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PROJECTIONS OF MEASURES ON NILPOTENT ORBITS AND ASYMPTOTIC MULTIPLICITIES OF K-TYPES IN RINGS OF REGULAR FUNCTIONS I

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Let G be the adjoint group of a real semi-simple Lie algebra $\underline{g}$ and let K be a maximal compact subgroup of G. K_C, the complexification of K, acts on $\underline{p}_C^*$, the complexified cotangent space of G/K at eK. If $\mathcal{O}$ is a nilpotent K_C orbit in $\underline{p}_C^*$, we study the asymptotic behavior of the K-types in the module $R[\overline{\mathcal{O}}]$, the regular functions on the Zariski closure of $\mathcal{O}$. We show that in many cases this asymptotic behavior is determined precisely by the canonical Liouville measure on a nilpotent G orbit in $\underline{g}^*$ which is naturally associated to $\mathcal{O}$. We provide evidence for a conjecture of Vogan stating that this relationship is true in general. Vogan's conjecture is consistent with the philosophy of the orbit method for representations of real reductive groups.

1. Introduction.

Let G be the adjoint group of a semi-simple Lie algebra $\underline{g}$, and let G_C be the adjoint group of $\underline{g}_C$, the complexification of $\underline{g}$. If $\underline{g} = \underline{k} \oplus \underline{p}$ is Cartan decomposition of $\underline{g}$, then $\underline{g}_C = \underline{k}_C \oplus \underline{p}_C$ is the corresponding vector space decomposition of $\underline{g}_C$. K is the connected subgroup of G with Lie algebra $\underline{k}$. K_C is the connected subgroup of G_C with Lie algebra $\underline{k}_C$. K is a maximal compact subgroup of G. T is a maximal torus in K with Lie algebra $\underline{t}$. $\underline{g}^* = \mathrm{Hom}_R(\underline{g}, R)$. Define $\underline{k}^*$ and $\underline{t}^*$ similarly. $\Delta^+ = \Delta_c^+$ is a positive system of roots for the pair $(\underline{k}_C, \underline{t}_C)$. $\hat{K}$ is the corresponding set of dominant integral weights (i.e. the set of equivalence classes of finite dimensional irreducible K modules) and ρ =half the sum of the roots in Δ^+. Using the Killing form, identify $\underline{g}$ and $\underline{g}^*$ and define the projection map $J : \underline{g}^* \to \underline{k}^*$.

Let $\Omega \subseteq \underline{g}^*$ be a nilpotent co-adjoint G orbit. Ω is a simplectic manifold with canonical Liouville measure β_Ω which is G invariant. The distribution $J_*(\beta_\Omega)$, the pushforward of β_Ω to $\underline{k}^*$, is well defined by formula: $J_*(\beta_\Omega)(f) = \beta_\Omega(f \circ J)$ if $f \in C_c^\infty(\underline{k}^*)$, because the set $\mathrm{supp}(f \circ J) \cap \Omega$ is bounded. Let $\mathcal{O} = \mathbf{c}(\Omega)$ be the K_C nilpotent orbit in $\underline{p}_C^*$ which is the Cayley transform of Ω. (See Section 2.3 for the definition of $\mathbf{c}$.) $\overline{\mathcal{O}}$ is the Zariski closure of

$\mathcal{O}$. $R[\overline{\mathcal{O}}]$, the co-ordinate ring of $\overline{\mathcal{O}}$, is a completely reducible K module. If $\mu \in \hat{K}, V_\mu$ is the irreducible K module with highest weight μ, and $m(\mu) =$ the multiplicity of V_μ in $R[\overline{\mathcal{O}}]$. β_μ is the canonical Liouville measure on the orbit $\Omega(\mu) = K \cdot \{-i(\mu+\rho)\}$ in $\underline{k}^*$. If $f \in C_c^\infty(\underline{k}^*)$ and $t > 0$, let $\tilde{f}_t(x) = f(t^{-1}x)$, and $\beta_\mu(f) = \int_{\Omega(\mu)} f\beta_\mu$.

Theorem 1. *For every $f \in C_c^\infty(\underline{k}^*), \lim_{t\to\infty} t^{-\dim_C \mathcal{O}} \sum_{\mu\in\hat{K}} m(\mu)\ \beta_\mu(\tilde{f}_t)$ exist and is finite. Let $\mathbf{M}_{\overline{\mathcal{O}}}$ be the distribution on $\underline{k}^*$ whose value at $f \in C_c^\infty(\underline{k}^*)$ is given by the previous limit. $\mathbf{M}_{\overline{\mathcal{O}}}$ is not identically zero.*

$\mathbf{M}_{\overline{\mathcal{O}}}$ is called the asymptotic multiplicity measure associated to $R[\overline{\mathcal{O}}]$.

Theorem 2. *Suppose that $\underline{g}$ is complex, so that K_C may be identified with G. If Ω is a Richardson nilpotent orbit, then there is a non-zero constant c_Ω such that*

$$J_*(\beta_\Omega) = c_\Omega \cdot \mathbf{M}_{\overline{\Omega}}.$$

Thus when $\underline{g}$ is complex, for many nilpotent orbits (all of them in case $G = SL(n, C)$), $\mathbf{M}_{\overline{\Omega}}$, which measures the asymptotic behavior of m, is determined by the canonical Liouville measure on Ω.

Let $\mathcal{N}[\underline{p}_C^*]$ denote the nilpotent cone in $\underline{p}_C^*$, the complex dual of $\underline{p}_C$. Now suppose that $\mathcal{O}$ is a K_C orbit in $\mathcal{N}[\underline{p}_C^*]$ which is the Cayley transform of the G orbit Ω in $\mathcal{N}[\underline{g}^*]$ (the nilpotent cone in $\underline{g}^*$). Then we have:

Theorem 3. *($\underline{g}$ is real) If $\mathcal{O}$ is K_C nilpotent orbit in $\mathcal{N}[\underline{p}_C^*]$ which is even, then there is a non-zero constant c_Ω such that $J_*(\beta_\Omega) = c_\Omega \cdot \mathbf{M}_{\overline{\mathcal{O}}}$.*

($\mathcal{O} = K_C \cdot e$ is said to be even if the semi-simple element in the normal triple parametrizing $\mathcal{O}$ (see Section 2.3) has only even eigenvalues on $\underline{g}_C^*$. For example, the K_C nilpotent orbits of maximal dimension are even.)

Theorem 2 and 3 and many other examples suggest the vailidity of the following conjecture:

Conjecture (Vogan). If $\mathcal{O}$ is a K_C nilpotent orbit in $\mathcal{N}[\underline{p}_C^*]$, which is the Cayley transform of the nilpotent G orbit Ω in $\mathcal{N}[\underline{g}^*]$ then for some non-zero constant c_Ω, $J_*(\beta_\Omega) = c_\Omega \cdot \mathbf{M}_{\overline{\mathcal{O}}}$.

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2. Preliminaries.

2.1. Notation and basic conventions. $\underline{g}$ is a real semisimple lie algebra, and let $\underline{k}, \underline{p}, \underline{k}_C, \underline{p}_C, G, K, G_C$, and K_C be as in section 1. Let θ be the Cartan involution giving rise to the decomposition $\underline{g} = \underline{k} \oplus \underline{p}$. Let σ be the conjugation of $\underline{g}_C$ relative to $\underline{g}$. If L is a Lie subgroup of G_C, then L_o will denote the connected component of the identity of L. θ and σ extend to automorphisms of G_C. So L^θ and L^σ will denote fixed point sets of θ and σ in L. If L acts on a vector space V, and $v \in V$, then L^v denotes the centralizer of v in L.

$\mathbf{B}$ denotes the killing form of $\underline{g}$. We fix a maximal torus $\underline{t} \subset \underline{k}$, the corresponding connected subgroup in K will be denoted by T. In general, if V is a real vector space we write $V^* = \mathrm{Hom}_R(V, R)$. But, we write $V_C^* = \mathrm{Hom}_C(V_C, C)$, the complex dual of the complexification of V.

We identify $\underline{g}^*$ with $\underline{g}$ by means of $\mathbf{B}$. We will make use of the projection map $J : \underline{g}^* \to \underline{k}^*$. J is K equivariant. Since $\mathbf{B}$ is negative definite on $\underline{k}$ and allows us to identify $\underline{k}$ and $\underline{k}^*$, we define a positive definite bilinear form (,) on $\underline{k}^*$ by setting $(\gamma, v) = -\mathbf{B}(\gamma, v), \forall \gamma, v \in \underline{k}^*$.

Let $\Delta_c = \Delta(\underline{k}_C, \underline{t}_C)$ denote the roots of the pair $(\underline{k}_C, \underline{t}_C)$, $\Delta_c^+ = \Delta_c^+(\underline{k}_C, \underline{t}_C)$ will denote a choice of a set of positive roots, and $|\Delta_c^+| =$ the cardinality of Δ_c^+. When there is no possibility of confusion, we will drop the subscript on Δ_c^+. For each α in Δ_c, h_α is the corresponding element of $i\underline{t}$ under the identification of $\underline{t}$ and $\underline{t}^*$ provided by the restriction of $\mathbf{B}$ to $\underline{k}$. $\mathbf{D}_\alpha$ is the differential operator defined on $\underline{t}^*$ by the formula: $\mathbf{D}_\alpha(g)(x) = \frac{d}{dt}(g(x - i\alpha t))_{|t=0}$. If $\nu \in \underline{t}^* - (0)$, and $f \in C_c(\underline{t}^*)$, then $\mathbf{H}_\nu = \int_0^\infty f(s\nu)\, ds$. $\mathbf{H}_\nu$ is the Heaviside distribution corresponding to ν.

2.2. Measures and integral formulae. We choose Lebesgue measures dX on $\underline{k}$ and dH on $\underline{t}$. We will use the same notation for the corresponding Lebesgue measures on $\underline{k}^*$ and $\underline{t}^*$. Whenever a Lebesgue measure is defined on a Lie subalgebra $\underline{c}$ of $\underline{g}$ we will choose the left invariant Haar measure on the corresponding group C so that the Lebesgue measure and the Haar measure correspond under the exponential map. We denote by $v(T)$ the integral $\int_T 1\, dt$ where dt is the Haar measure on T chosen according to the aforementioned convention.

If $\xi \in \underline{g}^*$, and $\Omega = G \cdot \xi$ is the corresponding co-adjoint orbit, we recall the definition of the canonical Liouville measure β_Ω on Ω. $T_\xi(\Omega)$, the tangent space to Ω at ξ, may be identified with $\underline{g}/\underline{g}^\xi$. $\underline{g}/\underline{g}^\xi$ in turn may be identified with its image $\underline{g} \cdot \xi = ad(\underline{g})(\xi)$ in $\underline{g}^*$ under the injection $\overline{u} \to u \cdot \xi$, where $u \in \underline{g}$ represents the coset $\overline{u} \in \underline{g}/\underline{g}^\xi$ and $u \cdot \xi = -\xi \circ ad(u)$ is the co-adjoint representation.

Now define the bilinear form b_ξ on $\underline{g} \cdot \xi$ by the formula: $b_\xi(u \cdot \xi,\ v \cdot \xi) = \xi([u, v])$. Then b_ξ is well-defined, skew-symmetric and non-degenerate. $\dim_R \underline{g} \cdot \xi = \dim_R(\underline{g}/\underline{g}^\xi) = 2k$. b_ξ^k, the wedge product of b_ξ with itself k times, is a non zero $2k$ form. Since $\underline{g} \cdot \xi = T_\xi(\Omega)$, $b_\Omega^k : \xi \to b_\xi^k$ defines a volume form on Ω which is G-invariant. We normalize the corresponding G-invariant measure by multiplication with $((2\pi)^k k!)^{-1}$ to get the canonical measure β_Ω. We will often write $d_\xi = d_{G/G^\xi}$ in place of $\beta_{G\cdot\xi}$. Suppose that $f \in C_c(\underline{g}^*)$ and $\operatorname{supp} f \cap G \cdot \xi$ and $\operatorname{supp} f \cap G \cdot t\xi$ are both compact. If we set $L_t(f)(x) = f(tx)$, then $\beta_{G\cdot\xi}$ and $\beta_{G\cdot t\xi}$ are related as follows:

$$t^k \int_{G\cdot\xi} L_t(f)\beta_{G\cdot\xi} = \int_{G\cdot t\xi} f\beta_{G\cdot t\xi} \tag{2.2.1}$$

where we recall that $k = \frac{1}{2}(\dim_R \Omega)$.

Let $\underline{k} = \underline{t}+\underline{r}$ where $\underline{r}$ is the orthogonal complement of $\underline{t}$ in $\underline{k}$. Let W denote the Weyl group of the pair $(\underline{k}, \underline{t})$. The dimension of $\underline{r} = |\Delta(\underline{k}_C, \underline{t}_C)|$ and so is even. If $\eta \in \underline{t}^*$, let j_η be the two form on $\underline{r}$ defined by $j_\eta(X,Y) = \eta([X,Y])$. If η is regular with respect to $\underline{k}$, then the form j_η is non-degenerate. Choose a nonvanishing form μ of degree $|\Delta(\underline{k}_C, \underline{t}_C)|$ on $\underline{r}$ such that $dX = |\mu| \cdot dH$. Define a polynomial function π on $\underline{t}^*$ by the formula:

$$j_\eta \wedge j_\eta \wedge \cdots \wedge j_\eta = \pi(\eta)(|\Delta^+|)!\mu.$$
$$(|\Delta^+| \text{ factors })$$

π is a W skew-invariant polynomial function on $\underline{t}^*$ depending on μ and is proportional to $\pi_c^+ = \prod_{\alpha\in\Delta_c^+} h_\alpha$. From now on assume that μ is chosen so that $\pi(\eta)$ is a positive multiple of $\pi_c^+(i\eta)$ for $\eta \in \underline{t}^*$ and denote the corresponding π by π^+.

Normalize the K invariant measure $d_{K/T}$ so that if $f \in C_c(K \cdot \underline{t}^*_{\text{reg}})$, we have

$$\int_{\underline{k}^*} f(X)dX = \int_{K/T}\int_{\underline{t}^*_{\text{reg}}} |\pi_c^+(H)|^2 f(Ad(k)H)d_{K/T}(\dot{k})dH.$$

We define a map A^+ from $C_c(\underline{k}^*)$ to $C_c(\underline{t}^*)$ by the prescription:

$$A^+\phi(\eta) = \frac{1}{|W|\operatorname{vol}(T)} \cdot \frac{\pi^+(\eta)}{(2\pi)^{|\Delta^+|}} \int_K \phi(k\cdot\eta)\, dk. \tag{2.2.2}$$

Note that $A^+\phi$ is W skew-invariant. A^+ gives an isomorphism between $C_v^\infty(\underline{k}^*)^{K-\text{inv.}}$ and $C_c^\infty(\underline{t}^*)^{W-\text{skew-inv}}$. $(A^+)^t$, the inverse of the transpose of A^+, gives an isomorphism between K invariant tempered distributions on $\underline{k}^*$ and W skew-invariant distributions on $\underline{t}^*$ ([**Sen2**]).

We say that a K invariant measure v on $\underline{k}^*$ corresponds to a W skew measure Υ on $\underline{t}^*$ provided the following equation holds for all $\phi \in C_c^\infty(\underline{k}^*)$:

$$\int_{\underline{k}^*} \phi(\zeta)\, dv(\zeta) = \int_{\underline{t}^*} A^+(\phi)(\xi)\, d\Upsilon(\xi). \tag{2.2.3}$$

Example 2.2.4. Let $\lambda \in \underline{t}^*$ be regular and set $v = \beta_{K\cdot\lambda}$ and $B_\lambda = \sum_{w\in W} \epsilon(w)w\cdot$

δ_λ. $\beta_{K\cdot\lambda}$ is the Liouville measure on $K\cdot\lambda$ as defined above. Then we have

$$\begin{aligned}\int_{\underline{t}^*} A^+(\phi)(\xi)dB_\lambda(\xi) &= \sum_{w\in W}\epsilon(w)\langle w\cdot\delta_\lambda, A^+(\phi)\rangle\\ &= \sum_{w\in W}\epsilon(w)A^+(\phi)(w\lambda)\\ &= \sum_{w\in W}\epsilon(w)\frac{1}{|W|\operatorname{vol}(T)}\cdot\frac{\pi^+(w\lambda)}{(2\pi)^{|\Delta^+|}}\int_K\phi(k\cdot w\lambda)dk\\ &= \frac{1}{\operatorname{vol}(T)}\cdot\frac{\pi^+(\lambda)}{(2\pi)^{|\Delta^+|}}\int_K\phi(k\cdot\lambda)dk,\end{aligned}$$

because $K\cdot w\lambda = K\cdot\lambda$ and π^+ is W skew-invariant. It can be shown that

$$\frac{1}{\operatorname{vol}(T)}\cdot\frac{\pi^+(\lambda)}{(2\pi)^{|\Delta^+|}}\int_K\phi(k\cdot\lambda)dk = c_{K,T}\int_{K\cdot\lambda}\phi\,\beta_{K\cdot\lambda},$$

where $c_{K,T}$ is a non-zero constant that does not depend on λ. So B_λ corresponds to $c_{K,T}\beta_{K\cdot\lambda}$ under map $(A^+)^t$.

Remark 2.2.5. It is well known that if λ is regular, $\int_{K\cdot\lambda} 1\, d\beta_{K\cdot\lambda} = C(i\lambda)$ where $C(i\lambda)$ is the "dimension function": $\prod_{\alpha\in\Delta^+}\frac{\mathbf{B}(i\lambda,\alpha)}{\mathbf{B}(\rho,\alpha)}$. Therefore, the constant $c_{K,T}$ (in Example 2.2.4) $= \frac{\operatorname{vol}(K)}{\operatorname{vol}(T)}\cdot\frac{\pi^+(\lambda)}{C(i\lambda)(2\pi)^{|\Delta^+|}}$ (which is independent of $\lambda\in\underline{t}^*_{\text{reg.}}$).

The Fourier transform on $\underline{k}^*$ can be defined as follows.

$$\hat{f}(\gamma) = \int_{\underline{k}^*} f(X)e^{iB(\gamma,X)}\,dX = \int_{\underline{k}^*} f(X)e^{-i(\gamma,X)}\,dX. \tag{2.2.6}$$

In the same way, we define the Fourier transform on $\underline{t}^*$. We record the following facts about the Fourier transforms of distributions on $\underline{t}^*$ for use later.

If $f\in C_c^\infty(\underline{t}^*)$, $\mathbf{T}$ is a distribution on $\underline{t}^*$, and $w\in W$, set $f^w(x) = f(w^{-1}x)$ and $\mathbf{T}^w(f) = \mathbf{T}(f^w)$. Then $\widehat{(f^w)} = (\hat{f})^w$ and if $\mathbf{T}$ is tempered, we have $\widehat{\mathbf{T}^w} = (\widehat{\mathbf{T}})^w$. ($\widehat{\mathbf{T}}$ is the Fourier transform of $\mathbf{T}$. See Section 3.) Thus a tempered distribution is W invariant (resp. skew invariant) if and only if its Fourier transform is W invariant (resp. skew invariant). Lastly, it is easy to verify that $\widehat{\delta}_{-iw(\mu+\rho)} = e^{\mathbf{B}(w(\mu+\rho),\cdot)}$.

2.3. Results on nilpotents. $\mathcal{N}[\underline{g}]$ (respectively $\mathcal{N}[\underline{p}_C]$) will denote the set of nilpotent elements of $\underline{g}$ (respectively $\underline{p}_C$). It is known that $\mathcal{N}[\underline{g}]$(respectively $\mathcal{N}[\underline{p}_C]$) is a finite union of G (respectively K_C) orbits.

An ordered triplet $\{Z_1, Z_2, Z_3\}$ of elements in $\underline{g}_C$ is said to be a triple if the following commutation relations are satisfied: $[Z_1, Z_2] = 2Z_2, [Z_1, Z_3] = -2Z_3$, and $[Z_2, Z_3] = Z_1$. Let us fix a triple $\{H, E, F\}$ in $\underline{g}$ with the property that $\theta(H) = -H$, and $\theta(E) = -F$. A triple $\{H, E, F\}$ in $\underline{g}$ with these properties will be called a $\underline{g}$ Cayley triple. Every nilpotent E' in $\underline{g}$ is G conjugate to a nilpotent E lying in such a triple. Now define a new triple $\{c(H), c(E), c(F)\}$, which we will call the Cayley transform of $\{H, E, F\}$, as follows: $c(H) = i(E - F), c(E) = (1/2)[H - i(E + F)]$, and $c(F) = (1/2)[H + i(E + F)]$. The triple $\{c(H), c(E), c(F)\}$ is normal in the sense of Kostant and Rallis [**KR**], i.e. $c(H) \in \underline{k}_C$, and $c(E), c(F) \in \underline{p}_C$.

Let $\Omega = \Omega[E]$ be the G orbit of E in $\underline{g}$ and $\mathcal{O} = \mathcal{O}[c(E)]$ be the K_C orbit of $c(E)$ in $\underline{p}_C$. The assignment of Ω to $\mathcal{O}$ defines a bijective map $\mathbf{c}$, also called the Cayley transform, from the set of G conjugacy classes in $\mathcal{N}[\underline{g}]$ to the set of K_C conjugacy classes in $\mathcal{N}[\underline{p}_C]$. Proofs of this result can be found in [**Sec1**] and [**Dj**]. (Note that $\Omega[E]$ and $\mathcal{O}[c(E)]$ lie in the same G_C orbit.)

