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**CHARACTERS OF SUPERCUSPIDAL REPRESENTATIONS OF
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Let Θ_π be the character of an irreducible supercuspidal representation π of the special linear group $SL_n(F)$, where F is a p -adic field of characteristic zero and residual characteristic greater than n . In this paper, we investigate the existence of a regular elliptic adjoint orbit \mathcal{O}_π such that, up to a nonzero constant, Θ_π (composed with the exponential map) coincides on a neighbourhood of zero with the Fourier transform of the invariant measure on \mathcal{O}_π . When such an orbit \mathcal{O}_π exists, the coefficients in the local expansion of Θ_π as a linear combination of Fourier transforms of nilpotent adjoint orbits are given as multiples of values of the corresponding Shalika germs at \mathcal{O}_π . Let q be the order of the residue class field of F . If n and $q - 1$ are relatively prime, we show that there is an elliptic orbit \mathcal{O}_π as above attached to every irreducible supercuspidal π . When n and $q - 1$ have a common divisor, necessary and sufficient conditions for existence of an orbit \mathcal{O}_π are given in terms of the number of representations in the Langlands L -packet of π .

1. Introduction.

Let $d(\pi)$ be the formal degree of π . Our aim is to determine the conditions under which there exists a regular elliptic element X_π in the Lie algebra of $SL_n(F)$ such that

$$(1.1) \quad \Theta_\pi(\exp X) = d(\pi) \widehat{\mu}_{\mathcal{O}(X_\pi)}(X)$$

for all regular elements X in some neighbourhood of zero in the Lie algebra. Here $\widehat{\mu}_{\mathcal{O}(X_\pi)}$ denotes the Fourier transform of the orbital integral associated to the $\text{Ad } SL_n(F)$ -orbit $\mathcal{O}_\pi = \mathcal{O}(X_\pi)$ of X_π . An $\text{Ad } SL_n(F)$ -orbit \mathcal{O} is said to be nilpotent if it consists of nilpotent elements. Harish-Chandra ([HC2]) proved that there exist constants $c_{\mathcal{O}}(\pi)$ such that

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \text{ nilpotent}} c_{\mathcal{O}}(\pi) \widehat{\mu}_{\mathcal{O}}(X),$$

for regular elements X in some neighbourhood of zero. If (1.1) holds, the coefficients in Harish-Chandra's expansion have the form

$$(1.2) \quad c_{\mathcal{O}}(\pi) = d(\pi) \Gamma_{\mathcal{O}}(X_{\pi}), \quad \mathcal{O} \text{ nilpotent}$$

where $\Gamma_{\mathcal{O}}$ is the Shalika germ associated to the orbit \mathcal{O} .

In an earlier paper ([Mu1]), under the assumption $p > n$, (1.1) and (1.2) were proved for all irreducible supercuspidal representations of $GL_n(F)$. As shown by Howe ([H]) and Moy ([Mo]), the equivalence classes of irreducible supercuspidal representations of $GL_n(F)$ correspond bijectively with the conjugacy classes of admissible characters of multiplicative groups of degree n extensions of F . If θ is such a character, π_{θ} denotes an element of the corresponding equivalence class of representations. An irreducible supercuspidal representation of $SL_n(F)$ is a component of the restriction π'_{θ} of some π_{θ} to $SL_n(F)$. Moy and Sally ([MS]) studied the decompositions of the representations π'_{θ} .

Moy and Sally realized certain (not necessarily irreducible) components of π'_{θ} as representations induced from finite-dimensional representations of open compact subgroups. The inducing data for one of these components $\bar{\pi}$ is the restriction of the inducing data for π_{θ} to $SL_n(F)$. If $X_{\pi_{\theta}}$ is the element of the Lie algebra of $GL_n(F)$ appearing in (1.1) for $\pi = \pi_{\theta}$, set

$$S_{\theta} = X_{\pi_{\theta}} - \frac{\text{tr}(X_{\pi_{\theta}})}{n} I_n,$$

where I_n is the $n \times n$ identity matrix. §2 is devoted to proving (Proposition 2.6)

$$f_{\theta}(1)^{-1} \int_K f_{\theta}(k^{-1} \exp Xk) dk = \int_K \psi_0(\text{tr } S_{\theta} \text{ Ad } k^{-1}(X)) dk,$$

where f_{θ} is a particular matrix coefficient of $\bar{\pi}$, ψ_0 is a nontrivial character of F , K is a certain open compact subgroup, and X is any nilpotent element in the Lie algebra of $SL_n(F)$. Many results in §2 are proved by modifying similar results in §3 of [Mu1].

