Proposition 3. *Let (A, G, α) be a separable covariant system such that A has continuous trace, G is abelian and α is pointwise unitary. Suppose further that N is a closed subgroup of G such tha G/N is compactly generated. Then*

$$(A \rtimes_\alpha G)^\frown \xrightarrow{\mathrm{res}^G_N} (A \rtimes_{\alpha_N} N)^\frown$$

is a principal $\widehat{G/N}$-bundle.

Proof. By passing from A to $A \otimes \mathcal{K}$ and from α to $\alpha \otimes \mathrm{id}$ we may assume that A is stable. Let β be the action of G/N on $B := A \rtimes_{\alpha_N} N$ as in the constructions above. Then it follows from Proposition 1 that $\mathrm{res}^G_N : (A \rtimes_\alpha G)^\frown \to \widehat{B}$ is identical to the map $\mathrm{res}^{G/N} : (B \rtimes_\beta G/N)^\frown \to \widehat{B}$. In particular, this implies that β is pointwise untary. By [**18**, Theorem 1.10] we know that B is isomorphic to $(\mathrm{res}^N)^*(A) = C_0(\widehat{B}) \otimes_{C(\widehat{A})} A$, the pull-back of A along res^N. Since A has continuous trace by assumption, the same is true for $(\mathrm{res}^N)^*(A)$, since this algebra is by definition a quotient of the continuous trace algebra $C_0(\widehat{B}) \otimes A$ (see [**25**] for the definition of pull-backs of C^*-algebras). Hence, since G/N is compactly generated, it follows from [**30**, Corollary 2.2] that β is locally unitary. Thus the results in [**22**] imply that $\mathrm{res}^{G/N}$, and hence also res^G_N, is a principal $\widehat{G/N}$-bundle. □

Let us now recall from [**24**] a certain pairing in Čech cohomology. For this let X be a locally compact space and G a locally compact abelian group. Furthermore, let $\mathcal{G}$ and $\widehat{\mathcal{G}}$ denote the sheaves of germs of continuous G- and $\widehat{G}$-valued functions, respectively. Then there is a natural pairing

$$\langle \cdot, \cdot \rangle : H^1(X, \mathcal{G}) \times H^1(X, \widehat{\mathcal{G}}) \to H^3(X, \mathbb{Z})$$

(Čech cohomology) which is given in the following way: If $c \in H^1(X, \mathcal{G})$ and $d \in H^1(X, \widehat{\mathcal{G}})$ are represented by cocycles

$$c_{ij} : N_{ij} \to G \text{ and } d_{ij} : N_{ij} \to \widehat{G}$$

with respect to an open cover $\{N_i\}$ of X, then we can define a cocycle $\gamma \in Z^2(X, \mathcal{T})$ by

$$\gamma_{ijk} : N_{ijk} \to \mathbb{T}; \ \gamma_{ijk}(x) = \langle c_{ij}(x), \ d_{jk}(x) \rangle.$$

Here $\mathcal{T}$ denotes the sheaf of germs of continuous $\mathbb{T}$-valued functions on X. The class $\langle c, d \rangle$ is now defined as the image of $[\gamma]$ under the natural isomorphism $H^2(X, \mathcal{T}) \to H^3(X, \mathbb{Z})$. Recall also that there is a canonical one-to-one correspondence between the classes in $H^1(X, \mathcal{G})$ and the isomorphism classes of principal G-bundles over X. We are now ready to state the main result of this section.

Theorem 3. *Let (A, G, α) be a separable covariant system such that A has continuous trace, G is abelian and the action of G on $\widehat{A}$ is trivial. Suppose further that all Mackey obstruction are similar to a fixed type I multiplier $\omega \in Z^2(G, \mathbb{T})$. Then $A \rtimes_\alpha G$ has continuous trace and $(A \rtimes_\alpha G)^\wedge$ is homeomorphic to $Y := (A \rtimes_{\alpha_S} S)^\wedge$, where S denotes the symmetry group of ω. Furthermore, if G/S is compactly generated, then there exist closed subgroups H and L of G containing S with the following properties:*

(1) *The actions α_H and α_L are pointwise unitary, $G/S = H/S \times L/S$, and h_ω^H, the composition of h_ω with the projection from $\widehat{G}$ onto $\widehat{H}$, maps L/S isomorphically onto $\widehat{H/S}$.*

(2) *If c denotes the class in $H^1(Y, \widehat{\mathcal{L}/S})$ corresponding to the $\widehat{L/S}$-bundle res_S^L, and d denotes the class in $H^1(Y, \mathcal{L}/S)$ corresponding to the $\widehat{H/S}$- and hence L/S-bundle res_S^H, then*

$$\delta(A \rtimes_\alpha G) = (\mathrm{res}^S)^*(\delta(A)) + \langle c, d\rangle.$$

For the proof we need the following lemma, which gives a very weak form of the decomposition results of Section 1 for multipliers on arbitrary locally compact abelian groups. For this let us introduce the following notion: If ω is a multiplier on the locally compact abelian group G, then a closed subgroup H of G is called **maximally ω-trivial** if H is a maximal subgroup with respect to the property that ω_H is trivial.

Lemma 4. *Let ω be any multiplier on the locally compact abelian group G. Then there exists a maximally ω-trivial subgroup H of G. Moreover, if ω is type I and totally skew, and if H is maximally ω-trivial, then h_ω^H, the composition of h_ω with the projection from $\widehat{G}$ onto $\widehat{H}$, factors through an isomorphism between G/H and $\widehat{H}$.*

Proof. Let $\mathcal{M}$ denote the set of all closed subgroups L of G such that ω_L is trivial, ordered by inclusion, and let $\mathcal{L}$ be a chain in $\mathcal{M}$. Then it follows easily from the continuity of h_ω that the restriction of ω to the closure K of $\bigcup_{L\in\mathcal{L}} L$ is trivial, too. Hence by Zorn's lemma there exists a maximal element H in $\mathcal{M}$, which just means that H is maximally ω-trivial.

Finally suppose that ω is type I and totally skew and H is maximally ω-trivial. Since h_ω is an isomorphism $G \to \widehat{G}$, composition with the projection $\widehat{G} \to \widehat{H}$ gives an open and surjective map h_ω^H. Clearly H is contained in the kernel of h_ω^H. Assume that there exists $x \notin H$ such that $h_\omega(x)(h) =$

$h_\omega(h)(x) = 1$ for all $h \in H$. Then $h_\omega(x^n)(h) = (h_\omega(x)(h))^n = 1$ for all $n \in \mathbb{Z}$, and we have also $h_\omega(x^n)(x^m) = (h_\omega(x)(x))^{n+m} = 1$ for all $n, m \in \mathbb{Z}$. The continuity of h_ω in both variables now implies that ω_L is trivial, if L denotes the closed subgroup generated by H and x. But this contradicts the maximally ω-trivial property of H and we conclude that H is the kernel of h_ω^H. □

Proof of Theorem 3. It was already shown in [**15**] that $(A \rtimes_{\alpha_S} S)\hat{}$ is homeomorphic to $(A \rtimes_\alpha G)\hat{}$. Since all assertions in the theorem are invariant under passing from A to $A \otimes \mathcal{K}$ and α to $\alpha \otimes \mathrm{id}$ we may asume that A is stable. We start the proof for the case where ω is totally skew, and hence S is trivial. By Lemma 4 we find a maximally ω-trivial subgroup H of G. Since ω_H is trivial it follows that α_H is pointwise unitary. Hence by [**18**, Theorem 1.3] we know that the dual action of $\widehat{H}$ on the Hausdorff space $(A \rtimes_{\alpha_H} H)\hat{}$ is free and proper. Since $A \rtimes_{\alpha_H} H \cong (\mathrm{res}^H)^*(A)$ by [**18**, Theorem 1.10] it follows also that $A \rtimes_{\alpha_H} H$ has continuous trace. Now let β denote the action of G/H on $A \rtimes_{\alpha_H} H$ defined by the Packer-Raeburn stabilization trick. We show that the corresponding action of G/H on $(A \rtimes_{\alpha_H} H)\hat{}$ is free and proper. Then it follows from [**24**] that $A \rtimes_\alpha G \cong (A \rtimes_{\alpha_H} H) \rtimes_\beta G/H$ has continuous trace. Hence, let $x \in G$ and $\rho \times V \in (A \rtimes_{\alpha_H} H)\hat{}$. By part (2) of Proposition 1 we know that $(\rho \times V) \circ \beta_{\dot{x}}$ is equivalent to $(\rho \times V) \circ \gamma_x$, where γ denotes the canonical action of G on $A \rtimes_{\alpha_H} H$. Now let $U : G \to \mathcal{U}(\mathcal{H}_\rho)$ be an ω-representation of G which implements α in ρ. Since ω_H is trivial, we may assume that $U|_H$ is a homomorphism, which implies that $(\rho, U|_H)$ is a covariant representation of (A, H, α_H). Hence, by multiplying U with an appropriate character of G, we may assume that $V = U|_H$. Then, for all f in the dense subalgebra $C_c(H, A)$ of $A \rtimes_{\alpha_H} H$, we get:

$$\begin{aligned}
(\rho \times V) \circ \gamma_x(x) &= (\rho \times U|_H)(\gamma_x(f)) \\
&= \int_H \rho(\alpha_x(f(h)))U_h \, dh \\
&= \int_H U_x \rho(f(h)) U_x^* U_h \, dh \\
&= U_x \left(\int_H \rho(f(h)) h_\omega(x)(h) U_h \, dh \right) U_x^* \\
&= U_x(\rho \times (U|_H \otimes h_\omega(x)))(f) U_x^* \\
&= U_x(\rho \times (V \otimes h_\omega(x)))(f) U_x^*,
\end{aligned}$$

where we have easily verifiable equation

$$U_x^* U_h = h_\omega(x)(h) U_h U_x^*, \text{ for all } x \in G \text{ and } h \in H.$$

Thus, since $(\rho \times V) \circ \gamma_x$ is equivalent to $(\rho \times V) \circ \beta_{\dot{x}}$, it follows that the latter is also equivalent to $\rho \times (V \otimes h_\omega(x))$. Since the composition of h_ω with the projection from $\widehat{G}$ onto $\widehat{H}$ is an isomorphism $G/H \to \widehat{H}$, we see that this isomorphism carries that the action of G/H on $(A \rtimes_{\alpha_H} H)\widehat{\ }$onto the dual action of $\widehat{H}$ on the same space. But we have already mentioned above that this action is proper.