$\mathbf{c}^{-1}$, the inverse of the Cayley transform, is obtained as follows. Let $\mathcal{O}[e']$ be a K_C conjugacy class in $\mathcal{N}[\underline{p}_C]$. Then $\mathcal{O}[e']$ contains an element e which lies in a normal triple $\{x, e, f\}$ with the additional property that $\sigma(e) = f$. A normal triple with these properties will be called a $\underline{p}_C$ Cayley triple. Now define a triple $\{c^{-1}(x), c^{-1}(e), c^{-1}(f)\}$ in $\underline{g}$ with $c^{-1}(x) = e + f, c^{-1}(e) = (i/2)[e - f - x]$ and $c^{-1}(f) = (i/2)[e - f + x]$. (Then $\{c^{-1}(x), c^{-1}(e), c^{-1}(f)\}$ is a $\underline{g}$ Cayley triple.) Thus $\mathbf{c}^{-1}$ assigns $\mathcal{O}[e']$ to $\Omega[c^{-1}(e)]$.

3. Homogeneous distributions, group actions and differential operators.

Let $f \in C_c^\infty(R^n)$ and $\mathbf{m} \in C_c^\infty(R^n)'$, the space of distributions. Then if $t > 0$, define the functions f_t as follows: $f_t(x) = t^{-n} f(t^{-1}x)$(for all $x \in R^n$). We define $\hat{f}$, the Fourier transform of f by $\hat{f}(\gamma) = \int_{R^n} f(x) e^{-i(\gamma, x)}\, \mathrm{dm}(x)$. Here $(\cdot, \cdot) =$ the scalar product on R^n and dm is Lebesgue measure on R^n. Under appropriate hypotheses on f, the Fourier inversion theorem holds. That is, $f(x) = (2\pi)^{-n} \int_{R^n} \hat{f}(\gamma) e^{i(x, \gamma)}\, \mathrm{dm}(\gamma)$. It is easy to show that $(\hat{f_t}) = t^{-n}(\hat{f})_{t^{-1}}$ or equivalently $(\hat{f})_{t^{-1}} = t^n \hat{f}_t$.

$\mathcal{S} = \mathcal{S}(R^n)$ is the Schwartz space of R^n. If $\mathbf{m}$ is tempered, we define $\hat{\mathbf{m}}$, the Fourier transform of $\mathbf{m}$, by $\hat{\mathbf{m}}(f) = \mathbf{m}(\hat{f})$ for all $f \in \mathcal{S}$.

A distribution $\mathbf{M}$ on R^n is said to be homogeneous of degree s if $\mathbf{M}(f_t) = t^s \mathbf{M}(f)$.

Remark 3.1.1. A homogeneous distribution on R^n is tempered. (See [**Do**].)

Lemma 3.1.2. *Let $\mathbf{M}$ be a distribution on R^n which is homogeneous of degree s, then $\hat{\mathbf{M}}$, the Fourier transform of $\mathbf{M}$, is homogeneous of degree*

$-n-s$.

Proof. This depends on the fact that $\mathbf{M}(f_t) = t^s\mathbf{M}(f)$ for $f \in \mathcal{S}$. □

Example 3.1.3(a). $\delta = \delta_0$, the Dirac delta function at 0, is homogeneous of degree $-n$. Its Fourier transform, a constant multiple of Lebesgue measure, is homogeneous of degree $0 = -n - (-n)$.

Example 3.1.3(b). The Heaviside distribution $\mathbf{H}_v$, corresponding to a nonzero vector $v \in R^n$, (see Section 2.1) is homogeneous of degree $-n+1$.

Lemma 3.1.4. *Let* $\mathbf{M}$ *be a homogeneous distribution on* R^n *of degree* s *and let* $\mathbf{D}$ *be a constant coefficient, homogeneous differential operator of order* ℓ *then* $D \cdot \mathbf{M}$ *is homogeneous of degree* $s - \ell$. (*Note that by definition* $\mathbf{D} \cdot \mathbf{M}(f) = (-1)^\ell \mathbf{M}(\mathbf{D}f)$.)

Proposition 3.1.5. *Let* $\mathbf{T}$ *and* $\mathbf{S}$ *be homogeneous distributions on* R^n *of degree* k *and* ℓ *respectively. If* $\mathbf{T} * \mathbf{S}$ (*the convolution of* $\mathbf{T}$ *and* $\mathbf{S}$) *is defined then* $\mathbf{T} * \mathbf{S}$ *is homogeneous of degree* $k + \ell + n$.

Proof. It suffices to show that $\mathbf{T}*\mathbf{S}$ satisfies the Euler equation $(k+\ell+n)\Phi = \sum_{j=1}^{n} x_j \dfrac{\partial \Phi}{\partial x_j}$. This is done by computing $\sum_{j=1}^{n} x_j \dfrac{\partial(\mathbf{T} * \mathbf{S})}{\partial x_j}$ using (1) the fact that for each j, where $1 \le j \le n$, we have the equality:

$$x_j \frac{\partial(\mathbf{T} * \mathbf{S})}{\partial x_j} = \left(x_j \frac{\partial \mathbf{T}}{\partial x_j}\right) * \mathbf{S} + \mathbf{T} * \left(x_j \frac{\partial \mathbf{S}}{\partial x_j}\right) + \mathbf{T} * \mathbf{S} \tag{3.1.6}$$

and (2) the homogeneity of $\mathbf{T}$ and $\mathbf{S}$. Formula (3.1.6) follows by taking the Fourier transform of both sides. □

Corollary 3.1.7. *Let* $\mathbf{T}_i, i = 1, 2, \dots, r$, *be homogeneous distributions on* R^n *such that* $\deg \mathbf{T}_i = k_i$. *Assume that the convolution* $\mathbf{T}_1 * \cdots * \mathbf{T}_r$ *is defined. Then* $\deg(\mathbf{T}_1 * \cdots * \mathbf{T}_r) = \sum_{i=1}^{r} k_i + (r-1)n$.

Let us now consider the action of $\mathbf{GL}(\mathbf{n}, \mathbf{R})$ (respectively $\mathbf{O}(\mathbf{n})$) on functions and constant coefficient differential operators (respectively distributions) on R^n.

Suppose f is a smooth function, $\mathbf{D}$ is a constant coefficient differential operator, and $\mathbf{T}$ is a C^∞ distribution on R^n. Let $a, b \in \mathbf{GL}(\mathbf{n}, \mathbf{R})$ and $q, s \in \mathbf{O}(\mathbf{n})$. Then we define actions of $\mathbf{GL}(\mathbf{n}, \mathbf{R})$ on functions and constant coefficient differential operators as follows: $(a \cdot f)(x) = f(a^{-1}x)$; and $[(a \cdot \mathbf{D})(f)](x) = [a^{-1} \cdot (\mathbf{D}(a \cdot f))](x)$. One then checks that $(ab) \cdot f = a \cdot (b \cdot f)$, and $(ab) \cdot \mathbf{D} = a \cdot (b \cdot \mathbf{D})$. One also defines an action of $\mathbf{O}(\mathbf{n})$ on smooth distributions as follows: $(q\cdot\mathbf{T})(f) = \mathbf{T}(q^{-1}\cdot f)$, and one checks that $(qs)\cdot\mathbf{T} = q\cdot(s\cdot\mathbf{T})$.

Now assume that $\mathbf{D}$ is homogeneous of degree d. Let us compute $[q\cdot(\mathbf{D}\cdot\mathbf{T})](f)$. Well $[q\cdot(\mathbf{D}\cdot\mathbf{T})](f) = [\mathbf{D}\cdot\mathbf{T}](q^{-1}\cdot f) = (-1)^d\mathbf{T}[\mathbf{D}(q^{-1}\cdot f)] = (-1)^d\mathbf{T}[q^{-1}\cdot(q\cdot\mathbf{D}(q^{-1}\cdot f))] = (-1)^d[q\cdot\mathbf{T}]\{(q^{-1}\cdot\mathbf{D})(f)\} = \{(q^{-1}\cdot\mathbf{D})\cdot(q\cdot\mathbf{T})\}(f)$. So $q\cdot(\mathbf{D}\cdot\mathbf{T}) = (q^{-1}\cdot\mathbf{D})\cdot(q\cdot\mathbf{T})$.

We also need a result on the action of differential operators on families of distributions depending on a parameter $\lambda\in R^n$.

Remark 3.1.8. Let $\mathbf{D}$ be a homogeneous differential operator with constant coefficients of degree d. Suppose $a\in\mathbf{GL(n,R)}$ and $a^{-1}\cdot\mathbf{D} = \zeta(a^{-1})\mathbf{D}$ where $\zeta(a^{-1})$ is a scalar. Then $\mathbf{D}_\lambda\delta_{a\lambda} = (-1)^d\zeta(a^{-1})\mathbf{D}\cdot\delta_{a\lambda}$. (If $\mathbf{T}_\lambda$ is a family of distributions depending on a parameter $\lambda\in R^n$, then $\mathbf{D}_\lambda\mathbf{T}_\lambda(f)$ is defined to be $\mathbf{D}$ applied to the expression $\mathbf{T}_\lambda(f)$ where differentiation is in the variable λ.) We argue as follows. First note that $\mathbf{D}_\lambda\delta_\lambda = (-1)^d\mathbf{D}\cdot\delta_\lambda$ because $\langle\mathbf{D}_\lambda\delta_\lambda, f\rangle =$ (by definition) $\mathbf{D}_\lambda(\langle\delta_\lambda, f\rangle) = \mathbf{D}_\lambda(f(\lambda)) = (\mathbf{D}f)(\lambda) = \langle\delta_\lambda, \mathbf{D}f\rangle = \langle(-1)^d\mathbf{D}\cdot\delta_\lambda, f\rangle$. Similarly, $\langle\mathbf{D}_\lambda\delta_{a\lambda}, f\rangle = \mathbf{D}_\lambda(f(a\lambda)) = \mathbf{D}_\lambda[(a^{-1}\cdot f)(\lambda)] = \{\mathbf{D}[(a^{-1}\cdot f)]\}(\lambda) = \{a^{-1}\cdot[(a^{-1}\cdot\mathbf{D})(f)]\}(\lambda) = \{a^{-1}\cdot[\zeta(a^{-1})\mathbf{D}(f)]\}(\lambda) = \zeta(a^{-1})[a^{-1}\cdot(\mathbf{D}f)](\lambda) = \zeta(a^{-1})(\mathbf{D}f)(a\lambda) = \zeta(a^{-1})\langle\delta_{a\lambda}, \mathbf{D}f\rangle = \zeta(a^{-1})\langle(-1)^d\mathbf{D}\cdot\delta_{a\lambda}, f\rangle$. In general, $\mathbf{D}_\lambda\delta_{a\lambda} = (-1)^d(a^{-1}\cdot\mathbf{D})\cdot\delta_{a\lambda}$.

The following lemma will be used in Section 6.

Lemma 3.1.9. *Let $f\in C_c^\infty(R^n), \mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\in R^n$ with all the $\mathbf{v}_i\neq\mathbf{0}$. Then,*

$$\lim_{t\to\infty} t^{n-m}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty} f_t(\mathbf{u}+j_1\mathbf{v}_1+j_2\mathbf{v}_2+\cdots+j_m\mathbf{v}_m)$$
$$= \mathbf{H}_{v_1}*\mathbf{H}_{v_2}*\cdots*\mathbf{H}_{v_m}(f).$$

Proof. If $m=1$, the lemma asserts that

$$\lim_{t\to\infty} t^{n-1}\sum_{j=0}^{\infty} f_t(\mathbf{u}+j\mathbf{v}) = \lim_{t\to\infty} t^{-1}\sum_{j=0}^{\infty} f[t^{-1}(\mathbf{u}+j\mathbf{v})] = \mathbf{H}_v(f).$$

This is a consequence of the usual Euler Maclaurin expansion. (See Theorem 6 and formulas (6.18)-(6.21) in [**W**].) For $m>1$, the lemma follows from the multi-dimensional analog of the Euler Maclaurin expansion. (See formula (1.6) in [**Ly1**] and formula (1.4) in [**Ly2**].)

Convolutions such as $\mathbf{H}_{v_1}*\mathbf{H}_{v_2}*\cdots*\mathbf{H}_{v_m}$ in (3.1.9) will appear frequently in formulas in sections 6 and 7. Here is an alternate description of this convolution in these cases. $\square$

Proposition 3.1.10 (Theorem B4 in [**GP**]). *Let $R_+^m = \{(s_1, \dots, s_m) | s_i \geq 0$, for all $i\}$, be the positive orthant in R^m, V be the span of the set of non-zero vectors $\{v_1, \dots, v_m\}$, and let $L : R_+^m \to V$ be the mapping which sends $(s_1, \dots, s_m)$ onto $s_1 v_1 + \cdots + s_m v_m$. Suppose that in addition for some ξ in real dual of $V, \xi(v_i) > 0$ for all i. Then the mapping L is proper and the convolution $\mathbf{H}_{v_1} * \mathbf{H}_{v_2} * \cdots * \mathbf{H}_{v_m} = L_*(ds_1 \dots ds_m)$, the push forward of the standard Lebesgue measure on R_+^m.*

4. The pushforward to k^* of the measure defined on a nilpotent orbit in $\underline{g}^*$.

Let $E \in \underline{g}^*$ be nilpotent and let $\Omega = \Omega(E)$ denote the orbit $G \cdot E$. Let β_Ω denote the canonical Liouville measure on Ω. Then β_Ω is tempered. This is true for any G orbit in $\underline{g}^*$. (See [**Rao**].) Moreover, β_Ω is a Radon measure on $\underline{g}^*$. Since Ω is nilpotent, β_Ω is homogeneous of degree $(1/2)\dim_R \Omega - \dim_R \underline{g}$. (This fact about homogeneity follows from equation (5) in Theorem 1 in [**Rao**].)

If $f \in C_c(\underline{g}^*)$, let$I_\Omega(f) = I(f)$ be the integral $\int_\Omega f(g \cdot E)\, d\beta_\Omega$. Recall the projection map $J : \underline{g}^* \to \underline{k}^*$. We define the distribution $J_*(\beta_\Omega)$, the pushforward of β_Ω to $\underline{k}^*$, as follows.

Definition 4.1.1. If $f \in C_c(\underline{k}^*)$, then $\langle J_*(\beta_\Omega), f\rangle = I(f \circ J)$.

Remark 4.1.2. The previous definition makes sense because if $f \in C_c(\underline{k}^*)$ and we consider $\tilde{f} \in C(\underline{g}^*)$ defined by $\tilde{f} = f \circ J$ then $\operatorname{supp} \tilde{f} \cap \Omega$ is bounded. For we argue as follows. Define norms $\|\cdot\|_1$ on $\underline{k}^*$ and $\|\cdot\|_2$ on $\underline{p}^*$ by $\|Z\|_1^2 = -\mathbf{B}(Z,Z)$ and $\|W\|_2^2 = \mathbf{B}(W,W)$ (where $\mathbf{B}$ is the Killing form on $\underline{g}^*$). Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are K invariant, and since $\forall Y \in \underline{g}^*, \mathbf{B}(Y,Y) = -\|Y_{\underline{k}}\|_1^2 + \|Y_{\underline{p}}\|_2^2$ (where $Y_{\underline{k}}$ and $Y_{\underline{p}}$ are the components of Y in $\underline{k}^*$ and $\underline{p}^*$), we have the equality $\mathbf{B}(E,E) = \mathbf{B}(g \cdot E, g \cdot E) = -(\|(g\cdot E)_{\underline{k}}\|_1)^2 + (\|(g \cdot E)_{\underline{p}}\|_2)^2$. Therefore $-(\|(g \cdot E)_{\underline{k}}\|_1)^2 + (\|(g \cdot E)_{\underline{p}}\|_2)^2$ is constant on Ω. Now suppose that $\operatorname{supp} f \subseteq \{W \in \underline{k}^* : \|W\|_1^2 \leq C\}$ where C is some positive constant, and suppose that $g \cdot E \in \operatorname{supp} \tilde{f}$. Then $J(g \cdot E)$ is in $\operatorname{supp} f$ which implies that $(\|(g \cdot E)_{\underline{k}}\|_1)^2 \leq C$, and hence we have $(\|(g \cdot E)_{\underline{k}}\|_1)^2 + (\|(g \cdot E)_{\underline{p}}\|_2)^2 < 2C + \mathbf{B}(E,E)$. Therefore $\operatorname{supp} \tilde{f} \cap \Omega$ is bounded.

Note that $J_*(\beta_\Omega)$ is K invariant in the following sense. If $f \in C_c(\underline{k}^*)$, and for each $x \in K, Z \in \underline{k}^*, f^x(Z) = f(x \cdot Z)$, then $\langle J_*(\beta_\Omega), f^x\rangle = \langle J_*(\beta_\Omega), f\rangle$. This follows from the G invariance of I and the fact that if $x \in K$, and $g \in G$ then $x \cdot (g \cdot E)_{\underline{k}} = (xg \cdot E)_{\underline{k}}$.

Lemma 4.1.3. (a) *If v is any homogeneous K invariant distribution of degree d on $\underline{k}^*$, then $(A^+)^t(v)$ is homogeneous of degree $d + |\Delta_c^+|$ on $\underline{t}^*$. (See Section* 2.2 *for definition of $(A^+)^t$).* (b) *If Ω is nilpotent then the* *distribution* $J_*(\beta_\Omega)$ *is homogeneous of degree* $(1/2)\dim_R \Omega - (1/2)\dim_R \underline{k}$ *and* $(A^+)^t(J_*(\beta_\Omega))$ *is homogeneous of degree* $-\dim_R \underline{t} - |\Delta_c^+| + (1/2)\dim_R \Omega$.

Proof. To prove (a), note that if $\phi \in C_c^\infty(\underline{k}^*)$ and if ξ is regular, then $A^+(\phi_t)(\xi) = t^{-|\Delta|}[A^+(\phi)]_t(\xi)$. Thus, on the one hand we have $\langle v, \phi_t\rangle = t^d\langle v, \phi\rangle = t^d\langle (A^+)^t(v), A^+(\phi)\rangle$. But also

$$\begin{aligned}\langle v, \phi_t\rangle &= \langle (A^+)^t(v), A^+(\phi_t)\rangle \\ &= t^{-|\Delta|}\langle (A^+)^t(v), [A^+(\phi)]_t\rangle.\end{aligned}$$

So the result follows.

To prove (b), suppose that $f \in C_c^\infty(\underline{k}^*)$, then $\tilde{f}_t = t^{\dim_R \underline{p}}(\tilde{f})_t$. Hence

$$\begin{aligned}\langle J_*(\beta_\Omega),\ f_t\rangle &= I[(\tilde{f}_t)] = I[t^{\dim_R \underline{p}}(\tilde{f})_t] \\ &= t^{\dim_R \underline{p}} I[(\tilde{f})_t] \\ &= t^{\dim_R \underline{p}} t^{(1/2)\dim_R \Omega - \dim_R \underline{g}} I(\tilde{f}) \\ &= t^{(1/2)\dim_R \Omega - \dim_R \underline{k}} \langle J_*(\beta_\Omega),\ f\rangle.\end{aligned}$$

□

Remark 4.1.4. Now note that if $c(E) \in \underline{p}_C^*$ denotes an element of the Cayley transform of Ω, then we have that $\dim_R \Omega(E) = 2\dim_C K_C \cdot c(E)$. Thus $J_*(\beta_\Omega)$ is homogeneous of degree $-\dim_C K_C + \dim_C K_C \cdot c(E) = -\dim_C K_C^{c(E)}$.

5. Asymptotic multiplicity distributions.

5.1. Definitions and basic properties. Let $\mathcal{R}$ be a semi-simple K module with finite K multiplicities. $\widehat{K}$ denotes the set of equivalence classes of irreducible representations of K. Let $\Delta^+(\underline{k}_C, \underline{t}_C) = \Delta_c^+$ be a fixed choice of positive roots, and $\rho =$ half the sum of the roots in Δ_c^+. If $\pi \in \widehat{K}$, let $\mu(\pi)$ denote the corresponding dominant integral weight relative to Δ_c^+, $\Theta_{\mu(\pi)}$ denote the character of π and $\mathcal{O}(\mu(\pi))$ denote the orbit $K\cdot\{-i(\mu(\pi)+\rho)\}$ in $\underline{k}^*$. $m_R(\pi)$ denotes the multiplicity of π in $\mathcal{R}$ and $d(\mu(\pi))$ is the dimension of π. r will denote the rank of K.

Definition 5.1.1. We define $\mathbf{m}_\mathcal{R}$, the multiplicity distribution (or formal K measure) on $\underline{k}^*$ associated to $\mathcal{R}$ as follows. If $f \in C_c^\infty(\underline{k}^*)$, then

$$\mathbf{m}_\mathcal{R}(f) = \sum_{\pi\in\hat{K}} m_\mathcal{R}(\pi)\beta_{\mu(\pi)}(f).$$

Here $\beta_{\mu(\pi)}$ is the canonical Liouville measure defined on the orbit $\mathcal{O}(\mu(\pi))$. We will usually identify $\widehat{K}$ with the set of dominant integral weights and write $\mathbf{m}_\mathcal{R} = \sum_{\mu\in\hat{K}} m_\mathcal{R}(\mu)\beta_\mu$.