In Theorem 3.2, using Proposition 2.6 and results of Harish-Chandra, we show that (1.1) holds for $\pi = \bar{\pi}^g$, g in $GL_n(F)$, with $X_{\pi} = \text{Ad } g(S_{\theta})$. It then follows that (1.2) also holds (Corollary 3.5). Necessary and sufficient conditions for the representations $\bar{\pi}^g$ to be irreducible are determined in ([MS]). When these conditions are satisfied, the irreducible components of π'_{θ} are all of the form $\bar{\pi}^g$ ([MS]), and thus (1.1) and (1.2) hold. These irreducible components make up an L-packet of supercuspidal representations, and the associated X_{π} 's make up a set of representatives for the orbits within the

stable orbit of S_θ . These results are summarized in Corollary 3.6. If n and $q - 1$ are relatively prime (recall that q is the order of the residue class field of F), the representations $\bar{\pi}^g$ are irreducible for all admissible characters θ ([MS]), and therefore (1.1) and (1.2) hold for all irreducible supercuspidal representations of $SL_n(F)$.

The case where $\bar{\pi}$ is reducible is considered in §4. The irreducible components of π'_θ still form an L-packet of supercuspidal representations, and we can associate the stable orbit of S_θ to this L-packet. However, as proved in Theorem 4.5, if π is an element of the L-packet, (1.1) does not hold for any X_π . As shown in §3, appropriate direct sums of elements in the L-packet (that is, the representations $\bar{\pi}^g$) satisfy (1.1) and (1.2) with X_π in the stable orbit of S_θ .

Suppose n is prime. Although (1.1) may not hold, modulo determination of the values of the Shalika germs on the regular elliptic set, the coefficients $c_{\mathcal{O}}(\pi)$ appearing in the local character expansion of an irreducible supercuspidal representation are known for all nilpotent orbits \mathcal{O} . For details, see remarks at the end of §4. In this case, Assem([As]) has obtained explicit formulas for the functions $\hat{\mu}_{\mathcal{O}}$.

Results of type (1.1) and (1.2) have also been proved for supercuspidal representations of the unramified 3×3 unitary group ([Mu2]) and other classical groups ([Mu3]).

2. Preliminary results.

Let $n \geq 2$ be an integer which is prime to the residual characteristic p of F . Let $G = GL_n(F)$ and $G' = SL_n(F)$. To each admissible character θ of a degree n extension of F , Howe ([H]) associated a finite-dimensional representation κ_θ of an open, compact mod centre subgroup K_θ of G . The induced representation $\pi_\theta = \text{Ind}_{K_\theta}^G \kappa_\theta$ is irreducible and supercuspidal. In this way, Howe defined an injection from the set of conjugacy classes of admissible characters of degree n extensions of F into the set of equivalence classes of irreducible supercuspidal representations of G . Moy ([Mo]) showed that this map is a bijection. That is, every irreducible supercuspidal representation of G is equivalent to some π_θ .

From this point onward, we assume that p is greater than n . The main result of this section, Proposition 2.6, is the analogue of Proposition 3.10 of [Mu1] for a certain (not necessarily irreducible) component of the restriction of π_θ to G' .

Let E be a finite extension of F such that the degree of E over F is prime to p . We shall write O_E for the ring of integers in E , \mathfrak{p}_E for the maximal prime ideal in O_E , and ϖ_E for a prime element in O_E . Let $N_{E/F}$ and $\text{tr}_{E/F}$

be the norm and trace maps from E to F .

Fix an additive character ψ_F of F having conductor \mathfrak{p}_F , that is, $\psi_F|_{\mathfrak{p}_F} \equiv 1$ and $\psi_F|_{O_F} \not\equiv 1$. In later sections, Fourier transforms will be taken relative to the additive character ψ_0 of F defined by $\psi_0(x) = \psi_F(\varpi x)$. Set $\psi_E = \psi_F \circ \text{tr}_{E/F}$.

If $\theta : E^\times \rightarrow C^\times$ is a continuous quasi-character of E^\times , the conductor exponent $f_E(\theta)$ of θ is the smallest non-negative integer i such that $1 + \mathfrak{p}_E^i$ is contained in the kernel of θ .