Now suppose that G is compactly generated, still assuming that ω is totally skew. Then it follows from Lemma 3 that G has a splitting $H \times L$ with ω_H and ω_L trivial. Since G splits topologically we can write $A \rtimes_\alpha G$ as the iterated crossed product $(A \rtimes_{\alpha_H} H) \rtimes_\gamma L$, where now γ denotes the restriction of the canonical action of G on $A \rtimes_{\alpha_H} H$ to L. Since h_ω defines an isomorphism between L and $\widehat{H}$, the calculations above show that this isomorphism carries the action, say $\widehat{\gamma}$, of L on $(A \rtimes_{\alpha_H} H)\widehat{\ }$onto the dual action of $\widehat{H}$ on this space. The description of the isomorphism between $(\mathrm{res}^H)^*(A)$ and $A \rtimes_{\alpha_H} H$ given in **[25]** (note that α_H is locally unitary since G is compactly generated) shows that γ is in fact the diagonal action $\widehat{\gamma} \otimes \alpha_L$. Hence it is now a consequence of **[24]** that

$$\delta(A \rtimes_\alpha G) = \delta(A) + \langle c, d \rangle.$$

Now let us finally assume that S is nontrivial. Then there exists an $\widetilde{\omega} \in Z^2(G/S', \mathbb{T})$ such that ω is lifted from $\widetilde{\omega}$ (modulo similarity). Let β be the action from G/S on $B := A \rtimes_{\alpha_S} S$ coming from Packer-Raeburn stabilization trick. By Proposition 2 we know that all Mackey obstruction of the system $(B, G/S, \beta)$ are similar to $\widetilde{\omega}$. Finally, if H and L are as in the theorem, then we know that $\widetilde{\omega}_{H/S}$ and $\widetilde{\omega}_{L/S}$ are trivial, and by part (1) of Proposition 1, we see that the maps res^H_S and res^L_S are identical to $\mathrm{res}^{H/S}$ and $\mathrm{res}^{L/S}$. Hence, the equation for the Dixmier-Douady class of $A \rtimes_\alpha G$ follows directly by applying the second part of this proof to $B \rtimes_\beta G/S$. □

Remark. One possible application of Theorem 3 is to give a new proof of an important special case of Theorem 1, that any type I non-commutative torus is strongly Morita-equivalent to the continuous functions on an ordinary torus. (This fact was not used in the proof above.) Namely, apply Theorem 3 with G free abelian, $A = \mathcal{K}$, and α an action of G on A with type I obstruction cocycle ω. Then as we've seen, S is of finite index in G and the space Y in the theorem is just $\widehat{S}$, which is a torus. Since G/S is finite, the classes c and d in Theorem 3 are necessarily torsion elements. Since the cohomology of a torus is torsion-free, they must vanish and hence $\delta(A \rtimes_\alpha G) = 0$, proving that $A \rtimes_\alpha G$ is Morita -equivalent to $C(\widehat{S})$. However, as proved below in Proposition 5, $A \rtimes_\alpha G$ is also Morita-equivalent to the non-commutative torus $C^*(G, \omega^{-1})$.

We are now going to show that the pairing $\langle c, d\rangle$ may be nontrivial. In order to do this we need the following lemma. Recall that two actions α and α' of G on A are called **exterior equivalent**, denoted $\alpha \sim \alpha'$, if there exists a strictly continuous function $x \mapsto u_x$ from G into $\mathcal{U}(A)$ such that

(1) $u_{xy} = u_x\alpha(x)(u_y)$,

(2) $\alpha'(x) = (\mathrm{Ad}\, u_x)\alpha(x)$ for all $x, y \in G$. It is well-known that the crossed products $A \rtimes_\alpha G$ and $A \rtimes_{\alpha'} G$ are isomorphic if α and α' are exterior equivalent, and that conversely, if G is abelian and there is a $\widehat{G}$-equivariant isomorphism of $A \rtimes_\alpha G$ and $A \rtimes_{\alpha'} G$ compatible with the injections of A into the two multiplier algebras, then α and α' are exterior equivalent (see [**24**, Theorem 0.10]).

Lemma 5. *Let G be a locally compact group and A and B C^*-algebras. Suppose further that α and α' are actions of G on A and that β and β' are actions of G on B such that α is exterior equivalent to α' and β is exterior equivalent to β'. Then the diagonal actions $\alpha\otimes\beta$ and $\alpha'\otimes\beta'$ on any C^*-tensor product $A\otimes B$, in particular on the minimal tensor product, are also exterior equivalent.*

Proof. We show first that $\alpha\otimes\beta \sim \alpha'\otimes\beta$. For this we define $v : G \to \mathcal{U}(A\otimes B)$ by

$$v_x = u_x \otimes \mathrm{id},\ \ x \in G.$$

Straightforward computations show that v satisfies the conditions (1) and (2) above, and hence that $\alpha\otimes\beta$ is exterior equivalent to $\alpha'\otimes\beta$. The Lemma follows now easily from the symmetry of this result and from the fact that $\sim$ is an equivalence relation. □

Now let X be any second-countable locally compact space, H a second-countable locally compact abelian group, and let $p : E \to X$ and $q : F \to X$ be any principal H- and $\widehat{H}$-bundles, respectively. Then it was shown in [**22**] that there exist locally unitary actions β and γ of $\widehat{H}$ and H on $A = C_0(X, \mathcal{K})$ such that the bundles $\mathrm{res}^{\widehat{H}} : (A \rtimes_\beta \widehat{H})^\frown \to X$ and $\mathrm{res}^H : (A \rtimes_\gamma H)^\frown \to X$ are identical to the original bundles p and q (these actions are actually given by the dual actions of $\widehat{H}$ and H on the stabilized transformation group C^*-algebras defined by p and q, respectively). Now let $G = H \times \widehat{H}$ and define actions $\tilde{\beta}$ and $\tilde{\gamma}$ of G on $A = C_0(X, \mathcal{K})$ by $\tilde{\beta}(x, y) = \beta(x)$ and $\tilde{\gamma}(x, y) = \gamma(y)$, for all $x \in H$ and $y \in \widehat{H}$. We define a multiplier ω on G by

$$\omega((x_1, y_1), (x_2, y_2)) = \langle x_1, y_2\rangle,$$

where here $\langle\cdot,\cdot\rangle$ denotes the natural pairing of H with $\widehat{H}$. Then ω is easily seen to be totally skew and type I. Let $U : G \to \mathcal{U}(\mathcal{H})$ be an ω-representation of G, and let $\delta = \mathrm{Ad}\, U$ denote the corresponding action of G on $\mathcal{K}$. We are going to define the action α of G on $C_0(X,\mathcal{K})$ in the following way: First of all let α' be the diagonal action $\widetilde{\beta}\otimes\widetilde{\gamma}$ of G on $C_0(X,\mathcal{K})\otimes C_0(X,\mathcal{K})$. This action induces an action α'' of G on the balanced tensor product $C_0(X,\mathcal{K}) \otimes_{C_0(X)} C_0(X,\mathcal{K})$, which is canonically isomorphic to $C_0(X,\mathcal{K})$. Then we define $\alpha = \alpha'' \otimes \delta$ as the diagonal action of G on $C_0(X,\mathcal{K}) \otimes \mathcal{K} \cong C_0(X,\mathcal{K})$.

It is easy to check that the Mackey obstructions associated to the system $(C_0(X,\mathcal{K}), G, \alpha)$ are similar to ω. Furthermore, if we identify $C_0(X,\mathcal{K})$ with $C_0(X,\mathcal{K}) \otimes \mathcal{K} \otimes \mathcal{K}$ coming from the above procedure, then we see easily that $\alpha_H = \beta \otimes \mathrm{id} \otimes \delta_H$ and $\alpha_{\widehat{H}} = \gamma \otimes \mathrm{id} \otimes \delta_{\widehat{H}}$. But δ_H and $\delta_{\widehat{H}}$ are exterior equivalent to id and we see from Lemma 5 that therefore $\alpha_H \sim \beta \otimes \mathrm{id} \otimes \mathrm{id}$ and $\alpha_{\widehat{H}} \sim \gamma \otimes \mathrm{id} \otimes \mathrm{id}$. Since the isomorphism which comes from exterior equivalence intertwines the dual actions by [**24**, Theorem 0.10] this implies that the bundles $\mathrm{res}^H : (A \rtimes_\beta H)\widehat{\ } \to X$ and $\mathrm{res}^H : (A \rtimes_{\alpha_H} H)\widehat{\ } \to X$ are the same, and an analogous result is true for the actions of $\widehat{H}$. Hence with Theorem 3 we get the following:

Proposition 4. *Let X be a second-countable locally compact space and H a second-countable locally compact abelian group. Suppose further that $c \in H^1(X,\mathcal{H})$ and $d \in H^1(X,\widehat{\mathcal{H}})$. Then there exists an action α of $G = H \times \widehat{H}$ on $A = C_0(X,\mathcal{K})$ such that all multipliers associated to the system (A,G,α) are similar to a constant totally skew type I multiplier ω on G and*

$$\delta(A \rtimes_\alpha G) = \langle c, d\rangle.$$

Corollary 2. *Let X be a second-countable locally compact space. Then every class in $H^3(X,\mathbb{Z})$ (Čech cohomology) which can be written as a cup-product of classes in $H^1(X,\mathbb{Z})$ and in $H^2(X,\mathbb{Z})$ arises as the Dixmier-Douady class of $C_0(X,\mathcal{K}) \rtimes (\mathbb{Z}\times\mathbb{T})$ for some acton α of $\mathbb{Z}\times\mathbb{T}$ on $A = C_0(X,\mathcal{K})$ which is trivial on X and for which the Mackey obstruction is constant, type I, and totally skew. In particular, if X is any compact orientable 3-manifold with non-zero first Betti number, every class in $H^3(X,\mathbb{Z})$ arises as such a Dixmier-Douady class. Except in the case of the 0-class, A is therefore* **not** *Morita equivalent to $A \rtimes_\alpha G$.*

Proof. The first statement is immediate from the fact that if $H = \mathbb{Z}$, then $\mathcal{H}$ is the constant sheaf $\mathbb{Z}$ and $H^j(X,\widehat{\mathcal{H}}) \cong H^{j+1}(X,\mathbb{Z})$ for $j \geq 1$. The second statement about compact orientable 3-manifolds follows from the non-degeneracy of the cup-product pairing between H^1 and the torsion-free part of H^2, which in turn follows from Poincaré duality. □

Locally projective unitary actions. Now we discuss the special case of projective unitary actions and locally projective unitary actions. For the moment we drop the requirement that our groups be abelian.

Definition. Let (A, G, α) be a covariant system. Then α is called **projective unitary** (resp. **locally projective unitary, pointwise projective unitary**) if there exists an action β of G on $\mathcal{K}$, the algebra of compact operators on the separable Hilbert space $\mathcal{H}$, such that the diagonal action $\alpha \otimes \beta$ on $A \otimes \mathcal{K}$ is unitary (resp. locally unitary, pointwise unitary).

Remark. (1) The pointwise projective unitary actions on a type I C^*-algebra A are just actions where all Mackey obstructions are similar to a constant multiplier $\omega \in Z^2(G, \mathbb{T})$. In order to see this, let ω be the multiplier defined by the action β of G on $\mathcal{K}$ and let ω_ρ be the Mackey obstruction of the system (A, G, α) at $\rho \in \widehat{A}$. Then the Mackey obstruction for the action $\alpha \otimes \beta$ at the representation $\rho \otimes \mathrm{id} \in (A \otimes \mathcal{K})\widehat{\ }$ is just given by $\omega_\rho \omega$. Hence $\alpha \otimes \beta$ is pointwise unitary if and only if ω_ρ is in the class of ω^{-1} in $H^2(G, \mathbb{T})$ for all $\rho \in \widehat{A}$, which clearly means that the Mackey obstructions are constant.

(2) It follows directly from the definition above, that for the same class of groups for which pointwise unitary actions on separable continuous-trace are automatically locally unitary, pointwise projective unitary actions are automatically locally projective unitary.

Crossed products for covariant system (A, G, α) such that α is projective unitary have a very simple description in terms of A and the Mackey obstruction ω.

Proposition 5. *Let (A, G, α) be a covariant system such that α is a projective unitary. Furthermore, let ω^{-1} be the Mackey obstruction of an action β of G on $\mathcal{K}$ such that $\alpha \otimes \beta$ is unitary. (Thus ω is the Mackey obstruction of α at each point of $\widehat{A}$.) Then $A \rtimes_\alpha G$ is Morita equivalent to $C^*(G, \omega^{-1}) \otimes_{max} A$. In particular, if G is abelian and ω is type I with symmetry group S, $A \rtimes_\alpha G$ is Morita equivalent to $C_0(\widehat{S}) \otimes A$.*

Proof. Since $\alpha \otimes \beta$ is unitary, it follows that the action $\alpha \otimes \beta$ of G on $A \otimes \mathcal{K}$ is exterior equivalent to the trivial action id. Let $\overline{\beta}$ be an action of G on $\mathcal{K}$ with Mackey obstruction ω. Then by Lemma 5, the action $\alpha \otimes \beta \otimes \overline{\beta}$ of G on $A \otimes \mathcal{K} \otimes \mathcal{K}$ is exterior equivalent to the action $\mathrm{id} \otimes \overline{\beta}$ on $(A \otimes \mathcal{K}) \otimes \mathcal{K}$. By the proof of [**14**, Proposition 14] we know that $(A \otimes \mathcal{K}) \otimes \mathcal{K} \rtimes_{\mathrm{id} \otimes \overline{\beta}} G$ is isomorphic to $(A \otimes \mathcal{K}) \otimes_{max} (\mathcal{K} \rtimes_{\overline{\beta}} G)$, which is clearly Morita equivalent to $C^*(G, \omega^{-1}) \otimes_{max} A$ by [**14**, Theorem 18]. (The appearance of ω^{-1} in place of ω is due to the fact that the irreducible representations of $(\mathcal{K} \rtimes_{\overline{\beta}} G)$ are obtained by first extending the standard representation of $\mathcal{K}$ to an ω-covariant representation of $(\mathcal{K}, G)$, and then tensoring with arbitrary ω^{-1}-representations of G to get genuine covariant representations.) On the other hand it is clear that the Mackey obstruction of the action $\beta \otimes \overline{\beta}$ on $\mathcal{K} \otimes \mathcal{K}$

is trivial, hence this action is unitary and exterior equivalent to the trivial action id. Hence, by Lemma 5 we see that the action $\alpha \otimes \mathrm{id}$ of G on $A \otimes (\mathcal{K} \otimes \mathcal{K})$ is exterior equivalent to $\mathrm{id} \otimes \overline{\beta}$ on $(A \otimes \mathcal{K}) \otimes \mathcal{K}$. Hence we find that $(A \rtimes_\alpha G) \otimes \mathcal{K} = (A \otimes \mathcal{K} \otimes \mathcal{K}) \rtimes_{\alpha \otimes \mathrm{id}} G$ is Morita equivalent to $C^*(G, \omega^{-1}) \otimes_{max} A$. The last statement now follows from Theorem 1. □

3. The structure of certain crossed products with constant Mackey obstruction.

In this section we want to investigate certain crossed products $A \rtimes_\alpha G$ such that A has continuous trace, G is abelian, and such that the stabilizer for each $\rho \in \widehat{A}$ is equal to a fixed subgroup N of G. We will refer to such systems as systems with constant stabilizer N. If in addition the Mackey obstructions of the system (A, N, α_N) are constant, we will say that (A, G, α) is a covariant system with constant stabilizer and constant Mackey obstructions. Such systems are the main object of this section. However, as we will see later, some of our results are still true for systems with the weaker assumption that all symmetry groups of the Mackey obstructions are constant, rather than the more restrictive assumption that all Mackey obstructions are constant. We will refer to these as systems with constant stabilizer N and constant symmetry group S. **If not otherwise stated, we will assume throughout this section that (A, G, α) is always separable, A has continuous trace, and G is abelian.**

If (A, G, α) has constant stabilizer N and symmetry group S, then we know from [**15**] that the map $\mathrm{ind}_S^G : (A \rtimes_{\alpha_S} S)\widehat{\ } \to \mathrm{Prim}(A \rtimes_{\alpha_N} N)$ defined by $\rho \mapsto \ker(\mathrm{ind}_N^G \rho)$ is a G- and $\widehat{G}$-equivariant homeomorphism with respect to the canonical actions of G and $\widehat{G}$ on these spaces. It follows from this and [**12**] that

$$\mathrm{ind} : (A \rtimes_{\alpha_S})\widehat{\ } \to \mathrm{Prim}(A \rtimes_\alpha G); \rho \mapsto \ker(\mathrm{ind}_S^G \rho)$$

is well defined surjective map. Hence, if $\mathcal{Q}(\widehat{A})$ denotes the quasi-orbit space for the action of G on $\widehat{A}$, we get the following commutative diagram of maps:

$$\begin{array}{ccc} (A \rtimes_{\alpha_S} S)\widehat{\ } & \xrightarrow{\mathrm{ind}} & \mathrm{Prim}(A \rtimes_\alpha G) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle p} \\ \widehat{A} & \xrightarrow[q]{} & \mathcal{Q}(\widehat{A}) \end{array}$$

where $q : \widehat{A} \to \mathcal{Q}(\widehat{A})$ is the quotient map. The definition of p is a little bit more complicated: If $J \in \mathrm{Prim}(A \rtimes_\alpha G)$ and $\pi \times U \in (A \rtimes_\alpha G)\widehat{\ }$ such that $J = \ker(\pi \times U)$, then there exists a unique quasi-orbit $\mathcal{O} \in \mathcal{Q}(\widehat{A})$ such that $\ker \pi = \bigcap_{\rho \in \mathcal{O}} \ker \rho$. The image of J under p is exactly this quasi-orbit. All maps in the diagram are continuous, open and surjective, which follows by routine arguments from the continuity of restricting and inducing representations. They are also equivariant under the various actions of G and $\widehat{G}$ on these spaces, which follows for instance from [**24**, Lemma 2.3]. Now suppose that the action of G/N on $\widehat{A}$ is proper. Then $\mathcal{Q}(\widehat{A}) = \widehat{A}/G$ is Hausdorff, and we can prove the following theorem:

Theorem 4. *Let (A, G, α) be a covariant system with constant stabilizer N and constant symmetry group S. Suppose further that $\widehat{A}$ is a proper G/N-space. Then induction induces a homeomorphism $(A \rtimes_{\alpha_S} S)\widehat{\ }/G \to \operatorname{Prim}(A \rtimes_\alpha G)$, the action of $\widehat{G}$ on $\operatorname{Prim}(A \rtimes_\alpha G)$ has constant stabilizer $S^\perp$, and $\operatorname{Prim}(A \rtimes_\alpha G)$ is a proper $\widehat{S}$- ($= \widehat{G}/S^\perp$)- space. Furthermore, the actions of $\widehat{G}$ and G on $\mathcal{W} = (A \rtimes_{\alpha_S} S)\widehat{\ }$ make $\mathcal{W}$ into a proper $\widehat{S}$- and G/N-space, respectively.*

Proof. Using the same arguments as in the proof of [**18**, Corollary 2.1], we see the actions of G and $\widehat{G}$ on $(A \rtimes_{\alpha_S} S)\widehat{\ }$ factor through commuting free and proper actions of G/N and $\widehat{S}$ on $(A \rtimes_{\alpha_S} S)\widehat{\ }$. By continuity of induction, ind gives a continuous open surjection from $(A \rtimes_{\alpha_S} S)\widehat{\ }$ to $\operatorname{Prim}(A \rtimes_\alpha G)$, and since the action of G on $(A \rtimes_{\alpha_S} S)\widehat{\ }$ factors through a proper action of G/N, the quotient space is Hausdorff and ind factors through this quotient space. The induced map $(A \rtimes_{\alpha_S} S)\widehat{\ }/G \cong (A \rtimes_{\alpha_N} N)\widehat{\ }/(G/N) \to \operatorname{Prim}(A \rtimes_\alpha G)$ is then a homeomorphism by [**14**, Theorem 24]. Since res and ind factor through homeomorphisms between $(A \rtimes_{\alpha_S} S)\widehat{\ }/G$ and $\operatorname{Prim}(A \rtimes_\alpha G)$ and between $(A \rtimes_{\alpha_S} S)\widehat{\ }/\widehat{S}$ and $\widehat{A}$, respectively, the theorem follows from the following lemma: □

Lemma 6. *Let H and L be two abelian locally compact groups acting properly on the locally compact space X. Suppose further, that the actions of H and L commute, and that the resulting action of $H \times L$ on X is free. Then the following statements are equivalent:*

(1) *$H \times L$ acts properly on X.*
(2) *H acts properly on X/L.*
(3) *L acts properly on X/H.*

Proof. By symmetry it is clearly enough to show (1) $\Leftrightarrow$ (2). (2) $\Rightarrow$ (1) is a special case of [**18**, Lemma 1.9]. In order to show (1) $\Rightarrow$ (2) let $(h_i, x_i)_{i \in I}$ be a net in $H \times X$ such that $(L(h_i x_i), L(x_i))$ converges in $X/L \times X/L$. Hence, by passing to a subnet if necessary, we may assume that there are elements $x_0, x_1 \in X$ and nets $(l_i^0)_{i\in I}, (l_i^1)_{i \in I} \subseteq L$ such that $l_i^0 x_i$ converges to x_0 and $l_i^1 h_i x_i$ converges to x^1. By replacing x_i by $l_i^0 x_i$ and l_i^1 by $l_i^1 (l_i^0)^{-1}$, it follows from (1) that, again by passing to a subnet, we may assume that $(l_i^1 h_i)_{i\in I}$ converges in $H \times L$, from which clearly follows that $(h_i)_{i\in I}$ converges in H. $\square$

If S is compactly generated, and hence also $\widehat{S}$ is a Lie group, and if G/N is also a Lie group, then it follows from Palais slice theorem [**21**, Theorem 4.1] that all spaces in the theorem above are actually principal $\widehat{S}$- and/or G/N-bundles. As we have mentioned before, such bundles can be classified by certain elements in Čech cohomology, and we are now going to describe the relation between the various topological invariants. Recall that if $p : X \to Z$ is a principal G-bundle with representative $c_{ij} : N_{ij} \to G$ in $Z^1(Z, \mathcal{G})$ and if $f : Y \to Z$ is a continuous map, then the pull-back $f^*(p)$ of p along f is the principal G-bundle with representative $\{c_{ij} \circ f\} : f^{-1}(N_{ij}) \to G$ in $Z^1(Y, \mathcal{G})$.

Theorem 5. *Let (A, G, α) be a covariant system with constant stabilizer N and constant symmetry group S. Suppose further that $\widehat{A}$ is a principal G/N-bundle and that α_S is locally unitary. Then all maps in the diagram*

$$\begin{tikzcd}
(A \rtimes_{\alpha_S} S)\widehat{} \arrow[r, "\mathrm{ind}"] \arrow[d, "\mathrm{res}"'] & \mathrm{Prim}(A \rtimes_\alpha G) \arrow[d, "p"] \\
\widehat{A} \arrow[r, "q"'] & \widehat{A}/G
\end{tikzcd}$$

are principal bundles. The horizontal arrows are principal G/N-bundles, and the vertical arrows are principal $\widehat{S}$-bundles. Furthermore, we have the relations

$$\mathrm{ind} = p^*q \quad \textit{and} \quad \mathrm{res} = q^*p.$$

In the case where $S = N$, i.e. if all Mackey obstructions are trivial, this result was proved in [**24**] using Takai's duality theorem. But this method doesn't work nicely in the situation above, at least if the Mackey obstructions of the system are not type I, since then $A \rtimes_\alpha G$ is in general not a type I C^*-algebra. Hence for the proof we will use the idea of constructing local Green twisting maps as given in [**27**] and [**28**] for the case of trivial Mackey obstruction. In fact, this construction gives even more information about the bundle $p : \mathrm{Prim}(A \rtimes_\alpha G) \to \widehat{A}/G$, since it gives the possibility to construct explicitly a representative for this bundle in $Z^1(\widehat{A}/G, \widehat{\mathcal{S}})$. Let us start by recalling the definition for Green twisting maps. For this let (A, G, α) be a covariant system such that G is abelian, and let H be a closed subgroup of G. A **Green twisting map** for (A, G, α) is a strictly continuous homomorphism $u : H \to \mathcal{U}(A)$ such that

(1) $\alpha_h(a) = u_h a u_h^*$ for all $a \in A$ and $h \in H$, and
(2) $\alpha_x(u_h) = u_h$ for all $x \in G$ and $h \in H$.

Furthermore, we say that a system (A, G, α) has **local Green twisting maps** if there exists a cover $\{N_i\}$ of open subsets of $\widehat{A}/G$ and maps $u^i : H \to \mathcal{U}(A|_{q^{-1}(N_i)})$ such that each u_i is a Green twisting map for $(A|_{q^{-1}(N_i)}, G, \alpha)$. If N_i and u_i are as above, then we will call $\{N_i, u^i\}$ a **system of local Green twisting maps with domain** H for (A, G, α).

The proof of the following proposition follows the lines of [**28**, Proposition 2.6], where a similar result is proved for the case of trivial Mackey obstructions.

Proposition 6. *Let (A, G, α) be a covariant system with constant stabilizer N and constant symmetry group S, such that $\widehat{A}/G$ is Hausdorff. Suppose further that there exists a system $\{N_i, u^i\}$ of local Green twisting maps with domain S for (A, G, α). Then $p : \mathrm{Prim}(A \rtimes_\alpha G) \to \widehat{A}/G$ is a principal $\widehat{S}$-bundle which is represented by the cocycle $\gamma_{ij} : N_{ij} \to \widehat{S}$ given by the relation*

$$\rho \circ u^i = \gamma_{ij}(q(\rho))(\rho \circ u^j),$$

with $\rho \in q^{-1}(N_{ij})$. Furthermore, if res *and* ind *are as in the diagram, then* res $= q^* p$.

Proof. Since $u^i : S \to \mathcal{U}(A|_{q^{-1}(N_i)})$ is a unitary map which implements α_S, it is well known that the map $(\rho, \chi) \mapsto \rho \times \chi(\rho \circ u^i)$ is an $\widehat{S}$-equivariant homeomorphism from $\widehat{A|_{q^{-1}(N_i)}} \times \widehat{S}$ onto $(A|_{q^{-1}(N_i)} \rtimes_{\alpha_S} S)\widehat{\ }$. The action of G on $(A|_{q^{-1}(N_i)} \rtimes_{\alpha_S} S)\widehat{\ }$ is given by

$$\begin{aligned} (\rho \times \chi(\rho \circ u^i)) \circ \gamma_x &= (\rho \circ \alpha_x) \times \chi(\rho \circ u^i) \\ &= (\rho \circ \alpha_x) \times \chi(\rho \circ (\alpha_x \circ u^i)) \\ &= (\rho \circ \alpha_x) \times \chi((\rho \circ \alpha_x) \circ u^i), \end{aligned}$$

where again γ denotes the canonical action of G on $A|_{q^{-1}(N_i)} \rtimes_{\alpha_S} S$. But this shows that the homeomorphism above is G-equivariant, from which follows that $(A|_{q^{-1}(N_i)} \rtimes_{\alpha_S} S)\widehat{\ }/G$ is homeomorphic to $N_i \times \widehat{S}$. Since the map ind in our diagram is $\widehat{G}$- and hence $\widehat{S}$-equivariant, we conclude that

$$(q(\rho), \chi) \mapsto \ker(\mathrm{ind}_S^G(\rho \times \chi(\rho \circ u^i)))$$

is an $\widehat{S}$-equivariant homeomorphism h_i between $N_i \times \widehat{S}$ and $\mathrm{Prim}(A|_{q^{-1}(N_i)} \rtimes_\alpha G)$. This shows that $p : \mathrm{Prim}(A \rtimes_\alpha G) \to \widehat{A}/G$ is a principal $\widehat{S}$-bundle.