Remark 5.1.2. $\mathbf{m}_{\mathcal{R}}$ is clearly K invariant. Assume that it is tempered. (This is true with mild restrictions on $m_{\mathcal{R}}(\mu)$. See Lemma 5.1.8 below.) Recall $(A^+)^t$, the isomorphism between K-invariant tempered distributions on $\underline{k}^*$ and W skew-invariant tempered distributions on $\underline{t}^*$ defined in section 2.2. It follows from Example 2.2.4 that

$$(A^+)^t(\mathbf{m}_{\mathcal{R}}) = c_{K,T}^{-1} \cdot \sum_{\mu \in \widehat{K}} m_{\mathcal{R}}(\mu) \sum_{w \in W} \epsilon(w) \delta_{-iw(\mu+\rho)}$$

(where $\delta_{-iw(\mu+\rho)}$ is the Dirac delta function at $-iw(\mu+\rho)$). We will sometimes abuse notation and write

$$\mathbf{m}_{\mathcal{R}} = \sum_{\mu \in \hat{K}} m_{\mathcal{R}}(\mu) \sum_{w \in W} \epsilon(w) \delta_{-iw(\mu+\rho)}.$$

If $\pi \in \widehat{K}$, and $\mu = \mu(\pi)$ then Θ_μ is a smooth function on K. If θ_μ is the lift of Θ_μ (see Theorem 3.1 in [**BV3**]), then it is a result of Harish Chandra that θ_μ is a distribution on $\underline{k}$ whose Fourier transform is $\beta_{K \cdot \{-i(\mu+\rho)\}} = \beta_\mu$. (In the notation of [**BV3**], we have $\theta_\mu(X) = \Theta_\mu(\exp(X))\xi(X)$, where $\xi(X)$ is the square root of the Jacobian of $\exp : \underline{g} \to G$.) For this reason we define two additional distributions related to $\mathbf{m}_{\mathcal{R}}$.

Definition 5.1.3. The distribution $\Theta^{\mathcal{R}} = \sum_{\mu \in \widehat{K}} m_{\mathcal{R}}(\mu) \Theta_\mu$ is called the formal K-character of $\mathcal{R}$ on K, and its lift $\theta^{\mathcal{R}} = \sum_{\mu \in \widehat{K}} m_{\mathcal{R}}\ (\mu) \theta_\mu$ is called the formal K-character of $\mathcal{R}$ on $\underline{k}$.

For the proof of the following lemma, recall the definitions of the Fourier transform on $\underline{k}^*$ and $\underline{t}^*$ in (2.2.6). Set $\operatorname{vol}(K/T) = \int_{K/T} 1\, d_{K/T}(\dot{x})$.

Lemma 5.1.4 (Compare Theorem 3.8 in [**Ch**]). *If $f \in C_c^\infty(\underline{t}^*)$, and λ is a regular element in* $\operatorname{Hom}_R(i\underline{t}, R)$, *then:*

(5.1.5)

$$\lim_{t \to \infty} t^{|\Delta_c^+|} \sum_{w \in W} \epsilon(w) f(t^{-1}(-iw\lambda)) = C(\lambda) \cdot \left\{ \left(\prod_{\alpha \in \Delta_c^+} D_\alpha \right) \cdot f \right\} (0)$$

where $C(\lambda) = \prod_{\alpha \in \Delta_c^+} \frac{(\alpha,\lambda)}{(\alpha,\rho)}$.

Proof. This is based on results of Harish Chandra in [**HC**]. We begin with the following analog of Theorem 2 in [**HC**] which holds for all Z, Z' in $\underline{t}_C^*$:

$$\text{(5.1.6)} \qquad \pi(Z)\pi(Z') \int_{K/T} \exp[B(Ad(x)(Z), Z')]\, d_{K/T}(\dot{x})$$
$$= \operatorname{vol}(K/T) \left(\prod_{\alpha \in \Delta_c^+} (\alpha, \rho) \right) \sum_{w \in W} \epsilon(w) \exp[B(wZ, Z')].$$

Let $Z = -t^{-1}\lambda$, and multiply both sides of (5.1.6) by $\hat{f}(Z')$ where Z' belongs to $\underline{t}^*$. If we then integrate in variable Z' we obtain:

$$\pi(-t^{-1}\lambda)\int_{K/T} d_{K/T}(\dot{x}) \int_{\underline{t}^*} \exp[B(Ad(x)(-t^{-1}\lambda), Z')]\pi(Z')\hat{f}(Z')\, dZ'$$
$$= \mathrm{vol}(K/T)\left(\prod_{\alpha\in\Delta_c^+}(\alpha,\rho)\right)\sum_{w\in W}\epsilon(w)\int_{\underline{t}^*}\exp[B(-wt^{-1}\lambda, Z')]\hat{f}(Z')\, dZ'.$$

Thus we have

(5.1.7)
$$\pi(-t^{-1}\lambda)\int_{K/T}\widehat{(\pi\hat{f})}[Ad(x)(it^{-1}\lambda)]\, d_{K/T}(\dot{x})$$
$$= \mathrm{vol}(K/T)\left(\prod_{\alpha\in\Delta_c^+}(\alpha,\rho)\right)\sum_{w\in W}\epsilon(w)\hat{\hat{f}}(iwt^{-1}\lambda).$$

If $g \in C_c^\infty(\underline{t}^*)$, then $\hat{\hat{g}}(Z) = \kappa g(-Z)$ where κ is independent of g. If we multiply both sides of (5.1.7) by $t^{|\Delta_c^+|}$ and factor $\pi(-\lambda)$ as $i^{|\Delta_c^+|}\pi(i\lambda)$, we get

$$i^{|\Delta_c^+|}\pi\lambda\int_{K/T}\widehat{(\pi\hat{f})}[Ad(x)(it^{-1}\lambda)]\, d_{K/T}(\dot{x})$$
$$= t^{|\Delta_c^+|}\kappa\,\mathrm{vol}(K/T)\left(\prod_{\alpha\in\Delta_c^+}(\alpha,\rho)\right)\sum_{w\in W}\epsilon(w)f(-iwt^{-1}\lambda).$$

Therefore,

$$\lim_{t\to\infty} t^{|\Delta_c^+|}\sum_{w\in W}\epsilon(w)f(t^{-1}(-iw\lambda))$$
$$= \lim_{t\to\infty}\frac{i^{|\Delta_c^+|}\pi(i\lambda)\int_{K/T}\widehat{\pi\hat{f}}[Ad(x)(it^{-1}\lambda)]\, d_{K/T}(\dot{x})}{\kappa\,\mathrm{vol}(K/T)\left(\prod_{\alpha\in\Delta_c^+}(\alpha,\rho)\right)}$$
$$= \frac{i^{|\Delta_c^+|}\pi(i\lambda)\widehat{\pi\hat{f}}(0)}{\kappa\left(\prod_{\alpha\in\Delta_c^+}(\alpha,\rho)\right)}.$$

Now $(\widehat{D_\alpha f})(\gamma) = [-B(\alpha,\cdot)\hat{f}](\gamma)$ and $\pi(Z) = (-i)^{|\Delta_c^+|}\prod_{\alpha\in\Delta_c^+}B(\alpha, Z)$. It follows that

$$\widehat{\pi\hat{f}}(0) = i^{|\Delta_c^+|}\kappa\cdot\left\{\left(\prod_{\alpha\in\Delta_c^+}D_\alpha\right)\cdot f\right\}(0)$$

and hence the desired limit equals

$$C(\lambda)\cdot\left\{\left(\prod_{\alpha\in\Delta_c^+}D_\alpha\right)\cdot\right\}(0)$$

where $C(\lambda) = (i)^{2|\Delta_c^+|}\dfrac{\pi(i\lambda)}{\pi(-i\rho)} = \prod_{\alpha\in\Delta_c^+} \dfrac{(\alpha,\lambda)}{\alpha,\rho)}$. □

Lemma 5.1.8. *Suppose there is a constant c such that $m_{\mathcal{R}}(\mu) \le cd(\mu)$ for all $\mu \in \hat{K}$, then* (a) $\mathbf{m}_{\mathcal{R}}, \Theta^{\mathcal{R}}$ *and* $\theta^{\mathcal{R}}$ *are tempered distributions,* (b) $\mathbf{m}_{\mathcal{R}}$ *is the Fourier transform of* $\theta^{\mathcal{R}}$ *(if $\underline{k}$ and $\underline{k}^*$ are identified).*

Proof. (a) The fact that $\Theta^{\mathcal{R}}$ is a distribution on K follows from Lemme III.2.1 of [**DHV**]. Since K is compact, $\Theta^{\mathcal{R}}$ is tempered . (For example, suppose that $\Theta^{\mathcal{R}} = \sum_{\mu\in\hat{K}} d(\mu)\Theta_\mu$, then the Plancherel theorem for K asserts that $\Theta^{\mathcal{R}}$ is the Dirac delta function at the identity $e \in K$.)

We know that $(A^+)^t(\mathbf{m}_{\mathcal{R}})$ is a constant multiple of

$$\sum_{\mu\in\hat{K}} m_{\mathcal{R}}(\mu) \sum_{w\in W} \epsilon(w)\delta_{-iw(\mu+\rho)}$$

and $(A^+)^t(\theta^{\mathcal{R}})$ is a constant multiple of $\sum_{\mu\in\hat{K}} m_{\mathcal{R}}(\mu) \sum_{w\in W} \epsilon(w)\, e^{\mathbf{B}(w(\mu+\rho),\cdot)}$. Because of the bound on $m_{\mathcal{R}}(\mu)$, we can find a positive integer s such that $\sum_{\mu\in\hat{K}} m_{\mathcal{R}}(\mu)(1+\|\mu\|)^{-s} < \infty$. It follows that for each $w \in W$, both $\sum_{\mu\in\hat{K}} m_{\mathcal{R}}(\mu)\delta_{-iw(\mu+\rho)}$ and $\sum_{\mu\in\hat{K}} m_{\mathcal{R}}(\mu)e^{\mathbf{B}(w(\mu+\rho),\cdot)}$ are tempered distributions. (See Part I, Section 3 of [**SW**] or Part II, Section 28 of [**Do**].) Thus $(A^+)^t(\mathbf{m}_{\mathcal{R}})$ and $(A^+)^t(\theta^{\mathcal{R}})$ are tempered. Hence so are $\mathbf{m}_{\mathcal{R}}$ and $\theta^{\mathcal{R}}$.

(b) follows from the fact that β_μ is the Fourier transform of θ_μ. □

In the remainder of this section $\mathcal{R}$ will usually be a finitely generated $S(\underline{p}_C)$ module with a compatible K_C action such that:

5.1.9(a). $\mathcal{R}$ is $S(\underline{p}_C)^{K_C}$ finite (i.e., there is an ideal $I \subseteq S(\underline{p}_C)^{K_C}$ with finite codimension which annihilates $\mathcal{R}$), and

5.1.9(b). $\mathcal{R}$ is K_C finite, and completely reducible as a K_C module.

Proposition 5.1.10. *If $\mathcal{R}$ is a finitely generated $S(\underline{p}_C)$ module with a compatible K_C action satisfying conditions* 5.1.9 (a) *and* 5.1.9 (b) *then all K_C multiplicities in $\mathcal{R}$ are finite and for some constant $c = c_{\mathcal{R}}$, we have $m_{\mathcal{R}}(\mu) \le cd(\mu)$ for all $\mu \in \widehat{K}$.*

Proof. By (5.1.9(a)), $I = \mathrm{Ann}(\mathcal{R}) \cap S(\underline{p}_C)^{K_C}$ has finite codimension in $S(\underline{p}_C)^{K_C}$. Here $\mathrm{Ann}(\mathcal{R})$ is annihilator of $\mathcal{R}$ in $S(\underline{p}_C)$. It follows that the ring $\dfrac{S(\underline{p}_C)^{K_C}}{I}$ is Artinian. So $I = \bigcap_{k=1}^m J_k$ where the J_k are the maximal ideals of $S(\underline{p}_C)^{K_C}$ which contain I. Therefore $\bigcap_{k=1}^m S(\underline{p}_C)\cdot J_k \subseteq S(\underline{p}_C)\cdot I \subseteq \mathrm{Ann}(\mathcal{R})$. It is clear that for each k, $\dfrac{S(\underline{p}_C)}{S(\underline{p}_C)\cdot J_k}$ has finite K_C multiplicities, since

$\dfrac{S(\underline{p}_C)}{S(\underline{p}_C)\cdot J_k}$ is the ring of regular functions on a finite union of closures of K_C orbits in $\underline{p}_C^*$ [**KR**]. Suppose, $\mathcal{O} = K_C \cdot z$ is one of these orbits, and $R[\mathcal{O}]$ (respectively $R[\overline{\mathcal{O}}]$) is the ring of regular functions on $\mathcal{O}$ (respectively $\overline{\mathcal{O}}$, the Zariski closure of $\mathcal{O}$). Then for all $\mu \in \hat{K}$, since $R[\overline{\mathcal{O}}] \subseteq R[\mathcal{O}]$, we have $m_{\mathcal{R}[\overline{\mathcal{O}}]}(\mu) \leq m_{\mathcal{R}[\mathcal{O}]}(\mu) = \dim_C V_\mu^{K_C^z}$ (where $V_\mu^{K_C^z}$ is the space of K_C^z invariants in the irreducible module V_μ). Thus $m_{\mathcal{R}[\overline{\mathcal{O}}]}(\mu) \leq d(\mu)$. So the multiplicity of μ in $\dfrac{S(\underline{p}_C)}{S(\underline{p}_C)\cdot J_k}$ is bounded by a constant times $d(\mu)$. Now $\mathcal{T} = \dfrac{S(\underline{p}_C)}{\bigcap_{k=1}^m S(\underline{p}_C)\cdot J_k}$ has finite K multiplicities and for all $\mu \in \hat{K}, m_{\mathcal{T}}(\mu) \leq d(\mu)$ because $\mathcal{T}$ is a K submodule of the direct sum $\oplus_{k=1}^m \dfrac{S(\underline{p}_C)}{S(\underline{p}_C)\cdot J_k}$. Because we have a surjective map of $(S(\underline{p}_C), K_C)$ modules: $\mathcal{T} \to \dfrac{S(\underline{p}_C)}{\mathrm{Ann}(\mathcal{R})} \to (0)$, the multiplicity of $\mu \in \hat{K}$ in $\dfrac{S(\underline{p}_C)}{\mathrm{Ann}(\mathcal{R})}$ is also bounded by a constant times $d(\mu)$. Finally (5.1.9(b)) implies that $\mathcal{R}$ is finitely generated over $\dfrac{S(\underline{p}_C)}{\mathrm{Ann}(\mathcal{R})}$. Thus we can find W, a finite dimensional K submodule of $\mathcal{R}$, such that the K module map $\dfrac{S(\underline{p}_C)}{\mathrm{Ann}(\mathcal{R})} \otimes W \to \mathcal{R}$, defined by $f \otimes w \to fw$, is surjective. This implies the desired bound on $m_{\mathcal{R}}(\mu)$. $\square$

We can investigate the asymptotics of $m_{\mathcal{R}}(\mu)$ by studying the behavior of the following functions of the positive variable t, as $t \to \infty$:

(5.1.11(a)) $$N_{\mathcal{R}}(t) = \sum_{\substack{\mu \in \widehat{K} \\ \|\mu+\rho\| \leq t}} m_{\mathcal{R}}(\mu)\, d(\mu)$$

(5.1.11(b)) $$N_{\mathcal{R}}^C(t) = \sum_{\substack{\mu \in \widehat{K} \\ \Omega_{\underline{k}}(\mu) \leq t^2}} m_{\mathcal{R}}(\mu)\, d(\mu)$$

(5.1.11(c)) $$N_{\mathcal{R}}^L(t) = \sum_{\substack{\mu \in \widehat{K} \\ L(\mu) \leq t}} m_{\mathcal{R}}(\mu)\, d(\mu).$$

Here $\|\cdot\|$ denotes the norm on $i\underline{t}^*$ coming from the Killing form, $\Omega_{\underline{k}}$ is the Casimir operator for $\underline{k}$, and $\Omega_{\underline{k}}(\mu)$ is the value of the Casimir on V_μ. $L(\mu)$ denotes the sum of the coefficients of the fundamental weights in the representation of μ as a sum of fundamental weights.

$N_{\mathcal{R}}^L(t)$ is closely related to the Borho-Kraft multiplicity function [**BK**], defined as follows when $\mathcal{R}$ is also a ring:

$$F\{\mathcal{R}\}(t) = \sum_{\substack{\mu \in \widehat{K} \\ L(\mu) \leq t}} m_{\mathcal{R}}(\mu). \tag{5.1.12(a)}$$

For convenience when $\mathcal{R}$ is a ring write:

$$F^B\{\mathcal{R}\}(t) = \sum_{\substack{\mu \in \widehat{K} \\ \|\mu+\rho\| \leq t}} m_{\mathcal{R}}(\mu) \tag{5.1.12(b)}$$

$$F^C\{\mathcal{R}\}(t) = \sum_{\substack{\mu \in \widehat{K} \\ \Omega_{\underline{k}}(\mu) \leq t^2}} m_{\mathcal{R}}(\mu). \tag{5.1.12(c)}$$

The function $F\{\mathcal{R}\}$ is a quasi-polynomial (in the sense of Definition 2.2 in [**Bo**]) and so are the functions $N_{\mathcal{R}}, N_{\mathcal{R}}^L, N_{\mathcal{R}}^C, F^B\{\mathcal{R}\}$, and $F^C\{\mathcal{R}\}$. (Note that the degree of the leading term of $F\{\mathcal{R}\}(t)$ is equal to the Krull dimension of the ring $\mathcal{R}^{\underline{n}_k}$, where $\underline{n}_k$ is the complex subalgebra of $\underline{k}_C$ spanned by the root spaces for Δ_c^+ ([**Bo**]).)

We show below that $N_{\mathcal{R}}^C$ and $N_{\mathcal{R}}$ each grow like a positive constant multiple of t^d where $d = \operatorname{K}\dim \mathcal{R}$, the Krull dimension of $\mathcal{R}$ as an $S(\underline{p}_C)$ module. We then use this fact to define a K invariant distribution on $\underline{k}^*$ which measures the asymptotic behavior of $m_{\mathcal{R}}(\mu)$.

Proposition 5.1.13. *Let $\mathcal{R}$ be a finitely generated $S(\underline{p}_C)$ module with a compatible K_C action satisfying* 5.1.9(a) *and* 5.1.9(b). *For each $t > 0$, let $\mathcal{R}_{(t)}$ denote the subspace of $\mathcal{R}$ spanned by eigenvectors of $\Omega_{\underline{k}}$ with eigenvalue $\leq t^2$ so that $N_{\mathcal{R}}^C(t) = \dim_C \mathcal{R}_{(t)}$. Let $d = \operatorname{K}\dim \mathcal{R}$ (the Krull dimension of $\mathcal{R}$ as an $S(\underline{p}_C)$ module). Then there are constants A and B (depending only on $\underline{g}$) and $c(\mathcal{R})$ (depending on $\mathcal{R}$) such that for t sufficiently large:*

$$A \cdot c(\mathcal{R}) \cdot t^d \leq N_{\mathcal{R}}^C(t) \leq B \cdot c(\mathcal{R}) \cdot t^d. \tag{5.1.14}$$

Proof. Modify the arguments of Proposition 5.4 and 5.5 in [**Vo2**] for $\mathcal{R}$ rather than a Harish Chandra module. □

Corollary 5.1.15. *Under the hypotheses of Proposition* 5.1.13 *there are constants A' and B' (depending only on $\underline{g}$) and $c'(\mathcal{R})$ (depending on $\mathcal{R}$) such that for t sufficiently large:*

$$A' \cdot c'(\mathcal{R}) \cdot t^d \leq N_{\mathcal{R}}(t) \leq B' \cdot c'(\mathcal{R}) \cdot t^d. \tag{5.1.16}$$

Proof. It is clear that $N_{\mathcal{R}}$ and $N_{\mathcal{R}}^C$ have similar asymptotic behavior, so the result follows from Proposition 5.1.13. □

Proposition 5.1.17. *Let $\mathcal{R}$ be a finitely generated $S(\underline{p}_C)$ module with a compatible K_C action satisfying* 5.1.9(a) *and* 5.1.9(b). *Let d be the Krull dimension of $\mathcal{R}$. Let $p(\mathbf{m}_{\mathcal{R}})$ be the largest integer p such that (i) for each $f \in C_c^\infty(\underline{k}^*)$, the following limit is finite and (ii) for some $f \in C_c^\infty(\underline{k}^*)$ it is not equal to zero:*

$$\lim_{t\to\infty} t^{-p}(\mathbf{m}_{\mathcal{R}})(f_t) = \lim_{t\to\infty} t^{-p} \sum_{\mu\in\widehat{K}} m_{\mathcal{R}}(\mu)\beta_\mu(f_t). \tag{5.1.18}$$

Then $p(\mathbf{m}_{\mathcal{R}}) = d - \dim_R \underline{k}$.