Let θ be an admissible character ([H] or [Mo]) of the multiplicative group of a degree n extension E of F . In §3 of [Mu1], an element of E was associated to each such θ . In this paper, we call that element X_θ . For completeness, we restate the definition here.

Lemma 2.1 ([H]). *There exists a unique tower of fields*

$$F = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

and quasi-characters $\chi, \phi_1, \dots, \phi_r$ of $F^\times, E_1^\times, \dots, E_r^\times$ respectively, with ϕ_s generic over E_{s-1} and such that

$$\theta = (\chi \circ N_{E/F})(\phi_1 \circ N_{E/E_1}) \cdots \phi_r.$$

The conductor exponents are unique and satisfy

$$f_E(\phi_1 \circ N_{E/E_1}) > \cdots > f_E(\phi_r).$$

For the definition of generic, see [Mo], [MS] or [Mu1]. Set

$$\ell(s) = \left\lceil \frac{f_{E_s}(\phi_s) + n - 1}{n} \right\rceil, \quad 1 \leq s \leq r - 1.$$

Because $p > n$, the function $x \mapsto \phi_s \left(\sum_{0 \leq m \leq n-1} x^m/m! \right)$ is a character of $\mathfrak{p}_{E_s}^{\ell(s)}$, $s = 1, \dots, r - 1$. Thus there exists $c_s \in E_s$ such that

$$\phi_s \left(\sum_{m=0}^{n-1} x^m/m! \right) = \psi_{E_s}(c_s x), \quad x \in \mathfrak{p}_{E_s}^{\ell(s)}.$$

If $f_E(\phi_r) > 1$, c_r is defined as are c_1, \dots, c_{r-1} . If $f_E(\phi_r) = 1$, c_r is taken to be a root of unity in O_E such that $c_r + \mathfrak{p}_E$ generates O_E/\mathfrak{p}_E over $O_{E_{r-1}}/\mathfrak{p}_{E_{r-1}}$. c_s is not defined the same way as the element c_s of [MS], though it does satisfy the definition in [MS]. X_θ is given by

$$X_\theta = \varpi_F^{-1}(c_1 + \cdots + c_r).$$

Lemma 2.2 ([Mu1], Lemma 3.4). $E = F[X_\theta]$.

Thus X_θ is a regular elliptic element of \mathfrak{g} .

Let $M_s = \text{End}_{E_s} E^+$. For $i \geq 0$, set

$$\mathcal{A}_s^i = \left\{ X \in M_s \mid X \mathfrak{p}_E^j \subset \mathfrak{p}_E^{j+i} \ \forall j \right\}.$$

This definition is extended to all integers via $\mathcal{A}_s^{e_s+i} = \varpi_{E_s} \mathcal{A}_s^i$, where e_s is the ramification degree of E_s over F . \mathfrak{p}_E^0 is understood to mean O_E .

Let $j_s = f_E(\phi_s \circ N_{E/E_s})$. If $j_s > 1$, set $i_s = \lfloor j_s/2 \rfloor$ and $m_s = \lfloor (j_s + 1)/2 \rfloor$. If $j_r = 1$, set $i_r = m_r = 1$. Define

$$\widetilde{K}_\theta = \begin{cases} (1 + \mathcal{A}_{r-1}^{m_r})(1 + \mathcal{A}_{r-2}^{m_{r-1}}) \cdots (1 + \mathcal{A}_0^{m_1}), & \text{if } j_r > 1; \\ (\mathcal{A}_{r-1}^0)^\times (1 + \mathcal{A}_{r-2}^{m_{r-1}}) \cdots (1 + \mathcal{A}_0^{m_1}) & \text{if } j_r = 1. \end{cases}$$

\overline{K}_θ is defined similarly, except with i_s replacing m_s . In [Mu1], the notation K'_θ was used instead of \overline{K}_θ . However, in this paper, A' denotes $A \cap G'$, where A is a subset of G . The inducing subgroup for π_θ is $K_\theta = E^\times \overline{K}_\theta$. $K_E = (\mathcal{A}_0^0)^\times$ is an open compact subgroup of G . If C is an open subset of K'_E and \mathfrak{g}' is the Lie algebra of G' , set

$$\mathcal{I}(X, Y; C) = \int_C \psi_0(\text{tr}(X \text{Ad } k^{-1}(Y))) dk, \quad X, Y \in \mathfrak{g}.$$