Now let $\rho \in q^{-1}(N_{ij})$, and let χ be the unique element in $\widehat{S}$ such that $\rho \circ u^i = \chi(\rho \circ u^j)$. Then it follows also from the computations above that

χ only depends on the G-orbit of ρ, which implies that the map γ_{ij} is well defined.

Finally, for $q(\rho) \in N_{ij}$ it follows from [**24**, Lemma 2.3] that

$$\begin{aligned} h_i(q(\rho), \chi) &= \ker(\mathrm{ind}_S^G(\rho \times \chi(\rho \circ u^i))) \\ &= \ker(\mathrm{ind}_S^G(\rho \times \chi\gamma_{ij}(q(\rho))(\rho \circ u^j))) \\ &= \ker(\gamma_{ij}(q(\rho)) \otimes \mathrm{ind}_S^G(\rho \times \chi(\rho \circ u^j))) \\ &= \gamma_{ij}(q(\rho))h_j(q(\rho), \chi), \end{aligned}$$

from which follows that $\gamma_{ij} : N_{ij} \to \widehat{S}$ is a representative for this bundle. Finally, since the u^i are unitary maps which implement α_S on $A|_{q^{-1}(N_i)}$ it follows from the Phillips-Raeburn construction [**22**] that the principal $\widehat{S}$-bundle res : $(A \rtimes_{\alpha_S} S)^\frown \to \widehat{A}$ is equal to q^*p. □

We are now going to show that the covariant systems of Theorem 4 all possess a system of local Green twisting maps with domain S. The next lemma deals with the case where $N = G$ and α_S is unitary.

Lemma 7. *Let (A, G, α) be a covariant system with constant symmetry group, such that G acts trivially on $\widehat{A}$. Suppose further that there exists a unitary map $u : S \to \mathcal{U}(A)$ such that u implements α_S (i.e. α_S is unitary). Then u is a Green twisting map for (A, G, α).*

Proof. Let $\rho \in \widehat{A}$, ω the Mackey obstruction of α in ρ, and let $U : G \to \mathcal{U}(\mathcal{H}_\rho)$ be an ω-representation which implements α in ρ. Then $(\rho, U|_S)$ is a covariant representation of (A, S, α_S). Hence, by multiplying U with an appropriate character of G, we may assume that $U|_S = \rho \circ u$. Using the identity $U_x^* = \omega(x, x^{-1})U_{x-1}$ for $x \in G$ we get

$$\begin{aligned} \rho(\alpha_x(u_s)) &= U_x\rho(u_s)U_x^* = U_xU_sU_x^* \\ &= \omega(x,s)^{-1}\omega(x,x^{-1})U_{xs}U_{x^{-1}} \\ &= \omega(x,s)^{-1}\omega(xs,x^{-1})^{-1}\omega(x,x^{-1})U_{xsx^{-1}}. \end{aligned}$$

But since G is abelian, we have $U_{xsx^{-1}} = U_s$ and

$$\begin{aligned} \omega(x,s)\omega(xs,x^{-1}) &= \omega(s,x)\omega(sx,x^{-1}) \\ &= \omega(s,e)\omega(x,x^{-1}) = \omega(x,x^{-1}), \end{aligned}$$

from which follows that $\omega(x,s)^{-1}\omega(xs,x^{-1})\omega(x,x^{-1}) = 1$. But this implies that $\rho(\alpha_x(u_s)) = U_s = \rho(u_s)$ for all $x \in G$ and $s \in S$. Since we can do this for all $\rho \in \widehat{A}$ it follows that $a_x(u_s) = u_s$ for all $x \in G$ and $s \in S$. Thus u is a Green twisting map for (A, G, α). □

Before we state our next result, let us recall the definition of induced covariant systems. For this let G be a locally compact group and H a closed subgroup of G such that there is an action β of H on a C^*-algebra D. The **induced C^*-algebra** $\mathrm{Ind}(D,\beta)$ is then defined by

$$\begin{aligned}\mathrm{Ind}(D,\beta) = \{f \in C^b(G,D);\ & f(xh) = \beta(h^{-1})f(x) \\ & \text{for all } x \in G,\ h \in H \text{ and } (\dot{x} \mapsto \|f(x)\|) \in C_0(G/H)\}.\end{aligned}$$

The **induced action** $\mathrm{Ind}\,\beta$ of G on $\mathrm{Ind}(D,\beta)$ is defined by

$$(\mathrm{Ind}\,\beta_x(f))(y) = f(x^{-1}y),\ x,y \in G.$$

The covariant system $(\mathrm{Ind}(D,\beta), G, \mathrm{Ind}\,\beta)$ is called the induced covariant system of (D,H,β). The proof of the following lemma is routine (see for instance [**32**, Section 4]).

Lemma 8. *Suppose that $S \subseteq N$ are closed subgroups of the abelian locally compact group G, and let (D,N,β) be a covariant system such that there exists a Green twisting map $u : S \to \mathcal{U}(D)$. Then* $\mathrm{Ind}\,u : S \to \mathcal{U}(\mathrm{Ind}(D,\beta))$ *defined by* $(\mathrm{Ind}\,u_s f)(x) = u_s(f(x))$ *is a Green twisting map for* $(\mathrm{Ind}(D,\beta), G, \mathrm{Ind}\,\beta)$.

Proposition 7. *Let (A,G,α) be as in Theorem 4, i.e. (A,G,α) has constant stabilizer N and constant symmetry group S such that in addition $\widehat{A}$ is a G/N-principal bundle and α_S is locally unitary. Then there exists a system $\{N_i, u^i\}$ of local Green twisting maps with domain S for (A,G,α).*

Proof. Let $\{N_i\}$ be an open cover of $\widehat{A}/G$ such that for each $i \in I$ there exists a section $d_i : N_i \to q^{-1}(N_i)$. Since α_S is locally unitary we may assume by possibly taking a refinement of $\{N_i\}$ that the actions β^i of N on $A|_{d(N_i)}$ which, for all $i \in I$, are canonically defined by α_N, have the property that β^i_S is unitary. Hence by Lemma 7 there exists a Green twisting map $u^i : S \to \mathcal{U}(A|_{d_i(N_i)})$ for each $(A|_{d_i(N_i)}, N, \beta^i)$. Now let $\varphi^i : q^{-1}(N_i) \to G/N$ be defined by $\varphi^i(\rho) = \dot{x}$ if and only if $\rho = \rho_0 \circ \alpha_{x^{-1}}$ for some $\rho_0 \in d_i(N_i)$. Then, for each $i \in I$, φ^i is obviously continuous and G-equivariant. Hence by [**5**, Theorem] there exist G-equivariant isomorphisms $\Phi^i : A|_{q^{-1}(N_i)} \to \mathrm{Ind}(A|_{d_i(N_i)}, \beta^i)$. Thus, by Lemma 8, $\mathrm{Ind}\,u^i : S \to \mathcal{U}(\mathrm{Ind}(A|_{d_i(N_i)}, \beta^i))$ is a Green twisting map for each system $(\mathrm{Ind}(A|_{d_i(N_i)}, \beta^i), G, \mathrm{Ind}\,\beta^i)$, which is carried to a Green twisting map $v^i : S \to \mathcal{U}(A|_{q^{-1}(N_i)})$ by Φ^i. Hence $\{N_i, v^i\}$ is a system of local Green twisting maps for (A,G,α) with domain S. $\square$

Proof of Theorem 5. Combining Proposition 7 with Proposition 6, the only thing which remains to prove is the fact that $\mathrm{ind} : (A \rtimes_{\alpha_S} S)\widehat{\ } \to$

$\mathrm{Prim}(A \rtimes_{\alpha_S} S)$ is a principal G/N-bundle such that $\mathrm{ind} = p^*q$. So let $c_{ij} : N_{ij} \to G/N$ be a representative for q in $Z^1(\widehat{A}/G, \mathcal{G}/\mathcal{N})$. By possibly taking a refinement of the cover $\{N_i\}$, it follows from Proposition 7 together with some arguments in the proof of Proposition 6 (we may assume the existence of local Green twisting maps $u^i : S \to \mathcal{U}(A|_{q^{-1}(N_i)})$) that we have $\widehat{G}$-equivariant homeomorphisms

$$(c_i(z), \chi) \mapsto \ker(\mathrm{ind}_S^G(c_i(z) \times \chi(c_i(z) \circ u^i)))$$

between $c_i(N_i) \times \widehat{S}$ and $p^{-1}(N_i)$, where $c_i : N_i \to q^{-1}(N_i)$, $i \in I$, are local sections with transition functions c_{ij}. Combined with the inclusions of $c_i(N_i) \times \widehat{S}$ into $q^{-1}(N_i) \times \widehat{S} \cong (A|_{q^{-1}(N_i)} \rtimes_{\alpha_S} S)\widehat{}$ we get sections $\widetilde{c}_i : \mathrm{res}^{-1}(N_i) \to \mathrm{ind}^{-1}(\mathrm{res}^{-1}(N_i))$, and if $\widetilde{c}_{ij}$ are the transition functions of the $\widetilde{c}_i$, then it follows directly from construction of the $\widetilde{c}_i$ that $\widetilde{c}_{ij} = c_{ij} \circ p$. □

We return now to the question of when a given crossed product $A \rtimes_\alpha G$ has continuous trace. For this we need a slight generalization of [**18**, Theorem 3.1]. This wil be the only place in this section where we investigate covariant systems (A, G, α) such that we do not assume *a priori* that A has continuous trace.

Proposition 8. *Suppose that (A, G, α) is a separable covariant system such that G is abelian and the action of G on* $\mathrm{Prim}(A)$ *is free. Then $A \rtimes_\alpha G$ has continuous trace if and only if A has continuous trace and G acts properly on $\widehat{A}$.*

Remark. In [**18**, Theorem 3.1] the same result is proved for the special case where A is assumed to have continuous trace. The reason for this restriction was the use of a lemma [**18**, Lemma 3.2], which says that if A has continuous trace and G acts freely on $\widehat{A}$, then $\mathrm{ind}_{\{e\}}^G \rho$ is irreducible for all $\rho \in \widehat{A}$. But we will now see that this lemma was not needed seriously.