Proof. We first show that $\lim_{t\to\infty} t^{-d+\dim \underline{k}}(\mathbf{m}_{\mathcal{R}})(f_t)$ exists (i.e. is finite) for all $f \in C_c^\infty(\underline{k}^*)$. For each such f, choose $L > 0$, so that $\operatorname{supp} f \subseteq \{x \in \underline{k}^* : (x,x) < L\}$, then $\operatorname{supp} f_t \subseteq \{x \in \underline{k}^* : (x,x) < tL\}$. So

$$\mathbf{m}_{\mathcal{R}}(f_t) = \sum_{\mu\in\widehat{K}} m_{\mathcal{R}}(\mu)\beta_\mu(f_t) = t^{-\dim \underline{k}} \sum_{\substack{\mu\in\widehat{K}\\ \|\mu+\rho\|\le tL}} m_{\mathcal{R}}(\mu)\beta_\mu[f(t^{-1}\cdot)].$$

Hence,

$$\begin{aligned}|(\mathbf{m}_{\mathcal{R}})(f_t)| &\le t^{-\dim \underline{k}} \sum_{\substack{\mu\in\widehat{K}\\ \|\mu+\rho\|\le tL}} m_{\mathcal{R}}(\mu)\operatorname{vol}(\mathcal{O}(\mu)) \sup|f| \\ &= t^{-\dim \underline{k}} \sup|f| \sum_{\substack{\mu\in\widehat{K}\\ \|\mu+\rho\|\le tL}} m_{\mathcal{R}}(\mu)\, d(\mu) = Ct^{-\dim \underline{k}} N_{\mathcal{R}}(tL),\end{aligned}$$

where C depends on f. By (5.1.15), $N_{\mathcal{R}}(tL)$ is bounded above by a constant (depending on $\mathcal{R}$) times $L^d t^d$. So it is clear that $\lim_{t\to\infty} t^{-d+\dim \underline{k}}|(\mathbf{m}_{\mathcal{R}})(f_t)|$ exists, and so the limit in (5.1.18) exists when $p = d - \dim_{\mathcal{R}} \underline{k}$.

We next show that if $p = d - \dim_{\mathcal{R}} \underline{k}$, the limit in (5.1.18) is non zero for some $f \in C_c^\infty(\underline{k}^*)$.

Choose a function $\phi \in C_c^\infty(\underline{k}^*)$ with the following properties: $\phi \ge 0, \phi \ge 1$ on $\{x \in \underline{k}^* | \ \|x\| \le 1\}$, and ϕ is K invariant. Note that $\phi_t(x) \ge t^{-\dim \underline{k}}$ if $\|x\| \le t$.

$$\mathbf{m}_{\mathcal{R}}(\phi_t) = \sum_{\mu\in\widehat{K}} m_{\mathcal{R}}(\mu)\beta_\mu(\phi_t) = t^{-\dim \underline{k}} \sum_{\mu\in\widehat{K}} m_{\mathcal{R}}(\mu)\, d(\mu)\phi[-it^{-1}(\mu+\rho)]$$

because of the K invariance of ϕ.

It follows that,

$$\begin{aligned} t^{-p}\mathbf{m}_{\mathcal{R}}(\phi_t) &= t^{-p-\dim \underline{k}} \sum_{\mu\in\widehat{K}} m_{\mathcal{R}}(\mu)\ d(\mu)\phi[-it^{-1}(\mu+\rho)] \\ &\geq t^{-p-\dim \underline{k}} \sum_{\substack{\mu\in\widehat{K} \\ \|\mu+\rho\|\leq t}} m_{\mathcal{R}}(\mu)\ d(\mu) \\ &= t^{-p-\dim \underline{k}} N_{\mathcal{R}}(t) = t^{-d} N_{\mathcal{R}}(t). \end{aligned}$$

Since by (5.1.15), $N_{\mathcal{R}}(t)$ is bounded below by a positive constant times t^d, we must have

$$\lim_{t\to\infty} t^{-d+\dim \underline{k}}(\mathbf{m}_{\mathcal{R}})(\phi_t) \neq 0.$$

On the other hand if $p > d - \dim \underline{k}$, then for all $f \in C_c^\infty(\underline{k}^*)$, we have:

$$\lim_{t\to\infty} t^{-p}(\mathbf{m}_{\mathcal{R}})(f_t) = \lim_{t\to\infty} t^{(-p+d-\dim \underline{k})} t^{(-d+\dim \underline{k})}(\mathbf{m}_{\mathcal{R}})(f_t) = 0.$$

So $p(\mathbf{m}_{\mathcal{R}}) = d - \dim \underline{k}$. □

Definition 5.1.19. Let $\mathcal{R}$ be a finitely generated $S(\underline{p}_C)$ module with a compatible K_C action satisfying 5.1.9(a) and 5.1.9(b). Let $\mathbf{M}_{\mathcal{R}}$ be the distribution on $\underline{k}^*$ defined by the limit in (5.1.18) for $p = p(\mathbf{m}_{\mathcal{R}}) = d - \dim_{\mathcal{R}} \underline{k}$, i.e. $\mathbf{M}_{\mathcal{R}}(f) = \lim_{t\to\infty} t^{-d+\dim \underline{k}}(\mathbf{m}_{\mathcal{R}})(f_t)$, for $f \in C_c^\infty(\underline{k}^*)$. It is clear that $\mathbf{M}_{\mathcal{R}}$ is K invariant and homogeneous of degree $-d + \dim_{\mathcal{R}} \underline{k}$. $\mathbf{M}_{\mathcal{R}}$ will called the *asymptotic multiplicity measure* of $\mathcal{R}$.

Remark 5.1.20. Assume $\mathcal{R}$ satisfies the same conditions in Definition 5.1.19 then we can also define the $\mathbf{C}_{\mathcal{R}}$, the *asymptotic K character* of $\mathcal{R}$, (in a similar way to $\mathbf{M}_{\mathcal{R}}$): $\mathbf{C}_{\mathcal{R}}(f) = \lim_{t\to 0^+} t^d\theta^{\mathcal{R}}(f_t)$, for all $f \in C_c^\infty(\underline{k}^*)$. It is clear that $\mathbf{C}_{\mathcal{R}}$ is also K invariant and homogeneous of degree d. Also $\mathbf{M}_{\mathcal{R}} = \widehat{\mathbf{C}}_{\mathcal{R}}$, the Fourier transform of $\mathbf{C}_{\mathcal{R}}$. If we set $\mathcal{R}' = \mathcal{R} \otimes_C F$, then $\mathbf{M}_{\mathcal{R}'} = (\dim_C F)\mathbf{M}_{\mathcal{R}}$. This is based on the fact that $\Theta^{\mathcal{R}'} = \Theta^{\mathcal{R}}\Theta^F$. This allows us to prove that $\mathbf{C}_{\mathcal{R}'} = (\dim_C F)\mathbf{C}_{\mathcal{R}}$, from which we obtain the desired result for $\mathbf{M}_{\mathcal{R}'}$. We omit the details.

Now suppose that $\mathcal{P}$ and $\mathcal{S}$ are $(S(\underline{p}_C), K_C)$ submodules of $\mathcal{R}$. Then it is easy to establish the following facts concerning $N_{\mathcal{R}}, N_{\mathcal{P}}$, and $N_{\mathcal{S}}$ which are analogues of facts proven in [**Bo**] concerning $F\{\mathcal{R}\}, F\{\mathcal{P}\}, and F\{\mathcal{S}\}$.

(5.1.21(a)) $$N_{\mathcal{R}/\mathcal{P}} = N_{\mathcal{R}} - N_{\mathcal{P}}$$

(5.1.21(b)) $$N_{\mathcal{P}\cap\mathcal{S}} = N_{\mathcal{P}} + N_{\mathcal{S}} - N_{\mathcal{P}+\mathcal{S}}$$

We also have the analogue of Lemma 2.6 in [**Bo**].

Proposition 5.1.22. *Suppose $\mathcal{R}$ is an integral domain containing C (the complex numbers) satisfying conditions* 5.1.9(a) *and* 5.1.9(b)*, and $\mathcal{S}$ is a K stable subring of $\mathcal{R}$ containing C such that $\mathcal{R}$ is the integral closure of $\mathcal{S}$ (in its field of fractions). Let* $\operatorname{K}\dim\mathcal{R} = d$*. Then as quasi-polynomials, $N_{\mathcal{R}}(t)$ and $N_{\mathcal{S}}(t)$ have the same leading terms and* $\deg N_{\mathcal{R}/\mathcal{S}} < \operatorname{K}\dim\mathcal{R}$*.*

Proof. We argue as in Lemma 2.6 of [**Bo**]. First since $\mathcal{S} \subseteq \mathcal{R}$, we have the inequality $N_{\mathcal{S}}(t) \leq N_{\mathcal{R}}(t)$ for all t. Let $Q(\mathcal{S})$ denote the field of fractions of $\mathcal{S}$. Since $\mathcal{R}$ is the integral closure of $\mathcal{S}$ in $Q(\mathcal{S}), \mathcal{R}$ is finitely generated as an $\mathcal{S}$ module. Consider the left $\mathcal{S}$-module $\mathcal{R}/\mathcal{S}$. Let $I = \operatorname{Ann}_{\mathcal{S}}(\mathcal{R}/\mathcal{S})$, the annihilator of $\mathcal{R}/\mathcal{S}$ in $\mathcal{S}$. $I \neq (0)$ because (1) for each $r \in \mathcal{R}$, there is an $s \in \mathcal{S}$ such that $sr \in \mathcal{S}$ (since $\mathcal{R} \subset Q(\mathcal{S})$) and (2) $\mathcal{R}/\mathcal{S}$ is finitely generated over $\mathcal{S}$. Clearly I is a K submodule of $\mathcal{S}$. Choose a highest weight vector $e \in I$. Let $\omega \in \operatorname{Hom}_R(i\underline{t}, R)$ be the weight of e. We have the inclusion $e\mathcal{R} \subseteq \mathcal{S}$, and hence $e\mathcal{R}^{\underline{n}_k} \subseteq \mathcal{S}^{\underline{n}_k}$ (since $e \in I^{\underline{n}_k}$). It follows that for any $\mu \in \widehat{K}$, we have the inequality: $m_{\mathcal{R}}(\mu) \leq m_{\mathcal{S}}(\mu + \omega)$. Since ω is dominant, $d(\mu + \omega) \geq d(\mu)$. This implies that: $m_{\mathcal{R}}(\mu) \leq m_{\mathcal{S}}(\mu + \omega)d(\mu + \omega)$. Hence,

$$
\begin{aligned}
(5.1.23)\quad N_{\mathcal{R}}(t) &= \sum_{\substack{\mu\in\widehat{K}\\ \|\mu+\rho\|\leq t}} m_{\mathcal{R}}(\mu)\, d(\mu) \leq \sum_{\substack{\mu\in\widehat{K}\\ \|\mu+\rho\|\leq t}} m_{\mathcal{S}}(\mu+\omega)d(\mu+\omega)\\
&= \sum_{\substack{\nu\in\omega+\widehat{K}\\ \|\nu-\omega+\rho\|\leq t}} m_{\mathcal{S}}(\nu)d(\nu) \leq \sum_{\substack{\nu\in\omega+\widehat{K}\\ \|\nu+\rho\|\leq t+\|\omega\|}} m_{\mathcal{S}}(\nu)d(\nu)\\
&\leq \sum_{\substack{\nu\in\widehat{K}\\ \|\nu+\rho\|\leq t+\|\omega\|}} m_{\mathcal{S}}(\nu)d(\nu) = N_{\mathcal{S}}(t+\|\omega\|).
\end{aligned}
$$

We have established that $N_{\mathcal{R}}(t) \leq N_{\mathcal{S}}(t + \|\omega\|)$. Therefore since $N_{\mathcal{S}}(t) \leq N_{\mathcal{R}}(t)$, we conclude that $N_{\mathcal{R}}(t)$ and $N_{\mathcal{S}}(t)$ have the same leading terms. This together with (5.1.21(a)) implies that $\deg N_{\mathcal{R}/\mathcal{S}} < \operatorname{K}\dim\mathcal{R}$. □

Proposition 5.1.24. *Assume the hypotheses of Proposition* 5.1.22*, then* $\mathbf{M}_{\mathcal{R}} = \mathbf{M}_{\mathcal{S}}$.

Proof. Let $d = \operatorname{K}\dim\mathcal{S}$. It follows from Proposition 5.1.22 that $\operatorname{K}\dim\mathcal{R} = d$. Suppose that $\mathbf{M}_{\mathcal{R}} - \mathbf{M}_{\mathcal{S}} \neq 0$. Then since $\mathbf{M}_{\mathcal{R}} - \mathbf{M}_{\mathcal{S}}$ is K invariant, there would exist a K invariant function $\phi \in C_c^\infty(\underline{k}^*)$ such that $\{\mathbf{M}_{\mathcal{R}} - \mathbf{M}_{\mathcal{S}}\}(\phi) \neq$

0. If we assume that $\operatorname{supp}\phi$ is contained in $\{x \in \underline{k}^* : (x,x) \leq L\}$, then

$$t^{-d+\dim \underline{k}}[\mathbf{m}_{\mathcal{R}} - \mathbf{m}_{\mathcal{S}}](\phi_t) = t^{-d+\dim \underline{k}} \sum_{\mu \in \widehat{K}} (m_{\mathcal{R}}(\mu) - m_{\mathcal{S}}(\mu))\beta_\mu(\phi_t)$$

$$= t^{-d} \sum_{\mu \in \widehat{K}} (m_{\mathcal{R}}(\mu) - m_{\mathcal{S}}(\mu)) d(\mu)\phi[-it^{-1}(\mu+\rho)]$$

$$= t^{-d} \sum_{\substack{\mu \in \widehat{K} \\ \|\mu+\rho\| \leq tL}} (m_{\mathcal{R}}(\mu) - m_{\mathcal{S}}(\mu)) d(\mu)\phi[-it^{-1}(\mu+\rho)].$$

The last sum above must go to zero as $t \to \infty$, because

$$\sum_{\substack{\mu \in \widehat{K} \\ \|\mu+\rho\| \leq tL}} (m_{\mathcal{R}}(\mu) - m_{\mathcal{S}}(\mu)) d(\mu) = N_{\mathcal{R}}(tL) - N_{\mathcal{S}}(tL)$$

is asymptotic to a constant times $L^{d'} t^{d'}$ with $d' < d$. But this contradicts $\{\mathbf{M}_{\mathcal{R}} - \mathbf{M}_{\mathcal{S}}\}(\phi) \neq 0$. □

Lemma 5.1.25. *If* $\operatorname{K}\dim(\mathcal{P}) = \operatorname{K}\dim(\mathcal{S}) > \operatorname{K}\dim(\mathcal{P} \cap \mathcal{S})$, *then* $\mathbf{M}_{\mathcal{P}+\mathcal{S}} = \mathbf{M}_{\mathcal{P}} + \mathbf{M}_{\mathcal{S}}$.

Proof. Apply (5.1.21(b)) and argue as in Prop. 5.1.24. □

5.2. Asymptotic multiplicity measures of K_C orbits in $\underline{p}^*$. If A is a quasi-projective variety in $\underline{p}_C^*$, let $R(A)$ denote the field of rational functions on A. $R[A]$ denotes the ring of everywhere regular rational functions on A. Thus, if A is closed (in the Zariski topology on $\underline{p}_C^*$) and irreducible, $R[A]$ is the co-ordinate ring of A. In this case A^n denotes the normalization of A.

Let $\mathcal{O}$ be a K_C orbit in $\underline{p}_C^*$, $\overline{\mathcal{O}}$ its Zariski closure and $I(\overline{\mathcal{O}})$ = the ideal of functions on $\underline{p}_C^*$ vanishing on $\overline{\mathcal{O}}$. Suppose that $\mathcal{O}$ is regular in $\underline{p}_C^*$, i.e. of maximal dimension among K_C orbits in $\underline{p}_C^*$. Then $I(\overline{\mathcal{O}}) \cap S(\underline{p}_C)^{K_C}$ has finite co-dimension in $S(\underline{p}_C)^{K_C}$ (see Chapter II, of [**KR**]). It is then clear that $R[\overline{\mathcal{O}}]$ is a finitely generated $S(\underline{p}_C)$ module which satisfies the conditions (5.1.9(a)) and (5.1.9(b)). If $\mathcal{O}$ is not regular, then $\mathcal{O}$ is contained in the closure of a regular K_C orbit in $\underline{p}_C^*$ (by Theorem 9 of [**KR**] and the discussion which precedes it). And again we can conclude that $R[\overline{\mathcal{O}}]$ is a finitely generated $S(\underline{p}_C)$ module which satisfies the conditions (5.1.9(a)) and (5.1.9(b)).

Definition 5.2.1. If $\mathcal{O}$ is any K_C orbit in $\underline{p}_C^*$, define $m_{\overline{\mathcal{O}}}(\mu) = m_{R[\overline{\mathcal{O}}]}(\mu)$ for all $\mu \in \widehat{K}$. Then $\mathbf{m}_{\overline{\mathcal{O}}}$, the multiplicity distribution of $\overline{\mathcal{O}}$, is the distribution on $\underline{k}^*$ defined from the multiplicity function $m_{\overline{\mathcal{O}}} = \sum_{\mu \in \widehat{K}} m_{\overline{\mathcal{O}}}(\mu)\beta_\mu$. We often identify $\mathbf{m}_{\overline{\mathcal{O}}}$ with $(A^+)^t(\mathbf{m}_{\overline{\mathcal{O}}}) = c_{K,T} \cdot \sum_{\mu \in \widehat{K}} m_{\overline{\mathcal{O}}}(\mu) \sum_{w \in W} \epsilon(w)\delta_{-iw(\mu+\rho)}$. Similarly define $\Theta^{\overline{\mathcal{O}}} = \Theta^{R[\overline{\mathcal{O}}]}$ and $\theta^{\overline{\mathcal{O}}} = \theta^{R[\overline{\mathcal{O}}]}$.

We can now apply Definition 5.1.19 to $\mathcal{R} = \mathcal{R}[\overline{\mathcal{O}}]$. Note that $\operatorname{K}\dim R[\overline{\mathcal{O}}] = \dim_C \overline{\mathcal{O}} = \dim_C \mathcal{O}$.

Definition 5.2.2. Let $\mathcal{O}$ and $\mathbf{m}_{\overline{\mathcal{O}}}$ be as above. Let $\mathbf{M}_{\overline{\mathcal{O}}} = \mathbf{M}_{R[\overline{\mathcal{O}}]}$. That is, for $f \in C_c^\infty(\underline{k}^*)$,

$$\mathbf{M}_{\overline{\mathcal{O}}}(f) = \lim_{t\to\infty} t^{-\dim_C \mathcal{O} + \dim_R \underline{k}}(\mathbf{m}_{\overline{\mathcal{O}}})(f_t).$$

The degree of homogeneity of $\mathbf{M}_{\overline{\mathcal{O}}}$ is $\dim_C \mathcal{O} - \dim_R \underline{k}$. $\mathbf{M}_{\overline{\mathcal{O}}}$ will be called the *asymptotic multiplicity measure* of $\overline{\mathcal{O}}$.

We will sometimes want to consider the W skew-distribution $(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ on $\underline{t}^*$ in place of the K invariant distribution $\mathbf{M}_{\overline{\mathcal{O}}}$ on $\underline{k}^*$. Note that for all $\phi \in C_c^\infty(\underline{t}^*), c_{K,T}^{-1} \cdot (A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})(\phi) =$

$$\lim_{t\to\infty} t^{-\dim_C \mathcal{O} + |\Delta_c^+| + r} \left\{ \sum_{\mu \in \widehat{K}} m_{\overline{\mathcal{O}}}(\mu) \sum_{w \in W} \epsilon(w) \delta_{-iw(\mu+\rho)} \right\} (\phi_t). \tag{5.2.3}$$

By Lemma 4.1.3, the degree of homogeneity of $(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ is $\dim_C \mathcal{O} - |\Delta_c^+| - r$. We set $p(\mathcal{O}) = \dim_C \mathcal{O} - |\Delta_c^+| - r$.

Example 5.2.4. Let us consider the case when $\mathcal{O} = \{0\}$ in $\underline{p}_C^*$. Then $\mathcal{O} = \overline{\mathcal{O}}, m_{\mathcal{O}}(\mu) = 1$ if $\mu = 0$, and $m_{\mathcal{O}}(\mu) = 0$ if $\mu \neq 0$. Also $\dim_C \mathcal{O} = 0$, so that $p(\mathcal{O}) = -r - |\Delta_c^+|$. We now apply the Lemma 5.1.4. Let $\psi \in C_c^\infty(\underline{t}^*)$, then

$$\begin{aligned} &c_{K,T}^{-1} \cdot (A^+)^t(\mathbf{M}_{\mathcal{O}})(\psi) \\ &\quad = \lim_{t\to\infty} t^{r+|\Delta_c^+|} \sum_{w\in W} \epsilon(w)\delta_{-iw\rho}(\psi_t) = \left\{ \left(\prod_{\alpha\in\Delta_c^+} D_\alpha \right) \cdot \delta_0 \right\} (\psi). \end{aligned}$$

Now $\mathbf{c}^{-1}(\mathcal{O}) = \Omega = \{0\}$ in $\underline{g}^*$. It is clear from another result of Harish Chandra that $(A^+)^t(J_*(\beta_\Omega))$ is given by a constant multiple of the same expression.

For by definition, $\langle J_*(\beta_\Omega), \phi\rangle = \langle \delta_0, J^*\phi\rangle = J^*\phi(0) = \phi(0)$ where $\phi \in C_c^\infty(\underline{k}^*)$. Recall the definition of Harish Chandra's invariant integral on K:

$$F_\phi^K(Z) = \pi(Z) \int_{K/T} \phi(Ad(x)(Z))\, d_{K/T}(\dot{x}) \quad (Z \in \underline{t}^*) \tag{5.2.5}$$

where $\pi(Z) = \prod_{\alpha\in\Delta_c^+}(i\alpha, Z)$. By Harish Chandra, we know that up to a constant we have the equation

$$\phi(0) = \lim_{H\to 0} \left\{ \left(\prod_{\alpha\in\Delta_c^+} D_\alpha \right) \cdot F_\phi^K \right\} (H).$$

Since $F_\phi^K(Z)$ is a constant times $A^+(\phi)(Z)$, we obtain the equation

$$\langle J_*(\beta_\Omega),\phi\rangle = \lim_{H\to 0}\left\{\left(\prod_{\alpha\in\Delta_c^+} D_\alpha\right)\cdot A^+(\phi)\right\}(H)$$
$$= \text{ a constant multiple of } \left\langle\left(\prod_{\alpha\in\Delta_c^+} D_\alpha\right)\cdot\delta_0,\ A^+(\phi)\right\rangle.$$

Thus $(A^+)^t(J_*(\beta_\Omega))$ is a constant multiple of $\left(\prod_{\alpha\in\Delta_c^+} D_\alpha\right)\cdot\delta_0$ which is equal to $c_{K,T}^{-1}\cdot(A^+)^t(\mathbf{M}_{\mathcal{O}})$. It follows that $J_*(\beta_\Omega)$ is a constant multiple of $\mathbf{M}_{\mathcal{O}}$. This is a special case of a conjecture of Vogan. (See (8.1.1).)

Remark 5.2.6. It is possible to define an asymptotic multiplicity measure for any K_C orbit $\mathcal{O} = K_C\cdot z$, not just for $\overline{\mathcal{O}}$. Consider $R[\mathcal{O}]$, the regular functions on $\mathcal{O}$. $R[\mathcal{O}]$ is an $(S(\underline{p}_C), K_C)$ module, which is completely reducible as a K_C module, and $\operatorname{K\dim} R[\mathcal{O}] = \dim_C\mathcal{O}$. If $\mu\in\widehat{K}$, set $m_{\mathcal{O}}(\mu) = m_{R[\mathcal{O}]}(\mu) = \dim_C V_\mu^z$ (by Frobenius reciprocity). Then $R[\mathcal{O}]$ satisfies the following asymptotic growth condition. For some positive constants $c_{\mathcal{O}}$ and $d_{\mathcal{O}}$, if t is sufficiently large:

$$c_{\mathcal{O}}t^{\dim_C\mathcal{O}} \le N_{R[\mathcal{O}]}(t) \le d_{\mathcal{O}}t^{\dim_C\mathcal{O}}. \tag{5.2.7}$$

We then define $\mathbf{M}_{\mathcal{O}}$, the *asymptotic multiplicity measure* of $\mathcal{O}$, by:

$$\mathbf{M}_{\mathcal{O}}(f) = \lim_{t\to\infty} t^{-\dim_C\mathcal{O}+\dim_R\underline{k}}(\mathbf{m}_{\mathcal{O}})(f_t),$$

for $f\in C_c^\infty(\underline{k}^*)$. (The degree of homogeneity of $\mathbf{M}_{\mathcal{O}}$ is $\dim_C\mathcal{O}-\dim_R\underline{k}$.)

Condition (5.2.7) holds for any K_C orbit $\mathcal{O}$ in $\underline{p}_C^*$ because the same sort of growth conditions are satisfied by $N_{R[\overline{\mathcal{O}}]}(t)$. For we can argue as follows.

Set $N_{\mathcal{O}}(t) = N_{R[\mathcal{O}]}(t)$ and $N_{\overline{\mathcal{O}}}(t) = N_{R[\overline{\mathcal{O}}]}(t)$. Certainly, $N_{\overline{\mathcal{O}}}\le N_{\mathcal{O}}$ which implies that $N_{\mathcal{O}}$ is bounded below by a positive multiple of t^d. On the other hand, if $m_{\overline{\mathcal{O}}\setminus\mathcal{O}}(\mu)$ is the number of copies of V_μ inside $R[\mathcal{O}]$ whose highest weight v_μ does not belong to $R[\overline{\mathcal{O}}]$, and $N_{\overline{\mathcal{O}}\setminus\mathcal{O}}$ is the sum of all products $m_{\overline{\mathcal{O}}\setminus\mathcal{O}}(\mu)d(\mu)$ for $\|\mu+\rho\|\le t$, then $N_{\mathcal{O}} = N_{\overline{\mathcal{O}}} + N_{\overline{\mathcal{O}}\setminus\mathcal{O}}$.

Let $\{f_1/g_1,\dots,f_r/g_r,h_{r+1},\dots,h_s\}$ be a set of generators of $R[\mathcal{O}]^{\underline{n}_k}$ chosen so that $f_1,\dots,f_r,g_1,\dots,g_r$, and $h_{r+1},\dots,h_s$ all belong to $R[\overline{\mathcal{O}}]^{\underline{n}_k}$. The corresponding (dominant) weights are $\xi_1,\dots,\xi_r,\gamma_1,\dots,\gamma_r$, and $\zeta_{r+1},\dots,\zeta_s$. Suppose V_μ is inside $R[\mathcal{O}]$ and v_μ does not belong to $R[\overline{\mathcal{O}}]$. Since v_μ is a polynomial in the generators of $R[\mathcal{O}]^{\underline{n}_k}$, for some i, $1\le i\le r$, a power of f_i/g_i appears in v_μ. Fix $t>0$. For each i between 1 and r, let $M_i(t)$ denote the largest power of f_i/g_i which appears in a highest weight vector $v_\mu\in R[\mathcal{O}]/R[\overline{\mathcal{O}}]$ such that $\|\mu+\rho\|\le t$. It is clear that for each i between 1 and r, there is a dominant weight ω_i such that $\|\omega_i + M_i(t)(\xi_i - \gamma_i) + \rho\| \leq t$. It follows that $M_i(t)\|(\xi_i - \gamma_i)\| = \|M_i(t)(\xi_i - \gamma_i)\| \leq t$. Hence there is a constant c_i, independent of t, such that $M_i(t)\|\gamma_i\| \leq c_i t$.

Now let $g_t =$ the product of all the powers $g_i^{M_i(t)}$ for all $1 \leq i \leq r$ and set $\nu_t =$ the weight of $g_t = \sum M_i(t)\gamma_i$. So if V_μ is inside $R[\mathcal{O}], v_\mu$ does not belong to $R[\overline{\mathcal{O}}]$ and $\|\mu + \rho\| \leq t$, then $g_t v_\mu \in R[\overline{\mathcal{O}}]$. Clearly we have the inequalities:

$$m_{\overline{\mathcal{O}}\setminus\mathcal{O}}(\mu) \leq m_{\overline{\mathcal{O}}}(\mu + \nu_t) \text{ and } d(\mu) \leq d(\mu + \nu_t)$$

for all μ such that $m_{\overline{\mathcal{O}}\setminus\mathcal{O}}(\mu) \neq 0$ and $\|\mu + \rho\| \leq t$. But the inequalities: $\|\mu + \rho\| \leq t$ and $M_i(t)\|\gamma_i\| \leq c_i t$ (where c_i is independent of t) imply that for some constant κ (independent of t) $\|\mu + \nu_t + \rho\| \leq \kappa t$. It follows that $N_{\overline{\mathcal{O}}\setminus\mathcal{O}}(t) \leq N_{\overline{\mathcal{O}}}(\kappa t)$ which implies that $N_{\mathcal{O}}(t)$ grows no faster than a positive multiple of t^d.

Suppose that $\mathcal{O} = K_C \cdot e$ is a nilpotent orbit in $\underline{p}_C^*$. Let $\overline{\mathcal{O}}^n$ denote the normalization of $\overline{\mathcal{O}}$. We may assume that K_C acts on $\overline{\mathcal{O}}^n$ with finitely many orbits, and the normalization map $\pi : \overline{\mathcal{O}}^n \to \overline{\mathcal{O}}$ is K_C equivariant. (See section 5.3 for a construction of $\overline{\mathcal{O}}^n$.) We can view $R[\overline{\mathcal{O}}^n]$ as a K_C submodule of $R[\mathcal{O}]$, so that $R[\overline{\mathcal{O}}^n]$ is a completely reducible K_C module. $R[\overline{\mathcal{O}}^n]$ is then a finitely generated $S(\underline{p}_C)$ module which satisfies the conditions (5.1.9(a)) and (5.1.9(b)). $\mathbf{M}_{\overline{\mathcal{O}}_n}$ is therefore well defined.

Proposition 5.2.8. *Let $\mathcal{O}$ be a nilpotent K_C orbit in $\underline{p}_C^*$ then $\mathbf{M}_{\overline{\mathcal{O}}} = \mathbf{M}_{\overline{\mathcal{O}}^n}$.*

Proof. $R[\overline{\mathcal{O}}^n]$ is the integral closure of $R[\overline{\mathcal{O}}]$ in $R(\overline{\mathcal{O}})$. Now apply Proposition 5.1.24. □

If $\underline{g}$ is complex, then we may identify K_C with G and $\underline{p}_C^*$ with $\underline{g}^*$. The following results are useful in computations.

Corollary 5.2.9. *Let $\underline{g}$ be a complex semi-simple Lie algebra and let $\mathcal{O}$ be a nilpotent G orbit in $\underline{g}^*$. Then $\mathbf{M}_{\mathcal{O}} = \mathbf{M}_{\overline{\mathcal{O}}}$.*

Proof. Apply the Proposition 5.2.8 and note the $R[\mathcal{O}] = R[\overline{\mathcal{O}}^n]$ since $R[\mathcal{O}]$ is the integral closure of $R[\overline{\mathcal{O}}]$ in $R(\overline{\mathcal{O}})$ (see Lemma 3.7 in [**BK**]). □

The following proposition will be useful later.

Proposition 5.2.10. *Suppose $X = \overline{G_C \cdot e \cap \underline{p}_C^*} = \overline{\mathcal{O}}_1 \cup \ldots \cup \overline{\mathcal{O}}_s$ (disjoint union) where each $\mathcal{O}_i$ is a nilpotent K_C orbit in $\underline{p}_C^*$ whose closure is a component of X. Then $\mathbf{M}_X = \sum_{i=1}^s \mathbf{M}_{\overline{\mathcal{O}}_i}$.*

Proof. We have $R[X] = \sum_{i=1}^s R[\overline{\mathcal{O}}_i]$. Note that $\operatorname{K\dim} R[X] = \operatorname{K\dim} R[\overline{\mathcal{O}}_i]$ which exceeds $\operatorname{K\dim}\{R[\overline{\mathcal{O}}_i] \cap R[\overline{\mathcal{O}}_j]\}$ for all $i \neq j$, and apply the obvious generalization of Lemma 5.1.25. □

Remark 5.2.11. Let X be as in the preceding proposition and assume in addition that $\overline{G_C \cdot e}$ does not contain a G_C orbit of co-dimension 2. Then for each $i, R[\overline{\mathcal{O}}_i^n] = R[\mathcal{O}_i]$, as we argued in Corollary 5.2.9, because each K_C orbit in $\overline{\mathcal{O}}_i$ has codimension at least 2 (since G_C orbits in $\overline{G_C \cdot e}$ are even dimensional). Therefore, $\mathbf{M}_{\mathcal{O}_i} = \mathbf{M}_{\overline{\mathcal{O}}_i^n} = \mathbf{M}_{\overline{\mathcal{O}}_i}$. Hence $\mathbf{M}_X = \sum_{i=1}^s \mathbf{M}_{\mathcal{O}_i}$.

Remark 5.2.12. (Orbit Covers) Let $\mathcal{O} = K_C \cdot e$ where $e \in \underline{p}_C^*$. Let $(K_C^e)'$ be a subgroup of K_C such that $(K_C^e)_o \subseteq (K_C^e)' \subseteq K_C^e$. Then the quotient $\widetilde{\mathcal{O}} = K_C/(K_C^e)'$ is a finite covering of $\mathcal{O} = K_C/K_C^e$. $\mathbf{D} = K_C^e/(K_C^e)'$ is the covering group and $R[\mathcal{O}] = R[\widetilde{\mathcal{O}}]^{\mathbf{D}}$. (The actions of K_C and $\mathbf{D}$ on $\widetilde{\mathcal{O}}$ commute. The $\mathbf{D}$ action is as follows: $\bar{a} \cdot \bar{x} = \overline{xa}$, where $\bar{a} = a(K_C^e)_o$ and $\bar{x} = x(K_C^e)_o$. K_C acts on the left.) This leads to the fact that $\mathbf{M}_{\widetilde{\mathcal{O}}} = d \cdot \mathbf{M}_{\mathcal{O}}$, where $d = |\mathbf{D}|$. Essentially, this is a consequence of the fact that $\deg(N_{\widetilde{\mathcal{O}}} - d \cdot N_{\mathcal{O}})$ is less than $\dim_C \widetilde{\mathcal{O}} = \dim_C \mathcal{O}$. (See Lemma 2.7 in [**Bo**].)

5.3. Multiplicity formulas for rings of regular functions. Let $\{x, e, f\}$ be a $\underline{p}_C^*$ Cayley triple, with $\mathcal{O} = K_C \cdot e$. In order to study $\mathbf{M}_{\overline{\mathcal{O}}}$, we will consider $X = X(e)$ a non-singular variety which is a desingularization of $\overline{\mathcal{O}}$ (see Definition 5.3.1 below). Set $m_X(\mu) =$ the multiplicity of the K-type μ in $R[X]$. $\mathbf{m}_X$ is the corresponding multiplicity distribution, and $\mathbf{M}_X$ is the corresponding asymptotic multiplicity measure. Using ideas of McGovern [**Mc**], we will show how to "approximate" $m_X(\mu)$ in order to compute $\mathbf{M}_X$.

Recall that $\mathbf{B}$ denotes the Killing form of $\underline{g}$. Let $\mathbf{B}_C$ denote the Killing form of $\underline{g}_C$. We want to make explicit the identification of $\underline{g}_C$ and $\underline{g}_C^*$ provided by $\mathbf{B}_C$. If $z \in \underline{g}_C$, then let $z^\sharp$ denote the element $\mathbf{B}_C(z, \cdot)$ in $\underline{g}_C^*$, and if $y \in \underline{g}_C^*$, let $y^\flat$ denote the element in $\underline{g}_C$ such that $y = \mathbf{B}_C(y^\flat, \cdot)$. If $w, y \in \underline{g}_C^*$, then define $[w, y]$ to be $[w^\flat, y^\flat]^\sharp$.

Definition 5.3.1. (Construction of a desingularization of $\overline{\mathcal{O}}$.) Let $\underline{g}_C^*(x; j) = \{z \in \underline{g}_C^* | [x, z] = jz\}$, the j-eigenspace of x. Likewise define $\underline{p}_C^*(x; j)$ and $\underline{k}_C^*(x; j)$. Set $V = V(e) = \sum_{j \geq 2} \underline{g}_C^*(x; j)$,

$$\underline{q}^* = \underline{q}^*(e) = \sum_{j \geq 0} \underline{g}_C^*(x; j), \quad \underline{u}^* = \underline{u}^*(e) = \sum_{j > 0} \underline{g}_C^*(x; j),$$

and $\underline{l}^* = \underline{l}^*(e) = \underline{g}_C^*(x; 0)$. Let $\underline{q}, \underline{l}$, and $\underline{u}$ denote respectively $(\underline{q}^*)^\flat$, $(\underline{l}^*)^\flat$, and $(\underline{u}^*)^\flat$. $\underline{q}, \underline{l}$, and $\underline{u}$ may be regarded as subalgebras of $\underline{g}_C$, so that $\underline{q}$ is a parabolic subalgebra with Levi decomposition $\underline{q} = \underline{l} \oplus \underline{u}$. Let Q, L, and U be the connected subgroups of G_C with Lie algebras $\underline{q}, \underline{l}$, and $\underline{u}$ respectively. It is well known that the morphism $\pi : G_{C \times_Q} V \to \overline{G_C \cdot e}$, defined by $\pi([g, v)] = g \cdot v$ is a desingularization of $\overline{G_C \cdot e}$. By similar arguments, if $\widetilde{V} = \sum_{j \geq 2} \underline{p}_C^*(x; j) = V \cap \underline{p}_C^*$, then the (restriction) mapping

$\pi : K_C \times_{Q\cap K_C} \widetilde{V} \to \overline{K_C \cdot e}$ is a desingularization (resolution of singularities) of $\overline{K_C \cdot e}$ in the sense of [**Slo**].

(Since $ad(e) : \underline{q}^* \to V$ is surjective, $ad(e) : \underline{q}^* \cap \underline{k}_C^* \to V \cap \underline{p}_C^*$ is surjective. This enables us to conclude that the orbit $Q \cap K_c \cdot e$ is dense in $\widetilde{V}$. The remainder of the argument is essentially the same as the one establishing that $\pi : G_C \times_Q V \to \overline{G_C \cdot e}$ is a desingularization of $\overline{G_C \cdot e}$ and may be found at the beginning of the proof of Theorem 3.1 in [**Mc**]. See Section 2 of [**Sek2**] for a discussion of the case when e is a principal nilpotent.)

Set $X = X(e) = K_C \times_{Q\cap K_C} \widetilde{V}$. We also consider X as a vector bundle over $K_C/Q\cap K_C$ with projection map $\tau : X \to K_C/Q\cap K_C$. $\mathbf{O}_X$ will denote the structure sheaf of X.

Throughout this section we will work with the fundamental Cartan subalgebra $\underline{h} = \underline{g}^{\underline{t}}$ of $\underline{g}$. Let $\underline{h} = \underline{t} + \underline{a}'$ be the Cartan decomposition of $\underline{h}$. Let $\Delta = \Delta(\underline{g}_C, h_C)$ and let $\Delta_{ci}, \Delta_{nci}, \Delta_{im}$, and Δ_{cplx} denote the subsets of Δ comprising respectively the compact imaginary, non-compact imaginary, imaginary and complex roots. Then $\underline{g}_C = \underline{k}_C + \underline{p}_C$ where $\underline{k}_C = \underline{t}_C \oplus \sum_{\alpha\in\Delta_{ci}} CX_\alpha \oplus \sum_{\alpha\in\Delta_{cplx}} C(X_\alpha + \theta X_\alpha)$ and $\underline{p}_C = \underline{a}' \oplus \sum_{\alpha\in\Delta_{cplx}} C(X_\alpha - \theta X_\alpha) \oplus \sum_{\alpha\in\Delta_{nci}} CX_\alpha$. Let $\mathcal{Q}$ be the set of non-zero weights of $\underline{t}_C$ in $\underline{p}_C$. Then $\mathcal{Q} = \Delta_{nci} \cup \{\alpha|_{\underline{t}_C} : \alpha \in \Delta_{cplx}\}$, and each weight in $\mathcal{Q}$ has multiplicity one.

If x and $x^\flat$ are as above, we define a positive system for Δ_{im} as follows. First note that since $\{x, e, f\}$ is a $\underline{p}_C^*$ Cayley triple, $x^\flat \in i\underline{t}$. Let $\Delta_{im}(x^\flat) = \{\alpha \in \Delta_{im} | \alpha(x^\flat) = 0\}$. Then $\Delta_{im}(x^\flat)$ is a root system. Choose any positive system $(\Delta_{im}(x^\flat))^+$ for $\Delta_{im}(x^\flat)$. Set $\Delta_{im}^+ = (\Delta_{im}(x^\flat))^+ \cup \{\alpha \in \Delta_{im} | \alpha(x^\flat) > 0\}$. Δ_{im}^+ is a positive root system for Δ_{im}. Now choose an ordering on the full set of non-zero $\underline{t}_C$ weights in $\underline{g}_C$ which is compatible with Δ_{im}^+. This choice gives an order in $\mathcal{Q}$ which is consistent with $\Delta_{nci}^+ = \Delta_{im}^+ \cap \Delta_{nci}$. Denote the resulting set of positive elements in $\mathcal{Q}$ by $\mathcal{Q}^+$. Write $\mathcal{Q}^+ = \{\mu_1, \ldots, \mu_\ell, \mu_{\ell+1}, \ldots, \mu_s\}$ where $\Delta_{nci}^+ = \{\mu_{\ell+1}, \ldots, \mu_s\}$. $\Delta_c^+ = \Delta_{ci}^+ \cup \{\mu_1, \ldots, \mu_\ell\}$ is a positive root system for $\Delta(\underline{k}_C, \underline{t}_C)$.

The action of $\underline{t}_C$ on $\underline{g}_C^*$ is the co-adjoint action. That is, if $w \in \underline{g}_C^*, H \in \underline{t}_C$, and $y \in \underline{g}_C$ then we have: $(H \cdot w)(y) = -w(H \cdot y) = -w([H, y]) = -\mathbf{B}_C(w^\flat, [H, y]) = \mathbf{B}_C([H, w^\flat], y)$. It follows that $H \cdot w = [H^\sharp, w]$.