Proof of Proposition 8. The "if" direction of the proposition is part of [**24**, Theorem 1.1]. As in [**18**, Theorem 3.1], it is enough to show that, if $A \rtimes_\alpha G$ has continuous trace, the dual action $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes_\alpha G$ is pointwise unitary. This will imply that $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))$ has continuous trace, and that the resulting action of G on $(A \otimes \mathcal{K}(L^2(G)))\widehat{}$ via $\widehat{\widehat{\alpha}}$ is proper, which of course implies the same properties for A and the action of G on $\widehat{A}$ via α. Let $\rho \in (A \rtimes_\alpha G)\widehat{}$. Then it follows from the assumptions and [**12**] that $\ker \rho = \ker(\mathrm{ind}_{\{e\}}^G \pi)$ for some $\pi \in \widehat{A}$. Now let $\chi \in \widehat{G}$. Then by [**24**, Lemma 2.3] we know that

$$\ker(\chi\rho) = \ker(\chi \,\mathrm{ind}_{\{e\}}^G \pi) = \ker(\mathrm{ind}_{\{e\}}^G \pi).$$

Hence it follows that the action $\widehat{\alpha}$ of $\widehat{G}$ on $\mathrm{Prim}(A \rtimes_\alpha G) = (A \rtimes_\alpha G)^\wedge$ is trivial.

We are now going to show that this action is in fact pointwise unitary. For this recall first that $(A \rtimes_\alpha G)^\wedge$ is homeomorphic to the quasi-orbit space $\mathcal{Q}(\mathrm{Prim}(A))$, which follows directly from the continuity of inducing and restricting representations. Since by assumption $A \rtimes_\alpha G$ has continuous trace, it follows in particular that each point in $(A \rtimes_\alpha G)^\wedge$ is closed. Hence, each quasi-orbit $\mathcal{O}$ is closed in $\mathcal{Q}(\mathrm{Prim}(A))$. Hence, by restricting α to $A|_{\mathcal{O}}$, we may assume for the moment that $(A \rtimes_\alpha G)^\wedge$ has only one element ρ and thus $A \rtimes_\alpha G \cong \mathcal{K}(\mathcal{H}_\rho)$. Let us now denote $\mathcal{K}(\mathcal{H}_\rho)$ simply by $\mathcal{K}$. Let ω be the Mackey obstruction for the dual action $\widehat{\alpha}$ of $\widehat{G}$ on $\mathcal{K}$, and let $\widetilde{S}$ be the symmetry group of ω. Then, for the double dual action $\widehat{\widehat{\alpha}}$ of G on $\mathcal{K} \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))$, it follows that the stability groups for the corresponding action on $\mathrm{Prim}(A \otimes \mathcal{K}(L^2(G)))$ are all given by $\widetilde{S}^\perp \subseteq G$. But since the action of G on $\mathrm{Prim}(A \otimes \mathcal{K}(L^2(G)))$ via the double action $\widehat{\widehat{\alpha}}$ is free if and only if the original action of G on $\mathrm{Prim}(A)$ is free, it follows by the assumptions that $\widetilde{S}^\perp$ is trivial, which implies that $\widetilde{S} = \widehat{G}$. But this shows that ω is trivial. Since we can do this for all $\rho \in (A \rtimes_\alpha G)^\wedge$ we see that $\widehat{\alpha}$ is pointwise unitary. □

From now on we will again assume that (A, G, α) is a separable covariant system such that A has continuous trace and G is abelian.

Theorem 6. *Let (A, G, α) be a covariant system with constant stabilizer N and constant Mackey obstruction ω and let H be a maximally ω-trivial subgroup of N as in Lemma 4. Then the following statements are equivalent:*

(1) *$A \rtimes_\alpha G$ has continuous trace.*

(2) *ω is type I and the action of G/N on $(A \rtimes_{\alpha_N} N)^\wedge$ is proper.*

(3) *The action of G/H on $A \rtimes_{\alpha_H} H$ is proper.*

If these conditions are true, and if $\mathrm{res}^H : (A \rtimes_{\alpha H})^\wedge \to \widehat{A}$ is the restriction map, and $\mathrm{ind}_H^G : (A \rtimes_{\alpha_H} H)^\wedge \to (A \rtimes_\alpha G)^\wedge$ is the induction map, then we have the following relation between the Dixmier-Douady classes of A and $A \rtimes_\alpha G$:

$$(\mathrm{res}^H)^*(\delta(A)) = (\mathrm{ind}_H^G)^*(\delta(A \rtimes_\alpha G)).$$

Proof. For the roof of (1) $\Leftrightarrow$ (2), let β be the action of G/N on $A \rtimes_{\alpha_N} N$ as in the Packer-Raeburn stabilization trick (assuming that A is stable). Then it follows from Proposition 8 that $A \rtimes_\alpha G \cong (A \rtimes_{\alpha_N} N) \rtimes_\beta G/N$ has continuous trace if and only if the corresponding action of G/N on $(A \rtimes_{\alpha_N} N)^\wedge$ is proper and $A \rtimes_{\alpha_N} N$ has continuous trace. By Proposition 1, the first assertion is true if and only if the canonical action of G/N on $(A \rtimes_{\alpha_N} N)^\wedge$ is proper, and, by Theorem 3, the second is true if ω is type I. On the other hand, it is clear that $A \rtimes_{\alpha_N} N$ is not type I if ω is not type I.

Since α_H is pointwise unitary, we know that $A \rtimes_{\alpha_H} H$ has continuous trace. Hence (3) $\Rightarrow$ (1) follows as above from Proposition 8 and the Packer-Raeburn stabilization trick. Furthermore, again by Proposition 8 it is enough to show that the action of G/H on $(A \rtimes_{\alpha_H} H)\widehat{\ }$ is free in order to show (1) $\Rightarrow$ (3). For this let $\rho \times V \in (A \rtimes_{\alpha_H} H)\widehat{\ }$ and $x \in G$ such that $x \notin H$. As we have seen in the proof of Theorem 3, if $x \in N$, then $(\rho \times V) \circ \gamma_x$ is equivalent to $\rho \times h^H_\omega(x)V$, which is not equivalent to $\rho \times V$ since the restriction $h^H_\omega(x)$ is a nontrivial element in $\widehat{H}$ if $x \notin H$. If $x \notin N$, then $(\rho \times V) \circ \gamma_x$ is equivalent to $(\rho \circ \alpha_x) \times W$, for an appropriate representation W of H. But this representation cannot be equivalent to $\rho \times V$, since for $x \notin N$, $\rho \circ \alpha_x$ is not equivalent to ρ.

Suppose now that (A, G, α) satisfies properties (1) to (3). Then, since G/H acts properly on $(A \rtimes_{\alpha_H} H)\widehat{\ }$, it follows easily from the Mackey-Green machine that ind_H^G is a well defined surjective map. Furthermore, ind_H^G factors through a homeomorphism between $(A \rtimes_\alpha H)\widehat{\ }/G$ and $(A \rtimes_\alpha G)\widehat{\ }$ with inverse map given by restricting representations. Now let β be the action of G/H on $A \rtimes_{\alpha_H} H$ given by stabilization trick. Then $\mathrm{ind}_{\{\dot e\}}^{G/H}$ also factors through the same homeomorphism between $(A \rtimes_{\alpha_H} H)\widehat{\ }/G$ and $(A \rtimes_\alpha G)\widehat{\ }$, since by part (2) of Proposition 1 the inverse map is the same. By the arguments used in the proof of Proposition 8 we know that the dual action $\widehat{\beta}$ of $H^\perp$ on $A \rtimes_\alpha G$ is pointwise unitary. Hence

$$\begin{aligned}(A \rtimes_{\alpha_H} H) \otimes \mathcal{K}(L^2(G/H)) &\cong (A \rtimes_\alpha G) \rtimes_{\widehat{\beta}} H^\perp \\ &\cong (\mathrm{res}^{H^\perp})^*(A \rtimes_\alpha G) \\ &= (\mathrm{ind}_H^G)^*(A \rtimes_\alpha G),\end{aligned}$$

since it is well known (see for instance the arguments in the proof of [**24**, Theorem 2.2]) that $\mathrm{res}^{H^\perp}(\pi \otimes \mathrm{id}) = \mathrm{ind}_{\{\dot e\}}^{G/H}(\pi)$ for all $\pi \in (A \rtimes_{\alpha_H} H)\widehat{\ }$, and since $\mathrm{ind}_{\{\dot e\}}^{G/H} = \mathrm{ind}_H^G$ by what we have seen above. Hence, if A is stable, then $A \rtimes_{\alpha_H} H$ is isomorphic to $(\mathrm{res}^H)^*(A)$ and also to $(\mathrm{ind}_H^G)^*(A \rtimes_\alpha G)$, from which follows that $(\mathrm{res}^H)^*(\delta(A)) = (\mathrm{ind}_H^G)^*(\delta(A \rtimes_\alpha G))$. □

Corollary 3. *Let (A, G, α) be a covariant system with constant stabilizer N and constant type I Mackey obstruction ω, such that G acts properly on $\widehat{A}$. Then $A \rtimes_\alpha G$ has continuous trace and we have the relation between $\delta(A \rtimes_\alpha G)$ and $\delta(A)$ stated in Theorem 6.*

Proof. The proof follows immediately from Theorem 6 since the fact that G/N acts properly on $\widehat{A}$ easily implies that G/N acts properly on $(A \rtimes_{\alpha_S} S)\widehat{\ } \cong (A \rtimes_{\alpha_N} N)\widehat{\ }$ (compare the arguments in the proof of [**18**, Corollary 2.1]). □

Remark. At this point we should remark that, as in the case of locally unitary actions on the stabilizer (see the remark on page 23 in [**24**]), the equation for the Dixmier-Douady class in the theorem does not determine $\delta(A \rtimes_\alpha G)$ uniquely. If this would be the case, then $\delta(A \rtimes_\alpha G)$ would be zero if $\delta(A)$ is zero. But by Corollary 2, there are quite a lot of examples which do not have this property.

In the proof of Corollary 3 we have used the fact that a proper action of G/N on $\widehat{A}$ also implies that the action of G/N on $(A \rtimes_\alpha N)^\wedge$ is proper, if (A, G, α) is a covariant system with constant stabilizer N and constant Mackey obstruction ω. If A is commutative, then it is well known that the converse is also true, i.e. if G/N acts properly on $(A \rtimes_{\alpha_N} N)^\wedge = \widehat{A} \times \widehat{N}$, then G/N must also act properly on $\widehat{A}$. In fact, by Williams's description of continuous-trace transformation group algebras [**33**], a necessary condition for $C_0(X) \rtimes G$ having continuous trace is that G act in a generalized sense properly on X. We will now see that an analogue is not true if A is a continuous-trace algebra even in the case where all Mackey obstructions vanish. In fact we will construct a covariant system (A, G, α) with constant stabilizer N such that α_N is pointwise unitary, A and $A \rtimes_\alpha G$ have continuous trace, but G doesn't even act smoothly on $\widehat{A}$. For this we need the following lemma:

Lemma 9. *Let (A, G, α) be a covariant system such that G is abelian, and let $\beta \in \operatorname{Aut} A$ such that β commutes with α_x for all $x \in G$. We define $\widehat{\beta} \in \operatorname{Aut}(A \rtimes_\alpha G)$ by $(\widehat{\beta}(f))(x) = \beta(f(x))$ for all $f \in C_c(G, A)$, and $\widehat{\widehat{\beta}} \in \operatorname{Aut}((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G})$ by $(\widehat{\widehat{\beta}}(g))(\chi) = \widehat{\beta}(g(\chi))$ for all $g \in C_c(\widehat{G}, A \rtimes_\alpha G)$, $\chi \in \widehat{G}$. Then the isomorphism $\Psi : (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \to A \otimes \mathcal{K}(L^2(G))$ of Takai duality transports $\widehat{\widehat{\beta}}$ into $\beta \otimes \mathrm{id}$.*

Proof. Let us identify $\mathcal{K}(L^2(G))$ with $C_0(G) \rtimes_\tau G$, and let

$$\Phi : (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \to C_0(G, A) \rtimes_{\tau \otimes \alpha} G$$

be the isomorphism described in [**14**, Proposition 30] (where τ denotes the action of G on $C_0(G)$ coming from left translation). Then it is easily seen that Φ transports $\widehat{\widehat{\beta}}$ into the automorphism $\widetilde{\beta}$ of $C_0(G, A) \rtimes_{\tau\otimes\alpha} G$ given on $f \in C_c(G \times G, A)$ by $(\widetilde{\beta}(f))(x, y) = \beta(f(x, y))$, $x, y \in G$, viewing $C_c(G \times G, A)$ canonically as as a dense subalgebra of $C_0(G, A) \rtimes_{\tau\otimes\alpha} G$. Now let $\widetilde{\Psi} : C_0(G, A) \rtimes_{\tau\otimes\alpha} G \to A \otimes \mathcal{K}(L^2(G))$ be the corresponding isomorphism (i.e. $\widetilde{\Psi} = \Psi \circ \Phi^{-1}$). Studying Raeburn's alternative proof of the Takai duality theorem [**23**, Theorem 6], we see that the inverse of $\widetilde{\Psi}$ is given on elementary tensors $a \otimes f \in A \otimes C_c(G \times G)$ by

$$\widetilde{\Psi}^{-1}(a \otimes f) = g_{a,f} \in C_c(G \times G, A), \qquad \text{where } g_{a,f}(x, y) = f(x, y)\alpha_y(a),$$

for all $x, y \in G$ (in fact, it is not hard to show directly that the map $a \otimes f \mapsto g_{a,f}$ extends to an isomorphism between $A \otimes (C_0(G) \rtimes_\tau G)$ and $C_0(G, A) \rtimes_{\tau\otimes\alpha} G$). Hence for all $a \in A$ and $f \in C_c(G \times G)$:

$$\widetilde{\Psi}^{-1}(\beta \otimes \mathrm{id}(a \otimes f)) = \widetilde{\Psi}^{-1}(\beta(a) \otimes f) = g_{\beta(a),f} = \widetilde{\beta}(g_{a,f})$$

since

$$\begin{aligned} g_{\beta(a),f}(x, y) &= f(x, y)\alpha_y(\beta(a)) \\ &= \beta(f(x, y)\alpha_y(a)) \\ &= (\widetilde{\beta}(g_{a,f}))(x, y), \end{aligned}$$

for all $x, y \in G$, which finishes the proof. □

Example 1. Let τ^1 and τ^2 be the actions of $\mathbb{Z}$ on $\mathbb{R}$ given by $\tau_n^1(t) = n + t$ and $\tau_n^2(t) = 2\pi n + t$, $n \in \mathbb{Z}$, $t \in \mathbb{R}$, and let us denote the corresponding actions of $\mathbb{Z}$ on $C_0(\mathbb{R})$ by the same letters. Let $A = C_0(\mathbb{R}) \rtimes_{\tau^1} \mathbb{Z}$, and let α be the action of $G = \mathbb{T} \times \mathbb{Z}$ on A given by

$$\alpha_{(z,n)} = \widehat{\tau^1}_z \widehat{\tau^2}_n,$$

$z \in \mathbb{T}$, $n \in \mathbb{Z}$, where $\widehat{\tau^1}$ denotes the dual action of $\mathbb{T} = \widehat{\mathbb{Z}}$ on A, and $\widehat{\tau^2}_n$ is defined as in Lemma 9, for each $n \in \mathbb{Z}$. Since τ^1 is a free and proper action of $\mathbb{Z}$ on $\mathbb{R}$, we know that A has continuous trace. In fact, it follows from Green's theorem [**13**] that A is isomorphic to $C(\mathbb{T}, \mathcal{K})$, where we have identified $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$. Furthermore, by the structure of G we may write $A \rtimes_\alpha G \cong (A \rtimes_{\widehat{\tau^1}} \mathbb{T}) \rtimes_{\widehat{\widehat{\tau^2}}} \mathbb{Z}$, where $\widehat{\widehat{\tau^2}}_n$ is also defined as in Lemma 9 (note that in our case $\widehat{\widehat{\tau^2}}$ is just the restriction of the canonical action of G on $A \rtimes_{\widehat{\tau^2}} \mathbb{T}$ to $\mathbb{Z}$). Hence Lemma 9 shows that $A \rtimes_\alpha G$ is isomorphic to $(C_0(\mathbb{R}) \otimes \mathcal{K}(l^2(\mathbb{Z}))) \rtimes_{\tau^2 \otimes \mathrm{id}} \mathbb{Z}$, which implies that $A \rtimes_\alpha G$ has continuous trace by the fact that τ^2 is also a free and proper action of $\mathbb{Z}$ on $\mathbb{R}$.

Finally, let us note that the action of $\mathbb{T}$ on $\widehat{A}$ is trivial, and that the action of $\mathbb{Z}$ on $\widehat{A}$ is just the irrational rotation given by the angle 2π. Hence (A, G, α) has constant stabilizer $N = \mathbb{T}$, A and $A \rtimes_\alpha G$ have continuous trace, but the action of $G/N = \mathbb{Z}$ on $\widehat{A}$ is not proper. In fact G doesn't even act smoothly on $\widehat{A}$.

Let us remark that this example gives also an answer to a question stated in [**11**] whether it is possible for a crossed product $A \rtimes_\alpha G$ to be of type I, while G does not act smoothly on $\widehat{A}$. Clearly, the group example of Auslander and Moore cited in [**11**] together with the Raeburn-Packer stabilization trick gives also such an example.

We now finish this paper by giving an example which shows that for crossed products $A \rtimes_\alpha G$ with A having continuos trace, it is not even necessary that the stability groups vary continuously in order for $A \rtimes_\alpha G$ to have continuous trace. This is also completely different from the transformation group case, where the condition of continuously varying stability groups is always necessary for $C_0(X) \rtimes G$ to have continuous trace [**33**] (see [**7**] for the case of nonabelian amenable groups).

Example 2. Let $G = \mathbb{Z} \times \mathbb{T}$, and let us define an action of G on $\mathbb{C}$ by $(n, z)w = zw$ for $n \in \mathbb{Z}$, $z \in \mathbb{T}$ and $w \in \mathbb{C}$. Furthermore, let ω be the Heisenberg multiplier on G, which is defined by $\omega((n,z),(m,w)) = z^m$, and let $U : G \to \mathcal{U}(L^2(G))$ be an ω-representation of G. Then, if we denote $\mathcal{K}(L^2(G))$ simply by $\mathcal{K}$, we define an action α of G on $A = C_0(\mathbb{C}) \otimes \mathcal{K}$ by $\alpha = \delta \otimes \operatorname{Ad} U$, where δ denotes the action of G on $C_0(\mathbb{C})$ coming from the action of G on $\mathbb{C}$. The stability groups for the action of G on $\widehat{A}$ are just given by stability groups of the action of G on $\mathbb{C}$, and we have $G_w = \mathbb{Z}$ if $w \neq 0$ and $G_0 = G$. Thus the stability groups do not vary continuously.