Now since $\underline{l}^* \supseteq \underline{t}_C^*$, it is clear that $\underline{q}^*, \underline{l}^*$, and $\underline{u}^*$ are $\underline{t}_C$ modules. Suppose that $w \in \underline{q}^*$ is of weight γ for the action of $\underline{t}_C$. Then $[x, w] = x^\flat \cdot w = \gamma(x^\flat)w$. It follows from the definition of $\underline{q}^*$ that $\gamma(x^\flat)$ is a non-negative integer. Define $\Delta(\underline{l}^* \cap \underline{k}_C^*, \underline{t}_C)$ to be the set of non-zero $\underline{t}_C$ weights of $\underline{l}^* \cap \underline{k}_C^*$. Define $\Delta(\underline{l}^* \cap \underline{p}_C^*, \underline{t}_C)$, $\Delta(\underline{u}^* \cap \underline{k}_C^*, \underline{t}_C)$ and $\Delta(\underline{u}^* \cap \underline{p}_C^*, \underline{t}_C)$ analogously. It is clear that $\Delta(\underline{l}^* \cap \underline{k}_C^*, \underline{t}_C)$ (respectively $\Delta(\underline{l}^* \cap \underline{p}_C^*, \underline{t}_C)$) consists of all β in Δ_c (respectively $\mathcal{Q}$) such that $\beta(x^\flat) = 0$. $\Delta(\underline{u}^* \cap \underline{k}_C^*, \underline{t}_C)$ (respectively $\Delta(\underline{u}^* \cap \underline{p}_C^*, \underline{t}_C)$) consists of all β in Δ_c^+ (respectively Q^+) such that $\beta(x^b) > 0$.

Remark 5.3.2. It is clear that $R[X]$ is a finitely generated $(S(\underline{p}_C), K_C)$ module satisfying the conditions (5.1.9(a)) and (5.1.9(b)) and that

$$\operatorname{K}\dim R[X] = \operatorname{K}\dim R[\overline{\mathcal{O}}] = \dim_C \mathcal{O}.$$

Also, $\mathbf{M}_X = \mathbf{M}_{\overline{\mathcal{O}}}$, by Prop. 5.2.8, since X is a normalization of $\overline{\mathcal{O}}$.

Theorem 5.3.3. (McGovern [**Mc**]) *Let* $d = \operatorname{K}\dim R[X] = \dim_C X = \dim_C \mathcal{O}$. *Let* $\mathcal{S} = \chi(H^{\cdot}(X, \mathbf{O}_X)) = \sum_{i=0}^{d}(-1)^i H^i(X, \mathbf{O}_X)$. *Then:*

(a) *For each* $p, H^p(X, \mathbf{O}_X)$ *is a finitely generated* $(S(\underline{p}_C), K_C)$ *module satisfying conditions* (5.1.9(a)) *and* 5.1.9(b)).

(b) *If* $p > 0$, *the support of* $H^p(X, \mathbf{O}_X)$ *is contained in* $\partial\overline{\mathcal{O}} = \overline{\mathcal{O}} \setminus \mathcal{O}$ *(the boundary of* $\overline{\mathcal{O}}$*).*

(c) $\mathbf{M}_X = \mathbf{M}_{\mathcal{S}}$.

(d) *As a* K *module,* $\mathcal{S}$ *is isomorphic to*

$$\sum_j (-1)^j \sum_i (-1)^i H^i\Big(K_C/Q \cap K_C, \quad K_C \times_{Q \cap K_C} [S(\underline{p}_C^*/(\underline{q}^* \cap \underline{p}_C^*)) \otimes \wedge^j \underline{p}_C^*(x;1)]\Big). \tag{5.3.4}$$

Proof. This is based on ideas in the proof of Theorem 3.1 in [**Mc**].

We would like to express the K structure of $R[X]$ in terms of more familiar K modules. Since $R[X] = \Gamma(X, \mathbf{O}_X) = H^0(X, \mathbf{O}_X)$, we begin by investigating $H^p(X, \mathbf{O}_X)$ for arbitrary p. Since $\pi : X \to \overline{\mathcal{O}}$ is proper, for each $p \geq 0$, the higher direct image sheaf $R^p\pi_*\mathbf{O}_X$ is a coherent sheaf on $\overline{\mathcal{O}}$. In addition by Théorème 3.7.3 of [**Grot2**], there is a Leray spectral sequence:

$$H^m(\overline{\mathcal{O}}, R^p\pi_*\mathbf{O}_X) \Rightarrow H^{m+p}(X, \mathbf{O}_X). \tag{5.3.5}$$

Since $\overline{\mathcal{O}}$ is affine, the cohomology $H^m(\overline{\mathcal{O}}, R^p\pi_*\mathbf{O}_X)$ with $m > 0$ vanishes. Hence by (5.3.5) there is an isomorphism: $H^0(\overline{\mathcal{O}}, R^p\pi_*\mathbf{O}_X) \simeq H^p(X, \mathbf{O}_X)$. Now since $R^p\pi_*\mathbf{O}_X$ is coherent on $\overline{\mathcal{O}}, \Gamma(\overline{\mathcal{O}}, R^p\pi_*\mathbf{O}_X) = H^0(\overline{\mathcal{O}}, R^p\pi_*\mathbf{O}_X)$ is finitely generated over $R[\overline{\mathcal{O}}]$. Thus $H^p(X, \mathbf{O}_X)$ satifies (5.1.9(a)). (It satisfies (5.1.9(b)) because of (5.3.8) below.)

Now let $U = \pi^{-1}(\mathcal{O})$ and $Z = X \setminus U$. To prove (b), consider the exact sequence of sheaves on X:

$$0 \to j_!(\mathbf{O}_X|_U) \to \mathbf{O}_X \to i_*(\mathbf{O}_X|_Z) \to 0 \tag{5.3.6}$$

where $i : Z \to X$ and $j : U \to X$ are inclusions. By applying the higher direct image functors $R^p\pi_*$, we obtain a long exact sequence:

$$(5.3.7)\quad \begin{aligned} 0 &\to \pi_*(j_!(\mathbf{O}_X|_U)) \to \pi_*\mathbf{O}_X \to \pi_*(i_*(\mathbf{O}_X|_Z)) \\ &\to R^1\pi_*(j_!(\mathbf{O}_X|_U)) \to R^1\pi_*\mathbf{O}_X \to R^1\pi_*(i_*(\mathbf{O}_X|_Z)) \\ &\to R^2\pi_*(j_!(\mathbf{O}_X|_U)) \to R^2\pi_*\mathbf{O}_X \to R^2\pi_*(i_*(\mathbf{O}_X|_Z)) \to \dots \end{aligned}$$

The mapping $\pi|_U : U \to \mathcal{O}$ is affine. So by Cor. 1.3.2 of [**Grot1**], if $p > 0$, then $R^p\pi_*(j_!(\mathbf{O}_X|_U)) = 0$. From the long exact sequence in (5.3.7) it follows that for all $p > 0$, we have $0 \to R^p\pi_*\mathbf{O}_X \to R^p\pi_*(i_*(\mathbf{O}_X|_Z))$. Therefore the support of the sheaf $R^p\pi_*\mathbf{O}_X$ is contained in the support of the sheaf $R^p\pi_*(i_*(\mathbf{O}_X|_Z))$ which is contained in $\pi[Z]$. It follows that $H^0(\overline{\mathcal{O}}, R^p\pi_*\mathcal{O}_X)$ and hence $H^p(X, \mathbf{O}_X)$ has support in $\pi[Z] = \overline{\mathcal{O}} \setminus \mathcal{O}$.

We now prove (c). For each $i > 0, \mathrm{K}\dim H^i(X, \mathbf{O}_X) < d$, since the support of $H^i(X, \mathbf{O}_X)$ lies in $\overline{\mathcal{O}} \setminus \mathcal{O}$ which has dimension strictly less than d. So if $\mathbf{m}_i$ denotes the multiplicity distribution of $H^i(X,\ \mathbf{O}_X)$, Prop. 5.1.17 implies that for all $f \in C_c^\infty(\underline{k}^*)$, $\lim_{t\to\infty} t^{-d+\dim \underline{k}}\ \mathbf{m}_i(f_t) = 0$. Now $R[X] - \mathcal{S} = \sum_{i=1}^{d}(-1)^i H^i(X, \mathbf{O}_X)$. Thus for all $f \in C_c^\infty(\underline{k}^*)$,

$$(\mathbf{M}_X - \mathbf{M}_{\mathcal{S}})(f) = \sum_{i=1}^{d}(-1)^i \lim_{t\to\infty} t^{-d+\dim \underline{k}}\mathbf{m}_i(f_t) = 0.$$

Hence $M_X = \mathbf{M}_{\mathcal{S}}$.

To prove (d) note that the mapping $\tau : X \to K_C/Q \cap K_C$, is an affine morphism of schemes with X noetherian. So by Chap. III, Cor. 1.3.3 of [**Grot1**],

$$(5.3.8)\qquad H^p(X, \mathbf{O}_X) \simeq H^p(K_C/Q \cap K_C, \tau_*\mathbf{O}_X),$$

where $\tau_*\mathbf{O}_X$ is the direct image of $\mathbf{O}_X$ under τ. So $\chi(H^\cdot(X, \mathbf{O}_X))$, the Euler characteristic of the cohomology groups $H^\cdot(X, \mathbf{O}_X)$, is equal to $\chi(H^\cdot(K_C/Q\cap K_C, \tau_*\mathbf{O}_X))$, the Euler characteristic of the cohomology groups $H^\cdot(K_C/Q \cap K_C, \tau_*\mathbf{O}_X)$. Furthermore, $\tau_*\mathbf{O}_X$ is the sheaf of sections of the bundle

$$\gamma : K_C \times_{Q\cap K_C} R[\widetilde{V}] \to K_C/Q \cap K_C.$$

So $H^p(K_C/Q \cap K_C, \tau_*\mathbf{O}_X) = H^p\left(K_C/Q \cap K_C, K_C \times_{Q\cap K_C} R[\widetilde{V}]\right)$.

Let $b = \dim_C \underline{p}_C^*(x;1)$. Then we also have the following Koszul type resolution of $R[\widetilde{V}]$ as a $Q \cap K_C$ module:

$$(5.3.9)\qquad \begin{aligned} &S(\underline{p}_C^*/(\underline{q}^* \cap \underline{p}_C^*)) \otimes \wedge^b \underline{p}_C^*(x;1) \\ &\to S(\underline{p}_C^*/(\underline{q}^* \cap \underline{p}_C^*)) \otimes \wedge^{b-1} \underline{p}_C^*(x;1) \to \cdots \\ &\to S(\underline{p}_C^*/(\underline{q}^* \cap \underline{p}_C^*)) \otimes \wedge^0 \underline{p}_C^*(x;1) \to R[\widetilde{V}] \to 0. \end{aligned}$$

(See Section 3.5 of [**AJ**].) We can obtain a resolution $K_C \times_{Q\cap K_C} R[\widetilde{V}]$ by replacing each term W of (5.3.9) by $K_C \times_{Q\cap K_C} W$. Using this resolution of $K_C \times_{Q\cap K_C} R[\widetilde{V}]$ and the additivity of the Euler-Poincaré characteristic we conclude that:

$$\begin{aligned}(5.3.10)\qquad &\sum_i (-1)^i H^i\left(K_C/Q\cap K_C, \tau_* \mathbf{O}_X\right)\\ &= \sum_j (-1)^j \sum_i (-1)^i H^i\Big(K_C/Q\cap K_C,\\ &\qquad K_C \times_{Q\cap K_C} [S(\underline{p}_C^*/(\underline{q}^*\cap \underline{p}_C^*)) \otimes \wedge^j \underline{p}_C^*(x;1)]\Big).\end{aligned}$$

This proves part (d) of the theorem since $\mathcal{S} =$

$$\sum_{i=0}^{d} (-1)^i H^i(X, \mathbf{O}_X) = \sum_i (-1)^i H^i\left(K_C/Q\cap K_C, \tau_* \mathbf{O}_X\right).$$

□

Corollary 5.3.11 (Notation(5.3.3)). *If e is even, $m_{\mathcal{S}}(\mu)$ is given by the following formula:*

$$(5.3.12)\qquad m_{\mathcal{S}}(\mu) = \sum_{w\in W} \epsilon(w) p(w(\mu+\rho)-\rho)$$

where p is the Kostant partition function for $\Delta(\underline{u}^*\cap \underline{p}_C^*, \underline{t}_C)$, that is $p(\nu) =$ the number of ways of expressing ν as a non-negative integral linear combination of weights in $\Delta(\underline{u}^*\cap \underline{p}_C^*, \underline{t}_C)$.

Proof. Since e is even, $\underline{p}_C^*(x;1) = 0$ and $\widetilde{V} = \underline{u}^*\cap \underline{p}_C^*$. So that (5.3.10) becomes:

$$(5.3.13)\quad \sum_i (-1)^i H^i\left(K_C/Q\cap K_C, K_C \times_{Q\cap K_C} [S(\underline{p}_C^*/(\underline{q}^*\cap \underline{p}_C^*))]\right).$$

The multiplicity formula (5.3.12) follows from (5.3.13) and the Bott-Borel-Weil Theorem.

The multiplicity formula (5.3.12) can be used to show that $(A^+)^t\,(\mathbf{M}_{\overline{\mathcal{O}}})$ is a sum of expressions of form $\sum_{w\in W} a_w w\cdot(\mathbf{D}\cdot\mathbf{Y}^+)$ where $\mathbf{D}$ is a homogeneous differential operator with constant coefficients on $\underline{t}^*$, $\mathbf{Y}$ is a convolution of Heaviside functions and the a_w are constants. (See Section 8.) □

Proposition 5.3.14. (Compare Theorem 7.2 in [**BK**].) *Let $\underline{g}$ be a complex semi-simple Lie algebra. Suppose that $\underline{s} = \underline{m}\oplus\underline{w}$ is a parabolic subalgebra,*

$h \in \underline{m}$ is a semisimple element, $\underline{m} = \underline{g}^h$ and $\underline{w}$ is the sum of the positive eigenspaces of $\mathrm{ad}(h)$ *acting on $\underline{g}$. Let S, M and W be the corresponding connected subgroups of G. Choose $e \in \underline{w}$, so that $\overline{G \cdot e} = G \cdot \underline{w}$, i.e. so that e is a Richardson element for $\underline{s}$. Let $X = G \times_S \underline{w}$, and $d = \dim_C X$. Set $\ell = [G^e : S^e]$, the degree of the moment map $X \to G \cdot \underline{w}$ (defined by $[(g, v)] \to g \cdot v$). Then N_X and $\ell N_{\overline{G \cdot e}}$ have the same leading terms. (It is clear from the proof that $N_X \geq \ell N_{\overline{G \cdot e}}$.)*

Proof. The moment map $\pi : G \times_S \underline{w} \to G \cdot \underline{w}$ is proper and of degree ℓ [**BB**]. Therefore π is finite (Exercise II.4.6 in [**Ha**]). Since π is finite, $R[X]$ is finitely generated as a module over $\pi^* R[G \cdot \underline{w}]$. Recall that $\underline{n}_k$ is the nilradical of $\underline{k}_C$ which in this case is the same as $\underline{g}_C$. By Corollary 10 of [**Gros**], $R[X]^{\underline{n}_k}$ is finitely generated over $(\pi^* R[G \cdot \underline{w}])^{\underline{n}_k}$. It follows that for some positive integer s we can find highest weight vectors $h_1, \ldots, h_s \in R[X]^{\underline{n}_k}$ such that:

$$R[X]^{\underline{n}_k} = \pi^*(R[G \cdot \underline{w}])^{\underline{n}_k} h_1 + \cdots + \pi^*(R[G \cdot \underline{w}])^{\underline{n}_k} h_s. \tag{5.3.15}$$

We may assume without loss of generality that each h_i belongs to a single weight space. Set $\lambda_i =$ the weight of h_i. Since π has degree $\ell, R(X)$ is a ℓ dimensional vector space over $(R(G \cdot \underline{w}))$. So we can find highest weight vectors $g_1, \ldots, g_\ell \in R[X]^{\underline{n}_k}$ constituting a basis for $R(X)$ over $\pi^*(R(G \cdot \underline{w}))$, i.e., such that:

$$R(X) = \pi^* R(G \cdot \underline{w}) g_1 + \cdots + \pi^* R(G \cdot \underline{w}) g_\ell. \tag{5.3.16}$$

Set $\xi_i =$ weight of g_i. Now consider each h_i appearing on the right hand side of (5.3.15). By (5.3.16) we can find functions $f_{i,1}, \ldots, f_{i,\ell}$ in $\pi^*(R(G \cdot \underline{w}))$ such that $h_i = f_{i,1} g_1 + \cdots + f_{i,\ell} g_\ell$. But since the g_j's are a basis for $R(X)$ over $\pi^* R(G \cdot \underline{w})$, and h_i and all the g_i are $\underline{n}_k$ invariants, the functions $f_{i,1}, \ldots, f_{i,\ell}$ must all belongs to $(\pi^* R(G \cdot \underline{w}))^{\underline{n}_k}$. So by a lemma in [**Ro**], each of these functions is a quotient of functions in $(\pi^* R[G \cdot \underline{w}])^{\underline{n}_k}$. Therefore by taking the product of the denominators of all the $f_{i,j}$ (for $1 \leq i \leq s$ and $1 \leq j \leq \ell$), we can find a function $b \in (\pi^* R[G \cdot \underline{w}])^{\underline{n}_k}$ (belonging to a single weight space) such that for all i, bh_i belongs to the sum:

$$(\pi^* R[G \cdot \underline{w}])^{\underline{n}_k} g_1 + \cdots + (\pi^* R[G \cdot \underline{w}])^{\underline{n}_k} g_\ell.$$

We now see that (5.3.15) implies that:

$$bR[X]^{\underline{n}_k} \subseteq (\pi^* R[G \cdot \underline{w}])^{\underline{n}_k} g_1 + \cdots + (\pi^* R[G \cdot \underline{w}])^{\underline{n}_k} g_\ell \subseteq R[X]^{\underline{n}_k}. \tag{5.3.17}$$

It is easy to deduce from (5.3.17) that N_X and $\ell N_{G \cdot u}$ have the same leading terms as in the argument for Prop. 5.1.22. $\square$

Proposition 5.3.18. *Assume the same hypotheses as in Prop.* 5.3.14 *and the same notation as in the proof. Then as G modules:*

$$R[X] \simeq \sum_i (-1)^i H^i(G/S, G \times_S R[\underline{w}]) \simeq \mathrm{Ind}_M^G(1).$$

Proof. The first isomorphism is established as follows. Let $\zeta : X \to G/S$ be the obvious projection mapping. (Note that X is the cotangent bundle of G/S.) $R[X] = \Gamma(X, \mathbf{O}_X) = H^0(X, \mathbf{O}_X)$. It is known that $H^i(X, \mathbf{O}_X) = 0$ if $i > 0$ (Lemma A2 in [**BK**]). So $R[X]$ is the Euler-Poincaré characteristic of $H^{\cdot}(X, \mathbf{O}_X)$. Since ζ is an affine morphism, for all $i > 0$,

$$H^i(X, \mathbf{O}_X) = H^i(G/S, \zeta_* \mathbf{O}_X) = H^i(G/S, G \times_S R[\underline{w}]).$$

Hence

$$R[X] \simeq \sum_i (-1)^i H^i(G/S,\ G \times_S R[\underline{w}]).$$

The fact that $\sum_i (-1)^i H^i(G/S, G \times_S R[\underline{w}]) \simeq \mathrm{Ind}_M^G(1)$ follows from Lemma 2.1 of [**Mc**]. □

Remark 5.3.19. It follows from (5.3.18) that if μ is an irreducible finite dimensional representation of G on the space V_μ, then $m_X(\mu) = \dim_C V_\mu^M$ and this function is given by the Heckman-Kostant formula. (See Section 6.)

Remark 5.3.20 (Notation 5.3.14). Suppose that either (a) $G \cdot \underline{w}$ is normal and that G^e is connected or (b) $G^e = S^e$; then $m_{G\cdot\underline{w}}(\mu) = m_{G\cdot e}(\mu) = m_{G\cdot h}(\mu) = \dim_C V_\mu^M$ for all $\mu \in \widehat{G}$. Explanation: If (a) holds then apply Theorem 6.3 of [**BK**]; if (b) holds then apply Theorem A1 of [**BK**].

6. Richardson orbits in complex semi-simple Lie algebras.

In this section, we will assume that $\underline{g}$ is a complex semi-simple Lie algebra. But we identify $\underline{g}$ with $\underline{g}^* = \mathrm{Hom}_R(\underline{g}, R)$. $\underline{k}$ is now a compact real form of $\underline{g}$. As usual $\underline{t}$ is a maximal torus of $\underline{k}$. Let $\underline{j}$ be the centralizer of $\underline{t}$ in $\underline{g}$ and let J be the Cartan subgroup of G corresponding to $\underline{j}$. K_C can be identified with G, and $\underline{p}_C^*$ can be identified with $\underline{g}$. Thus if Ω is a nilpotent G orbit in $\underline{g}$, $\mathbf{c}(\Omega)$, the Cayley transform of Ω will be identified with Ω.

Let us adopt the notation of Proposition 5.3.14 so that $\Omega = G \cdot e$ is the Richardson nilpotent orbit for the parabolic subalgebra $\underline{s} = \underline{m} \oplus \underline{w}$. (Recall that S, M and W are the connected subgroups of G corresponding to $\underline{s}, \underline{m}$ and $\underline{w}$). We will also assume that $\underline{j} \subseteq \underline{m}$. The main goal of this section is to show that:

Theorem 6.1.1. with assumptions as above *There is a nonzero constant* c_Ω, *such that* $J_*(\beta_\Omega) = c_\Omega \mathbf{M}_\Omega$. (*Recall from* (5.2.6) *that* $\mathbf{M}_\Omega = \mathbf{M}_{\overline{\Omega}}$.)

Now $\Delta(\underline{k}_C, \underline{t}_C) = \Delta(\underline{g}, \underline{j})$. Assume that $Z \in \underline{t}$ is regular and we define $\Delta^+(\underline{k}_C, \underline{t}_C) = \{\phi \in \Delta(\underline{k}_C, \underline{t}_C) | \phi(iZ) > 0\}$. $\Delta(\underline{w}, \underline{j}) = \Delta(\underline{w}, \underline{t}_C) \subseteq \Delta^+(\underline{k}_C, \underline{t}_C)$. Set $\Delta^+_{\underline{m}} = \Delta(\underline{m}, \underline{t}_C) \cap \Delta^+(\underline{k}_C, \underline{t}_C)$. W is, as usual, the Weyl group of $\Delta(\underline{k}_C, \underline{t}_C)$. $W_{\underline{m}}$ is the Weyl group of $\Delta(\underline{m}, \underline{t}_C)$. $W^1_{\underline{m}}$ is the "standard cross-section" to $W_{\underline{m}}$ in W. That is,

$$W^1_{\underline{m}} = \{\sigma \in W | \alpha \in \Delta^+(\underline{k}_C, \underline{t}_C) \text{ and } \sigma^{-1}\alpha \notin \Delta^+(\underline{k}_C, \underline{t}_C) \text{ implies } \alpha \in \Delta(\underline{w}, \underline{t}_C)\}.$$

We will write $\Delta^+(\underline{k}_C, \underline{t}_C) = \{\phi_1, \dots, \phi_n, \phi_{n+1}, \dots, \phi_{|\Delta^+|}\}$ where $\Delta^+_{\underline{m}} = \{\phi_1, \dots, \phi_n\}$ and $\Delta(\underline{w}, \underline{t}_C) = \{\phi_{n+1}, \dots, \phi_{|\Delta^+|}\}$. Set $\mathbf{Y}^+_{\underline{g}} = \mathbf{H}_{-i\phi_1} * \mathbf{H}_{-i\phi_2} * \cdots * \mathbf{H}_{i\phi_{|\Delta^+|}}$ (the convolution of the Heaviside functions of all the weights $-i\phi_j$), $\mathbf{Y}^+_{\underline{m}} = \mathbf{H}_{-i\phi_1} * \mathbf{H}_{-i\phi_2} * \cdots * \mathbf{H}_{-i\phi_n}$, and $\mathbf{Y}^+_{\underline{w}} = \mathbf{H}_{-i\phi_{n+1}} * \mathbf{H}_{-i\phi_{n+2}} * \cdots * \mathbf{H}_{i\phi_{|\Delta^+|}}$. So that $\mathbf{Y}^+_{\underline{g}} = \mathbf{Y}^+_{\underline{m}} * \mathbf{Y}^+_{\underline{w}}$.

Theorem 6.1.1 is a consequence of:

Theorem 6.1.2. $(A^+)^t(J^*(\beta_{G \cdot e}))$ *and* $(A^+)^t(\mathbf{M}_{G \cdot e})$ *are each a constant multiple of*

$$\mathbf{T}_{\underline{s}} = \sum_{\tau \in W/W_{\underline{m}}} \varepsilon(\tau)\tau \cdot \left\{ \left(\prod_{\alpha \in \Delta^+_{\underline{m}}} \mathbf{D}_\alpha \right) \cdot \mathbf{Y}^+_{\underline{w}} \right\}. \tag{6.1.3}$$

(Recall that $\mathbf{D}_\alpha$ is the directional derivative in the direction $-i\alpha$.)

Remark 6.1.4. Before proving (6.1.2), we note that it is easy to show that (a) $\mathbf{T}_{\underline{s}}$ does not depend on the choice of coset representatives for W/W_m; (b) $\mathbf{T}_{\underline{s}}$ is W skew invariant; and (c) $\mathbf{T}_{\underline{s}}$ has the right degree of homogeneity, namely $-r + |\Delta(\underline{w}, \underline{t}_C)| - |\Delta^+_{\underline{m}}|$. (According to Definition 5.2.2, $p(\Omega) = \dim_C(\Omega) - |\Delta^+(\underline{k}_C, \underline{t}_C)| - r = 2|\Delta(\underline{w}, \underline{t}_C)| - |\Delta^+(\underline{k}_C, \underline{t}_C)| - r = -r + |\Delta(\underline{w}, \underline{t}_C)| - |\Delta^+_{\underline{m}}|$.)

Proof of (6.1.2). We first show that $(A^+)^t(J_*(\beta_\Omega))$ is a constant multiple of $\mathbf{T}_{\underline{s}}$. There are two main points in the calculation. The first is an unpublished result of Harish Chandra that if $f \in C^\infty_c(\underline{g}^*)$, then (up to a constant) $\langle \beta_\Omega, f \rangle$ is the "value" at 0 (actually a limit) of the function defined by applying the differential operator $\left(\prod_{\alpha \in \Delta^+_{\underline{m}}} \mathbf{D}_\alpha\right)^2$ to F_f. (This result has been greatly generalized. See [**BV1**], [**BV2**] and [**Gi**].) Here, F_f is the invariant integral defined as follows:

$$F_f(Z) = \pi_c(Z) \int_{G/J} f(\mathrm{Ad}(x)(Z))d_{G/J}(\dot{x}) \quad (Z \in \underline{j}^*_{\mathrm{reg}})$$

where $d_{G/J}(\dot{x})$ is a G invariant measures on G/J.

($d_{G/J}$ is defined as follows. Let σ denote conjugation with respect to the real form $\underline{k}$ of $\underline{g}$. Then $(X,Y)_\sigma = -\mathbf{B}(X, \sigma Y)$ is scalar product on $\underline{g}$. Then define a Lebesgue measure $d_{\underline{g}}$ on $\underline{g}$ and $\underline{g}^*$ such that the hypercube determined by an orthonormal basis has unit volume. In the same way define a canonical Lebesgue measure $d_{\underline{j}}$ on $\underline{j}$ and $\underline{j}^*$. Now normalize $d_{G/J}$ by the requirement that $d_{\underline{g}} = |\pi_c(H)|^2 d_{G/J}(\dot{x})d_{\underline{j}}H$.)

The second main ingredient in the calculation of $(A^+)^t(J_*(\beta_\Omega))$ is a result of Sengupta ([**Sen1**] and [**Sen2**]), which says that up to a constant multiple independent of Z:

$$(6.1.5) \qquad (A^+)^t(J_*(\beta_{G\cdot Z})) = \sum_{w\in W} \varepsilon(w) w \cdot (\delta_Z * \mathbf{Y}^+_{\underline{g}}).$$

Here are the details of the computation of $(A^+)^t(J_*(\beta_\Omega))$. Note that the "equations" appearing below will usually only be true up to multiplication by non-zero constants which depend on such things as the normalization of measures. Throughout the computation we will use results from section 3 on the actions of groups and differential operators on homogeneous distributions.

Let $\mathbf{D} = \left(\prod_{\alpha\in\Delta^+_{\underline{m}}} \mathbf{D}_\alpha\right)$. Suppose that $\phi \in C^\infty_c(\underline{k}^*)$ and $f = J^*\phi$. Then by definition $\langle J_*(\beta_\Omega), \phi\rangle = \langle \beta_\Omega, f\rangle =$ (by Harish Chandra) $\lim_{t\to 0^+}(\mathbf{D}^2 \cdot F_f)(tZ)$. By the chain rule and the homogeneity of $\mathbf{D}$, we have that $\mathbf{D}^2_Z \cdot \{F_f(tZ)\} = t^{2|\Delta^+_{\underline{m}}|}(\mathbf{D}^2 \cdot F_f)(tZ)$ (The subscript Z indicates differentiation in that variable.) Rewrite this equality as $(\mathbf{D}^2 \cdot F_f)(tZ) = t^{-2|\Delta^+_{\underline{m}}|}\mathbf{D}^2_Z \cdot \{F_f(tZ)\}$. Then

$$(6.1.6) \qquad \langle \beta_\Omega, f\rangle = \lim_{t\to 0^+} t^{-2|\Delta^+_{\underline{m}}|}\mathbf{D}^2_Z \cdot \{F_f(tZ)\}.$$

Since Z is regular, $\beta_{G\cdot Z} = C\pi_c(Z)d_{G/J}$ where C depends on the connected component of $\underline{j}^*_{\mathrm{reg}}$ containing Z. See [**Sen1**]. By formula (2.2.1), $F_f(tZ) = \langle \beta_{G\cdot tZ},\ f\rangle$ (up to a constant independent of Z).

Thus (6.1.6) becomes $\lim_{t\to 0^+} t^{-2|\Delta^+_{\underline{m}}|}\mathbf{D}^2_Z\{\langle \beta_{G\cdot tZ},\ f\rangle\} =$ (by (6.1.5))

$$(6.1.7) \qquad \lim_{t\to 0^+} t^{-2|\Delta^+_{\underline{m}}|}\mathbf{D}^2_Z \left\{ \left\langle \sum_{w\in W} \varepsilon(w) w \cdot (\delta_{tZ} * \mathbf{Y}^+_{\underline{g}}),\ A^+(\phi) \right\rangle \right\} =$$

$$\lim_{t\to 0^+} t^{-2|\Delta^+_{\underline{m}}|} \left\langle \sum_{w\in W} \varepsilon(w) \mathbf{D}^2_Z \left(w \cdot (\delta_{tZ} * \mathbf{Y}^+_{\underline{g}})\right),\ A^+(\phi) \right\rangle.$$

We can rewrite the sum appearing in (6.1.7).

$$(6.1.8)\ \sum_{w\in W}\varepsilon(w)\mathbf{D}_Z^2\left(w\cdot(\delta_{tZ}*\mathbf{Y}_{\underline{g}}^+)\right)$$
$$=\sum_{w\in W}\varepsilon(w)\mathbf{D}_Z^2\left(w\cdot(\delta_{tZ})*w\cdot\mathbf{Y}_{\underline{g}}^+\right)$$
$$=\sum_{w\in W}\varepsilon(w)\left(\mathbf{D}_Z^2(w\cdot(\delta_{tZ}))*w\cdot\mathbf{Y}_{\underline{g}}^+\right)$$
$$=\sum_{w\in W}\varepsilon(w)\left((\mathbf{D}_Z^2(\delta_{twZ}))*w\cdot\mathbf{Y}_{\underline{g}}^+\right)$$

(by Remark 3.1.8)

$$=\sum_{w\in W}\varepsilon(w)\left((-1)^{2|\Delta_{\underline{m}}^+|}t^{2|\Delta_{\underline{m}}^+|}(w^{-1}\cdot\mathbf{D}^2)\cdot(\delta_{twZ})\right)*w\cdot\mathbf{Y}_{\underline{g}}^+$$
$$=(-1)^{2|\Delta_{\underline{m}}^+|}t^{2|\Delta_{\underline{m}}^+|}\sum_{w\in W}\varepsilon(w)\left((w^{-1}\cdot\mathbf{D}^2)\cdot(\delta_{twZ})\right)*w\cdot\mathbf{Y}_{\underline{g}}^+$$
$$=t^{2|\Delta_{\underline{m}}^+|}\sum_{w\in W}\varepsilon(w)\left((w^{-1}\cdot\mathbf{D}^2)\cdot(w\cdot\mathbf{Y}_{\underline{g}}^+)\right)*w\cdot\delta_{tZ}$$
$$=t^{2|\Delta_{\underline{m}}^+|}\sum_{w\in W}\varepsilon(w)\left(w\cdot(\mathbf{D}^2\cdot\mathbf{Y}_{\underline{g}}^+)\right)*w\cdot\delta_{tZ}.$$

Using the fact that $W_{\underline{m}}^1$ is a cross-section for $W_{\underline{m}}$, we rewrite (6.1.8) as:

(6.1.9.)

$$t^{2|\Delta_{\underline{m}}^+|}\sum_{\tau\in W_{\underline{m}}^1}\varepsilon(\tau)\tau\cdot\left\{\sum_{w'\in W_{\underline{m}}}\varepsilon(w')w'\cdot\left((\mathbf{D}^2\cdot\mathbf{Y}_{\underline{g}}^+)*\delta_{tZ}\right)\right\}$$
$$=t^{2|\Delta_{\underline{m}}^+|}\sum_{\tau\in W_{\underline{m}}^1}\varepsilon(\tau)\tau\cdot\left\{\sum_{w'\in W_{\underline{m}}}\varepsilon(w')\left(w'\cdot(\mathbf{D}^2\cdot\mathbf{Y}_{\underline{g}}^+)*w'\cdot\delta_{tZ}\right)\right\}$$
$$=t^{2|\Delta_{\underline{m}}^+|}\sum_{\tau\in W_{\underline{m}}^1}\varepsilon(\tau)\tau\cdot\left\{\sum_{w'\in W_{\underline{m}}}\varepsilon(w')\left([(w')^{-1}\cdot\mathbf{D}^2]\cdot[w'\cdot\mathbf{Y}_{\underline{g}}^+]*w'\cdot\delta_{tZ}\right)\right\}$$

since $\mathbf{D}$ is $W_{\underline{m}}$ skew

$$=t^{2|\Delta_{\underline{m}}^+|}\sum_{\tau\in W_{\underline{m}}^1}\varepsilon(\tau)\tau\cdot\left\{\sum_{w'\in W_{\underline{m}}}\varepsilon(w')\left(\mathbf{D}^2\cdot[w'\cdot\mathbf{Y}_{\underline{g}}^+]*w'\cdot\delta_{tZ}\right)\right\}.$$

Now write $\mathbf{Y}_{\underline{g}}^+=\mathbf{Y}_{\underline{m}}^+*\mathbf{Y}_{\underline{w}}^+$ and observe that if $w'\in W_m$, then $w'\cdot\mathbf{Y}_{\underline{g}}^+=$

$(w' \cdot \mathbf{Y}^+_{\underline{m}}) * \mathbf{Y}^+_{\underline{w}}$. So (6.1.9) becomes:

(6.1.10)
$$t^{2|\Delta^+_{\underline{m}}|} \sum_{\tau \in W^1_{\underline{m}}} \varepsilon(\tau)\tau \cdot \left\{ \sum_{w' \in W_{\underline{m}}} \varepsilon(w') \left(\mathbf{D} \cdot [w' \cdot \mathbf{Y}^+_{\underline{m}}]\right) * w' \cdot \delta_{tZ} * \mathbf{D} \cdot \mathbf{Y}_{\underline{w}} \right\}.$$

Replace the sum in (6.1.7) by (6.1.10). It is clear that $\langle \beta_\Omega,\ f\rangle =$

$$\text{(6.1.11)} \quad \lim_{t \to 0^+} \left\langle \sum_{\tau \in W^1_{\underline{m}}} \varepsilon(\tau)\tau \cdot \left\{ \sum_{w' \in W_{\underline{m}}} [w' \cdot (\mathbf{D} \cdot \mathbf{Y}^+_{\underline{m}})] * w' \cdot \delta_{tZ} * \mathbf{D} \cdot \mathbf{Y}_{\underline{w}} \right\}, A^+(\phi) \right\rangle.$$

Since $\mathbf{D} \cdot \mathbf{Y}^+_{\underline{m}} = \delta_0$,

$$\sum_{w' \in W_{\underline{m}}} [w' \cdot (\mathbf{D} \cdot \mathbf{Y}^+_{\underline{m}})] * w' \cdot \delta_{tZ} = \sum_{w' \in W_{\underline{m}}} w' \cdot \delta_{tZ}.$$

It follows that (6.1.11) equals $\langle \sum_{\tau \in W^1_{\underline{m}}} \varepsilon(\tau)\tau \cdot [\mathbf{D} \cdot \mathbf{Y}^+_{\underline{w}}],\ A^+(\phi)\rangle$. This establishes the first half of Theorem 6.1.2.

We will now compute $(A^+)^t(\mathbf{M}_{\overline{\Omega}}) = (A^+)^t(\mathbf{M}_\Omega)$ and show that it is a constant multiple of $\mathbf{T}_{\underline{s}}$. Set $p = -r + |\Delta(\underline{w}, \underline{t}_C)| - |\Delta^+_{\underline{m}}|$. Choose $f \in C^\infty_c(\underline{t}^*)$. By definition,

$$\text{(6.1.12)} \quad (A^+)^t(\mathbf{M}_{\overline{\Omega}})(f) = c^{-1}_{K,T} \lim_{t \to \infty} t^{-p} \left\{ \sum_{\mu \in \widehat{K}} m_{\overline{\Omega}}(\mu) \sum_{\tau \in W} \varepsilon(\tau)\delta_{-ir(\mu+\rho)} \right\} (f_t).$$

It follows from Props. 5.3.14 that if $X = G \times_S \underline{w}$, then $\mathbf{M}_X$ is a scalar multiple of $\mathbf{M}_{\overline{\Omega}}$. So by Prop. 5.3.18 and Remark 5.3.19, we can replace $m_{\overline{\Omega}}(\mu)$ in (6.1.12) by some multiple, say κ (which depends on Ω) of $m_X(\mu) = \dim_C V^M_\mu$. There is an explicit expression for $\dim_C V^M_\mu$ which is due to Heckman and Kostant:

$$\text{(6.1.13)} \qquad \dim_C V^M_\mu = \sum_{\sigma \in W} \varepsilon(\sigma) p(\sigma(\mu + \rho) - \rho)$$

where p is the Kostant partition function defined relative to $\Delta(\underline{w}, \underline{t}_C)$. (See Lemma 3.1 in [**He**].)

Let $\wedge^+$ denote the lattice of dominant weights in $\underline{t}^*_C$ and let $\mathbf{Q}$ denote the root lattice. Then

(6.1.14)

$$\left\{\sum_{\mu\in\widehat{K}} m_X(\mu)\sum_{\tau\in W}\varepsilon(\tau)\delta_{-ir(\mu+\rho)}\right\}(f_t)$$

$$=\sum_{\tau\in W}\sum_{\sigma\in W}\sum_{\mu\in\wedge^+\cap\mathbf{Q}}\varepsilon(\tau)\varepsilon(\sigma)p(\sigma(\mu+\rho)-\rho)f_t(-i\tau(\mu+\rho))$$

$$=\sum_{\tau\in W}\sum_{\sigma\in W}\sum_{\mu\in\wedge^+\cap\mathbf{Q}}\varepsilon(\tau)\varepsilon(\sigma)\sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \mu=\sigma^{-1}(\rho+\sum k_\alpha\alpha)-\rho}} f_t\left(-i\tau\left(\mu+\rho\right)\right)$$

$$=\sum_{\tau\in W}\sum_{\sigma\in W}\varepsilon(\tau)\varepsilon(\sigma)\sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \sigma^{-1}(\rho+\sum k_\alpha\alpha)-\rho\in\wedge^+\cap\mathbf{Q}}} f_t\left(-i\tau\sigma^{-1}\left(\rho+\sum k_\alpha\alpha\right)\right)$$

$$=\sum_{\tau\in W}\sum_{\sigma\in W}\varepsilon(\tau)\varepsilon(\sigma^{-1})\sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \sigma^{-1}(\rho+\sum k_\alpha\alpha)-\rho\in\wedge^+\cap\mathbf{Q}}} f_t\left(-i\tau\sigma^{-1}\left(\rho+\sum k_\alpha\alpha\right)\right).$$

Set $w=\tau\sigma^{-1}$ and note that $\varepsilon(w)=\varepsilon(\tau)\varepsilon(\sigma^{-1})$. Then (6.1.14) becomes:

$$\sum_{w\in W}\sum_{\sigma\in W}\varepsilon(w)\sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \sigma^{-1}(\rho+\sum k_\alpha\alpha)-\rho\in\wedge^+\cap\mathbf{Q}}} f_t\left(-iw\left(\rho+\sum k_\alpha\alpha\right)\right).$$

Let us now take advantage of the fact that $W^1_{\underline{m}}$ is a cross section for $W_{\underline{m}}$, to write the previous sum as:

(6.1.15)

$$\sum_{\tau\in W^1_{\underline{m}}}\varepsilon(\tau)\sum_{\sigma\in W}\sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \sigma^{-1}(\rho+\sum k_\alpha\alpha)-\rho\in\wedge^+\cap\mathbf{Q}}}\sum_{w'\in W_{\underline{m}}}\varepsilon(w')f_t\left(-i\tau w'\left(\rho+\sum k_\alpha\alpha\right)\right)$$

$$=\sum_{\tau\in W^1_{\underline{m}}}\varepsilon(\tau)\sum_{\sigma\in W}\sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \sigma^{-1}(\rho+\sum k_\alpha\alpha)-\rho\in\wedge^+\cap\mathbf{Q}}}\sum_{w'\in W_{\underline{m}}}\varepsilon(w')(\tau^{-1}\cdot f)_t\Big(-iw'\Big(\rho+$$

$$\sum k_\alpha\alpha\Big)\Big)$$

$$=\sum_{\tau\in W^1_{\underline{m}}}\varepsilon(\tau)\sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \rho+\sum k_\alpha\alpha\text{ is regular}}}\sum_{w'\in W_{\underline{m}}}\varepsilon(w')(\tau^{-1}\cdot f)_t\left(-iw'\left(\rho+\sum k_\alpha\alpha\right)\right).$$

This is clear because if for some σ in W, $\sigma^{-1}(\rho+\sum k_\alpha\alpha)-\rho\in\wedge^+\cap\mathbf{Q}$, then $\rho+\sum k_\alpha\alpha$ is regular. On the other hand if $\rho+\sum k_\alpha\alpha$ is regular then for a unique σ in W, $\sigma^{-1}(\rho + \sum k_\alpha \alpha) - \rho \in \Lambda^+$. Furthermore, since $\sigma^{-1}\rho - \rho$ is a sum of roots, $\sigma^{-1}(\rho + \sum k_\alpha \alpha) - \rho$ must also lie in $\mathbf{Q}$.