However, $A \rtimes_\alpha G$ has continuous trace. To see this, write $A \rtimes_\alpha G$ as $(A \rtimes_{\alpha_\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{T}$. The point is that the action of $\mathbb{T}$ on $(A \rtimes_{\alpha_\mathbb{Z}} \mathbb{Z})\hat{}$ is free, and also proper, since $\mathbb{T}$ is compact and $(A \rtimes_{\alpha_\mathbb{Z}} \mathbb{Z})\hat{}$ is Hausdorff, which follows from fact that $\alpha_\mathbb{Z}$ is pointwise unitary. The fact that $\mathbb{T}$ acts freely on $(A \rtimes_{\alpha_\mathbb{Z}} \mathbb{Z})\hat{}$ can easily be seen from the fact that $\mathbb{T}$ acts freely on $(\mathrm{res}^\mathbb{Z})^{-1}(\mathbb{C} \setminus \{0\})$, since $\mathbb{T}$ acts freely on $\mathbb{C} \setminus \{0\}$, and that $\mathbb{T}$ acts freely on $(\mathrm{res}^\mathbb{Z})^{-1}(\{0\})$, since the Mackey obstruction of α at 0 is given by ω and $\mathbb{Z}$ is a maximally ω-trivial subgroup of G (compare the arguments in the proof of Theorem 3). Hence $A \rtimes_\alpha G$ has continuous trace by Proposition 8. In fact, its spectrum is easily seen to be homeomorphic to $\mathbb{C}$, and $A \rtimes_\alpha G \cong A$ (non-canonically).

References

[1] L. Baggett and A. Kleppner, *Multiplier representations of abelian groups*, J. Funct. Anal., **14** (1973), 299-324.

[2] L. Baggett and T. Sund, *The Hausdorff dual problem for connected groups*, J. Funct. Anal., **43** (1981), 60-68.

[3] M. De Brabanter, *The classification of rational rotation C^*-algebras*, Arch. Math. (Basel), **43** (1984), 79-83.

[4] J. Dixmier and A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France, **91** (1963), 227-284.

[5] S. Echterhoff, *On induced covariant systems*, Proc. Amer. Math. Soc., **108** (1990), 703-706.

[6] ———, *Regularizations of twisted covariant systems and crossed products with continuous trace*, J. Funct. Anal., **116** (1993), 277-313.

[7] ———, *On transformation group C^*-algebras with continuous trace*, Trans. Amer. Math. Soc., **343** (1994), 117-133.

[8] ———, *Morita equivalent twisted actions and a new version of the Packer-Raeburn stabilization trick*, J. London Math. Soc., **50** (1994), 170-186.

[9] S. Echterhoff and E. Kaniuth, *Certain group extensions and twisted covariance algebras with generalized continuous trace*, Harmonic Analysis (Proceedings, 1987) (P. Eymard and J.-P. Pier, eds.), Lecture Notes in Math., vol. **1359**, Springer-Verlag, Berlin, Heidelberg, and New York, 1988, pp. 159-169.

[10] S. Echterhoff, E. Kaniuth, and A. Kumar, *A qualitative uncertainty principle for certain locally compact groups*, Forum Math., **3** (1991), 355-369.

[11] E.C. Gootman, *On certain properties of crossed products*, Operator Algebras and Applications (Kingston, Ont., 1980) (R.V. Kadison, ed.), Proc. Symp. Pure Math., vol. **38**, part 1, Amer. Math. Soc., Providence, R.I., 1982, pp. 311-321.

[12] E.C. Gootman and J. Rosenberg, *The structure of crossed product C^*-algebras: A Proof of the generalized Effros-Hahn conjecture*, Invent. Math., **52** (1979), 283-298.

[13] P. Green, *C^*-algebras of transformation groups with smooth orbit space*, Pacific J. Math., **72** (1977), 71-97.

[14] ———, *The local structure of twisted covariance algebras*, Acta Math., **140** (1978), 191-250.

[15] S. Hurder, D. Olesen, I. Raeburn, and J. Rosenberg, *The Connes spectrum for actions of abelian groups on continuous-trace algebras*, Ergodic Theory and Dynam. Sys., **6** (1986), 541-560.

[16] E. Kaniuth, *Primitive ideal spaces of groups with relatively compact conjugacy classes*, Arch. Math., **32** (1979), 16-24.

[17] J. Liukkonen, *Dual spaces of groups with precompact conjugacy classes*, Trans. Amer. Math. Soc., **180** (1973), 85-108.

[18] D. Olesen and I. Raeburn, *Pointwise unitary automorphism groups*, J. Funct. Anal., **93** (1990), 278-309.

[19] J. Packer and I. Raeburn, *Twisted crossed products of C^*-algebras*, Math. Proc. Camb. Phil. Soc., **106** (1989), 293-311.

[20] ———, *On the structure of twisted group C^*-algebras*, Trans. Amer. Math. Soc., **334** (1992), 685-718.

[21] R.S. Palais, *On the existence of slices for actions of non-compact Lie-groups*, Ann. of Math. (2), **73** (1961), 295-323.

[22] J. Phillips and I. Raeburn, *Crossed products by locally unitary automorphism groups and principal bundles*, J. Operator Theory, **11** (1984), 215-241.

[23] I. Raeburn, *On crossed products and Takai duality*, Proc. Ed. Math. Soc., **31** (1988), 321-330.

[24] I. Raeburn and J. Rosenberg, *Crossed products of continuous-trace C^*-algebras by smooth actions*, Trans. Amer. Math. Soc., **305** (1988), 1-45.

[25] I. Raeburn and D. Williams, *Pull-backs of C^*-algebras and crossed products by certain diagonal actions*, Trans. Amer. Math. Soc., **287** (1985), 755-777.

[26] ———, *Crossed products by actions which are locally unitary on the stabilizers*, J. Funct. Anal., **81** (1988), 385-431.

[27] ———, *Moore cohomology, principal bundles, and actions of groups on C^*-algebras*, Indiana U. Math. J., **40** (1991), 707-740.

[28] ———, *Topological invariants associated with the spectrum of crossed product C^*-algebras*, J. Funct. Anal., **116** (1993), 245-276.

[29] J. Rosenberg, *Homological invariants of extensions of C^*-algebras*, Operator Algebras and Applications (Kingston, Ont., 1980) (R.V. Kadison, ed.), Proc. Symp. Pure Math., vol. **38**, part 1, Amer. Math. Soc., Providence, R.I., 1982, pp. 35-75.

[30] ———, *Some results on cohomology with Borel cochains, with applications to group actions on operator algebras*, Advances in invariant subspaces and other results of operator theory (Proceedings, Timişoara and Herculane, Romania, 1984) (R.G. Douglas, C.M. Pearcy, B.Sz.-Nagy, F.H. Vasilescu, and D. Voiculescu, eds.), Operator Theory: Advances and Applications, vol. **17**, Birkhäuser, Basel, Boston, and Stuttgart, 1986, pp. 301-330.

[31] I.E. Schochetman, *The topology of certain group extensions*, Adv. in Math., **35** (1980), 113-128.

[32] A. Wassermann, *Equivariant K-theory I: Compact transformation groups with one orbit type*, Preprint, 1981.

[33] D. Williams, *Transformation group C^*-algebras with continuous trace*, J. Funct. Anal., **41** (1981), 40-76.

Received December 7, 1992 and revised February 15, 1994. The first author was partially supported by a research fellowship from the Germany Research Foundation. The second author was partially supported by NSF Grant DMS-90-02642.

FACHBEREICH MATHEMATIK-INFORMATIK
UNIVERSITÄT-GESAMTHOCHSCHULE PADERBORN
D-33098 PADERBORN
GERMANY
E-MAIL ADDRESS: ECHTER@UNI-PADERBORN.DE

AND

UNIVERSITY OF MARYLAND
COLLEGE PARK, MD 20742
E-MAIL ADDRESS: JMR@WINIFRED.UMD.EDU

PACIFIC JOURNAL OF MATHEMATICS

Founded by E. F. Beckenbach (1906-1982) and F. Wolf (1904-1989)

Manuscripts must be prepared in accordance with the instructions provided on the inside back cover.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $215.00 a year (10 issues). Special rate: $108.00 a year to individual members of supporting institutions.

Subscriptions, orders for back issues published within the last three years, and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at the University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 6143, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,
Berkeley, CA 94720, A NON-PROFIT CORPORATION
This publication was typeset using AMS-LATEX,
the American Mathematical Society's TEX macro system.

PACIFIC JOURNAL OF MATHEMATICS

Volume 170 No. 1 September 1995

Generalized generalized spin models (four-weight spin models) 1
EIICHI BANNAI and ETSUKO BANNAI

Fine structure of the Mackey machine for actions of abelian groups with constant Mackey obstruction 17
SIEGFRIED ECHTERHOFF and JONATHAN ROSENBERG

The corestriction of valued division algebras over Henselian fields. I 53
YOON SUNG HWANG

The corestriction of valued division algebras over Henselian fields. II 83
YOON SUNG HWANG

The cohomology of expansive $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups 105
ANATOLE KATOK and KLAUS SCHMIDT

The Anosov theorem for exponential solvmanifolds 143
EDWARD KEPPELMANN and CHRISTOPHER K. MCCORD

Projections of measures on nilpotent orbits and asymptotic multiplicities of K-types in rings of regular functions. I 161
DONALD RAYMOND KING

On almost-everywhere convergence of inverse spherical transforms 203
CHRISTOPHER MEANEY and ELENA PRESTINI

Characters of supercuspidal representations of $SL(n)$ 217
FIONA ANNE MURNAGHAN

The cohomology of higher-dimensional shifts of finite type 237
KLAUS SCHMIDT

On Gorenstein surface singularities with fundamental genus $p_f \geqq 2$ which satisfy some minimality conditions 271
TADASHI TOMARU