It now follows that the limit in (6.1.12)

$$(6.1.16) \quad = \lim_{t\to\infty} t^{-p} \sum_{\tau\in W^1_{\underline{m}}} \varepsilon(\tau) \sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \rho+\sum k_\alpha\alpha \text{ is regular}}} \sum_{w'\in W_{\underline{m}}} \varepsilon(w') (\tau^{-1}\cdot f)_t\left(-iw'\left(\rho+\sum k_\alpha\alpha\right)\right)$$

$$= \lim_{t\to\infty} \sum_{\tau\in W^1_{\underline{m}}} \varepsilon(\tau) t^{-|\Delta(\underline{w})|} \sum_{\substack{k_\alpha\in\mathbf{Z}^+,\alpha\in\Delta(\underline{w})\\ \rho+\sum k_\alpha\alpha \text{ is regular}}} t^{r+|\Delta^+_{\underline{m}}|} \sum_{w'\in W_{\underline{m}}} \varepsilon(w') (\tau^{-1}\cdot f)_t\left(-iw'\left(\rho+\sum k_\alpha\alpha\right)\right)$$

$$= \sum_{\tau\in W^1_{\underline{m}}} \varepsilon(w) w \cdot \left\{ \left(\prod_{\alpha\in\Delta^+_{\underline{m}}} \mathbf{D}_\alpha \right) \cdot \mathbf{Y}^+_{\underline{w}} \right\} (f_t)$$

(up to a constant multiple).

This last assertion follows from Lemma 3.1.9 and Lemma 5.1.4. The proof of Theorem 6.1.2 is now complete. □

7. Even nilpotent orbits in real semisimple Lie algebras.

We again assume that $\underline{g}$ is real. In this section, $\{x, e, f\}$ is an **even** $\underline{p}^*_C$ Cayley triple. The corresponding $\underline{g}^*$ Cayley triple is $\{H, E, F\}$. (See Section 2.3.) $\mathcal{O} = K_C \cdot e$ and $\Omega = G \cdot E$, so that $\mathcal{O} = \mathbf{c}(\Omega)$. Assume the notation from section 5.3. In addition, set $\lambda = -ix = E - F$, and $\Psi = G \cdot \lambda$. Recall that we have chosen a positive root system $\Delta^+ = \Delta^+_c$ for $\Delta(\underline{k}_C, \underline{t}_C)$ such that $x^\flat = i\lambda^\flat$ is dominant Δ^+. We will show:

Theorem 7.1.1 (with assumptions as above). *There is a non-zero constant c_Ω such that $J_*(\beta_\Omega) = c_\Omega \mathbf{M}_{\overline{\mathcal{O}}}$.*

Theorem 7.1.1 will follow from:

Theorem 7.1.2. *$(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ and $(A^+)^t(J_*(\beta_\Omega))$ are each multiples of the sum:*

$$(7.1.3) \qquad \mathbf{T}_{\underline{q}} \sum_{w\in W/W_{L\cap K}} \epsilon(w) w \cdot (\mathbf{D}_{L\cap K} \cdot \mathbf{Y}^+),$$

where $\mathbf{D}_{L\cap K} = \prod_{\alpha\in\Delta^+_{L\cap K}} \mathbf{D}_\alpha$. ($\Delta^+_{L\cap K}$ *is a positive system for* $\Delta(\underline{l}\cap\underline{k}_C,\underline{t}_C)$), *and* $\mathbf{Y}^+ = \mathbf{H}_{-i\mu_1} * \cdots * \mathbf{H}_{-i\mu_n}$ *with* $\{\mu_1,\mu_2,\dots,\mu_n\} = \mathcal{Q}^+\cap\Delta(\underline{u}^*\cap\underline{p}^*_C,\underline{t}_C)$.

Remark 7.1.4. Since x is even, e is also a Richardson element of $\underline{u}$. Thus $\dim_C G_C\cdot e = 2\dim_C\underline{u}$. It follows that $\dim_C \mathcal{O} = \frac{1}{2}(\dim_C G_C\cdot e) = \dim_C\underline{u}$. Now it is easy to see that $\mathbf{T}_{\underline{q}}, (A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$, and $(A^+)^t(J_*(\beta_\Omega))$ all have the same degree of homogeneity, namely $-r-|\Delta^+_{L\cap K}|+|\Delta(\underline{u}\cap\underline{p}^*_C,\underline{t}_C)| = -r-|\Delta^+_{L\cap K}|+\dim_C(\underline{u}^*\cap\underline{p}^*_C)$.

Proof of 7.1.2. The fact that $(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ is a constant multiple of (7.1.3) follows from Prop. (5.3.3) and the multiplicity formula in Corollary (5.3.11) as in our calculation of $(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ (in the second half of the proof of Theorem 6.1.2.) when $\mathcal{O}$ is the Richardson orbit of a complex semi-simple Lie algebra. (This is based on the similarity between the sums appearing in (5.3.12) and (6.1.13).)

The fact that $(A^+)^t(J_*(\beta_\Omega))$ is a constant multiple of the sum $\mathbf{T}_{\underline{q}}$ (defined in (7.1.3)) follows from two basic results. The first is that $\beta_\Omega = \lim_{t\to0^+}\beta_{t\Psi}$. This is an unpublished result of Rao (see [**Ba**]) which is a consequence of the fact that the even nilpotent orbit Ω is a deformation of the elliptic orbit Ψ. The second fact is a formula of Duflo and Vergne [**DV**] for the Fourier transform of $J_*(\beta_\Psi)$. These two results allow us to compute the Fourier transform of $J_*(\beta_\Omega)$, and hence $J_*(\beta_\Omega)$.

Here are the details of the computation of $(A^+)^t(J_*(\beta_\Omega))$. Suppose that $\phi\in C_c^\infty(\underline{k}^*)$ and $f\in C^\infty(\underline{k})$ is such that $\phi=\widehat{f}$. We have:

$$\langle J_*(\beta_\Omega),\phi\rangle = \lim_{t\to0^+}\langle J_*(\beta_{G\cdot t\lambda}),\phi\rangle = \lim_{t\to0^+}\langle J_*(\beta_{t\Psi}),\widehat{f}\rangle. \tag{7.1.4}$$

In the notation of [**DV**], $F_{t\Psi}|_{\underline{k}}$ is the Fourier transform of $J_*(\beta_{t\Psi})$. Therefore,

$$\lim_{t\to0^+}\langle J_*(\beta_{t\Psi}),\widehat{f}\rangle = \lim_{t\to0^+}\langle F_{t\Psi}|_{\underline{k}},\ f\rangle.$$

By definition,

$$\langle F_{t\Psi}|_{\underline{k}},\ f\rangle = \langle (A^+)^t(F_{t\Psi}|_{\underline{k}}),\ (A^+)(f)\rangle.$$

Théorème 41 of [**DV**] gives an integral formula for $(A^+)^t(F_{t\Psi}|_{\underline{k}})$. It follows from this formula that $\lim_{t\to0^+}\langle J_*(\beta_{t\Psi}),\widehat{f}\rangle =$

$$\int_{\underline{t}}\int_{R_n^+}(-1)^n F^{K/L\cap K}\left(\sum_{j=1}^n s_j\mu_j\right)(H)\left(\prod_{\alpha\in\Delta^+}\alpha(H)\right) \cdot (A^+)(f)(H)\,ds_1\dots ds_n\,dH. \tag{7.1.5}$$

In (7.1.5), $F^{K/L\cap K}\left(\sum_{j=1}^{n} s_j\mu_j\right)(\cdot)$ is the function on $\underline{t}_{\text{reg}}$ defined as follows. (See the formula for $F^{K/H}(\mu, X)$ following Corollarie 28 in [**DV**].) We set $\Delta_{K/L\cap K} = \{\alpha \in \Delta | \alpha(i\lambda^\flat) \neq 0\}$, $\Delta^+_{K/L\cap K} = \{\alpha \in \Delta_{K/L\cap K} | \alpha(i\lambda^\flat) > 0\}$ and $\nu = \sum_{j=1}^{n} s_j\mu_j$. Then,

$$F^{K/L\cap K}(\nu)(H) = \frac{1}{|W_{L\cap K}|}\sum_{w\in W}\frac{e^{\nu(wH)}}{\prod_{\alpha\in\Delta^+_{K/L\cap K}}\alpha(wH)}. \tag{7.1.6}$$

(Note that although Théorème 41 is proven under the assumption that rank $\underline{g}$ = rank $\underline{k}$, it easily extends to general $\underline{g}$.)

To complete the proof of (7.1.2) we need to rewrite the product

$$F^{K/L\cap K}(\nu)(H)\left(\prod_{\alpha\in\Delta^+}\alpha(H)\right)$$

in the integrand of (7.1.5). First note that by (7.1.6) and the W skew invariance of $(\prod_{\alpha\in\Delta^+}\alpha(H))$, we have

(7.1.7)
$$\begin{aligned}
&F^{K/L\cap K}(\nu)(H)\left(\prod_{\alpha\in\Delta^+}\alpha(H)\right)\\
&\quad= \frac{1}{|W_{L\cap K}|}\sum_{w\in W}\left\{\frac{e^{\nu(wH)}}{\prod_{\alpha\in\Delta^+_{K/L\cap K}}\alpha(wH)}\epsilon(w)\prod_{\alpha\in\Delta^+}\alpha(wH)\right\}\\
&\quad= \frac{1}{|W_{L\cap K}|}\sum_{w\in W}\epsilon(w)e^{\nu(wH)}\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(wH).
\end{aligned}$$

Let $W^1_{L\cap K}$ be the set of all $w \in W$ such that whenever $\mu \in \underline{t}^*_C$ is dominant relative to $\Delta^+, w\mu$ is dominant relative to $\Delta^+_{L\cap K}$. $W^1_{L\cap K}$ is the standard cross-section to $W_{L\cap K}$ in W. Then each element $w \in W$ has a unique decomposition $w = \sigma\tau$ where $\tau \in W^1_{L\cap K}$ and $\sigma \in W_{L\cap K}$ and length w = length $\sigma+$ length τ. Identify $W/W_{L\cap K}$ with $W^1_{L\cap K}$.

Now rewrite (7.1.7)as:

$$\frac{1}{|W_{L\cap K}|}\sum_{\tau\in W^1_{L\cap K}}\epsilon(\tau)\sum_{\sigma\in W_{L\cap K}}\epsilon(\sigma)e^{\nu(\sigma\tau H)}\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\sigma\tau H). \tag{7.1.8}$$

Since $\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\sigma\tau H) = \epsilon(\sigma)\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\tau H)$, (7.1.8) becomes:

$$\frac{1}{|W_{L\cap K}|}\sum_{\tau\in W^1_{L\cap K}}\epsilon(\tau)\sum_{\sigma\in W_{L\cap K}}e^{\nu(\tau H)}\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\tau H). \tag{7.1.9}$$

If we substitute (7.1.9) for $F^{K/L\cap K}(\nu)(H)\left(\prod_{\alpha\in\Delta^+}\alpha(H)\right)$ inside the integrand of (7.1.5), we obtain:

$$(7.1.10)\qquad \sum_{\tau\in W^1_{L\cap K}}\epsilon(\tau)\int_{\underline{t}}\sum_{\sigma\in W_{L\cap K}}\int_{R_n^+}(-1)^n e^{\nu(\sigma\tau H)} \cdot \prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\tau H)(A^+)(f)(H)ds_1\dots ds_n\, dH.$$

Since $e^{\nu(\sigma\tau H)} = e^{(\sigma^{-1}\cdot\nu)(\tau H)}$, $\nu = \sum_{j=1}^n s_j\mu_j$ and $W_{L\cap K}$ permutes $\{\mu_1, \mu_2, \dots, \mu_n\}$, the integral:

$$\int_{R_n^+}(-1)^n e^{\nu(\sigma\tau H)}\cdot\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\tau H)(A^+)(f)(H)\, ds_1\dots ds_n,$$

has the same value for all $\sigma\in W_{L\cap K}$, namely:

$$(7.1.11)\qquad \int_{R_n^+}(-1)^n e^{\nu(\tau H)}\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\tau H)(A^+)(f)(H)\, ds_1\dots ds_n.$$

In light of (7.1.11), (7.1.10) can be written as:

$$(7.1.12)\quad |W_{L\cap K}|\sum_{\tau\in W^1_{L\cap K}}\epsilon(\tau)\int_{\underline{t}}\int_{R_n^+}(-1)^n e^{\nu(\tau H)} \cdot\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(\tau H)(A^+)(f)(H)\, ds_1\dots ds_n\, dH.$$

If we reverse the order of integration in each summand in (7.1.12), and then make the change of variable $H' = \tau H$, we obtain:

$$(7.1.13)\quad |W_{L\cap K}|\sum_{\tau\in W^1_{L\cap K}}\epsilon(\tau)\int_{R_n^+}\int_{\underline{t}}(-1)^n e^{\nu(H')} \cdot\prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(H')(A^+)(f)(\tau^{-1}H')\, ds_1\dots ds_n\, dH'.$$

Let $g(H) = \prod_{\alpha\in\Delta^+_{L\cap K}}\alpha(H)(A^+)(f)(H)$. Since $(A^+)(f)$ is W skew invariant, (7.1.13) is the same as

$$(7.1.14)\qquad (-1)^n|W_{L\cap K}||W^1_{L\cap K}|\int_{R_+^n}\widehat{g}\left(\sum_{j=1}^n(-is_j\mu_j)\right)\, ds_1\dots ds_n$$

which equals (up to a constant)

$$(7.1.15)\qquad |W|\langle\mathbf{D}_{L\cap K}\cdot(\mathbf{H}_{-i\mu_1}*\dots*\mathbf{H}_{-i\mu_n}), [\widehat{(A^+)(f)}]\rangle.$$

Since $\mathbf{T}_{\underline{q}}$ is W skew invariant and $\langle \mathbf{T}_{\underline{q}}, [\widehat{(A^+)(f)}]\rangle$ is equal to (7.1.15), we conclude that $(A^+)^t(J_*(\beta_\Omega))$ is a constant multiple of $\mathbf{T}_{\underline{q}}$. The proof of Theorem 7.1.2 is now complete. $\square$

Remark 7.1.14. Proposition 7.1.1 can be shown to hold for each K_C nilpotent orbit $\mathcal{O}$ in $\mathcal{N}[\underline{p}_C^*]$ in the case $\underline{g} = su(p,q)$ because the analogue of Proposition 7.1.2 can be established. Since the argument is essentially the same as that for Theorem 6.1.2, we will only sketch it.

Suppose for the moment that $\underline{g}$ is arbitrary, $\mathcal{O}$ is an arbitrary K_C nilpotent orbit in $\mathcal{N}[\underline{p}_C^*]$ and that there is a θ stable parabolic subalgebra $\underline{q} = \underline{l} \oplus \underline{u}$ (Levi decomposition) such that $\overline{\mathcal{O}} = K_C \cdot (\underline{u}^* \cap \underline{p}_C^*)$ (where $\underline{u}^* \subseteq \underline{g}_C^*$ and $\underline{u}^*$ corresponds to $\underline{u}$ as in Section 5.3). Then as in Prop. 5.3.14, if $Y = K_C \times_{Q\cap K_C} (\underline{u}^* \cap \underline{p}_C^*)$, we have a proper morphism $Y \to \overline{\mathcal{O}}$. By the analogue of (5.3.14), N_Y and $bN_{\overline{\mathcal{O}}}$ have the same leading terms, where $b = [K_C^e : Q\cap K_C^e]$. This allows us to conclude that $\mathbf{M}_{\overline{\mathcal{O}}}$ is a constant multiple of $\mathbf{M}_Y$. For the purposes of calculating $\mathbf{M}_Y, m_Y(\mu)$ can be "approximated" by an expression like (5.3.12) (just as for X in (5.3.3), m_X is approximated by (5.3.12)). Noting the similarity between (5.3.12) and (6.1.13), we conclude as in section 6 that $(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ is a multiple of $\mathbf{T}_{\underline{q}}$. ($\mathbf{T}_{\underline{q}}$ is defined as in (7.1.3).)

Now assume that $\underline{g} = su(p,q)$. For each nilpotent K_C orbit $\mathcal{O}$ in $\mathcal{N}[\underline{p}_C^*]$, there is a θ stable parabolic $\underline{q}$ such that $\overline{\mathcal{O}} = K_C \cdot (\underline{u}^* \cap \underline{p}_C^*)$. Barbasch and Vogan give a formula for β_Ω (Theorem 4.2 of [**BV4**]) where Ω is the inverse Cayley transform of $\mathcal{O}$. This formula is similar to (6.1.6). That is, if $\mathbf{D} = \mathbf{D}_L = \prod_{\alpha\in\Delta^+(\underline{l},\underline{t}_C)} \mathbf{D}_\alpha$, then $\beta_\Omega =$ the limit of $\mathbf{D}$ applied to a regular elliptic orbital integral, as the elliptic parameter approaches zero while remaining in a fixed Weyl chamber. We then compute $J_*(\beta_\Omega)$ by using the results of [**DHV**] for the pushforward of the regular elliptic orbit.

These computation lead to an expression for $(A^+)^t(J_*(\beta_\Omega))$ as a sum over W of terms of the form $\epsilon(w)w\cdot(\mathbf{D}\cdot\mathbf{Y}_Q^+)$. Here $\mathbf{Y}_Q^+$ is a convolution of Heaviside functions $H_{-i\nu}$ with ν in Q^+. Since $Q^+ = \Delta^+(\underline{l}^* \cap \underline{p}_C^*, \underline{t}_C) \cup \Delta^+(\underline{u}^* \cap \underline{p}_C^*, \underline{t}_C)$, and $\Delta_L^+ = \Delta^+(\underline{l}^* \cap \underline{k}_C^*, \underline{t}_C) \cup \Delta^+(\underline{l}^* \cap \underline{p}_C^*, \underline{t}_C)$, we can write $\mathbf{D} = \mathbf{D}_{L\cap K} \cdot \mathbf{D}'$ and $\mathbf{Y}_Q^+ = \mathbf{Y}^+ * \mathbf{Y}'$ where $\mathbf{D}'$ is the product of the directional derivatives $\mathbf{D}_\nu$ with ν in $\Delta^+(\underline{l}^* \cap \underline{p}_C^*, \underline{t}_C)$, and $\mathbf{Y}'$ is the convolution of the Heaviside distributions $H_{-i\nu}$ with ν in $\Delta^+(\underline{l}^* \cap \underline{p}_C^*, \underline{t}_C)$. Therefore, $\mathbf{D}\cdot\mathbf{Y}_Q^+ = \mathbf{D}_{L\cap K}\cdot\mathbf{Y}^+ * \mathbf{D}'\cdot\mathbf{Y}' = \mathbf{D}_{L\cap K}\cdot\mathbf{Y}^+ * \delta_0 = \mathbf{D}_{L\cap K}\cdot\mathbf{Y}^+$. A few more elementary manipulations allow us to conclude that $(A^+)^t(J_*(\beta_\Omega))$ is a multiple of $\mathbf{T}_{\underline{q}}$.

8. A conjecture of Vogan.

Theorems 6.1.1 and 7.1.1 and many other examples suggest the validity of the following conjecture:

Conjecture 8.1.1 (Vogan). *If $\mathcal{O}$ is a K_C nilpotent orbit in $\mathcal{N}[\underline{p}_C^*]$, which is the Cayley transform of the nilpotent G orbit Ω in $\mathcal{N}[\underline{g}^*]$ then there is a non-zero constant c_Ω such that $J_*(\beta_\Omega) = c_\Omega \cdot \mathbf{M}_{\overline{\mathcal{O}}}$.*

In addition to the examples presented here which support the conjecture, it clear that $\mathbf{M}_{\overline{\mathcal{O}}}$ and $J_*(\beta_\Omega)$ must have the same general form, because $(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ and $(A^+)^t(J_*(\beta_\Omega))$ have the same general form. That is each of these expressions must be a finite sum of terms like:

$$\sum_{w \in W} a_w w \cdot (\mathbf{D} \cdot \mathbf{Y}^+), \tag{8.1.2}$$

where $\mathbf{D}$ is some homogeneous constant coefficient differential operator on $\underline{t}^*$, and $\mathbf{Y}$ is some convolution of Heaviside functions of real valued weights on $i\underline{t}$, and the a_w are constants. Because of Props. 5.3.3, 5.3.4, and Corollary 5.3.10 it is clear that ultimately $(A^+)^t(\mathbf{M}_{\overline{\mathcal{O}}})$ can be computed in terms of partition functions which must lead, as the computations in Theorem 6.1.2 reveal, to a sum of expressions like (8.1.2). On the other hand, by unpublished results of Harish Chandra, β_Ω must be computable as a limit of some constant coefficient differential operator applied to F_f, Harish Chandra's invariant integral. By applying the results of Sengupta, as in Theorem 6.1.2, we see that $(A^+)^t(J_*(\beta_\Omega))$ must be expressible as a sum of terms like (8.1.2).

So we see that there is a good reason to believe in the validity of Conjecture 8.1.1, although clearly the observations above and the methods of this paper do not suffice to establish it in general.

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