Here, tr denotes trace. As in [Mu1], given X in \mathfrak{g} define

$$H_X = \left\{ k \in K_E \mid 1 + \text{Ad } k^{-1}(X) \in \widetilde{K}_\theta \right\}$$

$$H_X^0 = \left\{ k \in K_E \mid 1 + \text{Ad } k^{-1}(X) \in \overline{K}_\theta \right\}.$$

It is easily seen from our description of \mathcal{A}_0^1 in §3 of [Mu1] that $\det(1 + \mathcal{A}_0^1) \subset 1 + \mathfrak{p}_F$. Because p does not divide n ($p > n$), given $x \in 1 + \mathfrak{p}_F$, there exists a unique $y \in 1 + \mathfrak{p}_F$ such that $y^n = x$ ([Ha], p. 217). Given $h \in 1 + \mathcal{A}_0^1$, let $d(h)$ be the scalar matrix y times the identity matrix, where $y \in 1 + \mathfrak{p}_F$ is such that $y^n = \det h^{-1}$. Thus $\det d(h) \det h = 1$. Viewing \mathcal{A}_s^m , $m \geq 1$, as a subset of \mathcal{A}_0^1 , define

$$B_s^m = \{ d(h)h \mid h \in 1 + \mathcal{A}_s^m \}.$$

Let \mathcal{N} be the nilpotent subset of \mathfrak{g} . Since a nilpotent matrix has trace zero, \mathcal{N} is also the nilpotent subset of \mathfrak{g}' .

Lemma 2.3. Assume $X \in \mathcal{N}$.

(1) If $j_r > 1$, then $\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(X_\theta, X; H'_X)$.

- (2) If $j_r = 1$ and $X \in \mathcal{A}_0^1$, then $\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(X_\theta, X; H'_X)$.
- (3) If $j_r = 1$ and $X \notin \mathcal{A}_0^1$, then $\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(X_\theta, X; H_X^{0'})$.

Proof. The proofs of Lemmas 3.7–9 of [Mu1] can be modified slightly to obtain a proof of this lemma.

First, assume that $r = 1$. In this case $\widetilde{K}_\theta = 1 + \mathcal{A}_0^{m_1}$. $X \in \mathcal{A}_0^i - \mathcal{A}_0^{i+1}$ for some integer i . If $i \geq m_1$, then $H'_X = K'_E$. If $j_1 = 1$ and $i = 0$, then $H_X^{0'} = K'_E$. Therefore, we assume that $i < m_1$ if $j_1 > 1$, and $i < 0$ if $j_1 = 1$. Since $H'_X = \emptyset$ if $j_1 > 1$, and $H_X^{0'} = \emptyset$ if $j_1 = 1$, we must show that $\mathcal{I}(X_\theta, X; K'_E) = 0$. Let $\ell = [(j_1 - i + 1)/2]$. At this point, in [Mu1], an extra integration over $1 + \mathcal{A}_0^\ell$ was introduced. Since $1 + \mathcal{A}_0^\ell$ is not a subset of K'_E , we introduce an integration over the subgroup B_0^ℓ of K'_E . $\mathcal{I}(X_\theta, X; K'_E)$ is a nonzero multiple of

$$\int_{K'_E} \int_{B_0^\ell} \psi_0(\text{tr}(X_\theta \text{Ad}(kb)^{-1}(X))) \, db \, dk.$$

It suffices to show that the inner integral vanishes for all $k \in K'_E$. Given $b \in B_0^\ell$, write $b = d(h)h$, $h \in 1 + \mathcal{A}_0^\ell$. Since $d(h)$ is a scalar matrix, $\text{Ad}(kb)^{-1}(X) = \text{Ad}(kh)^{-1}(X)$ for all $k \in K'_E$. Therefore, the inner integral equals

$$\int_{1 + \mathcal{A}_0^\ell} \psi_0(\text{tr}(X_\theta \text{Ad}(kh)^{-1}(X))) \, dh,$$

which, as shown in the proofs of Lemmas 3.7–9 of [Mu1], equals zero.

Assume $r \geq 2$. When $i \geq 1$, this case is argued as in the proof of Lemma 3.7 of [Mu1], except that the integrals over K_E and $1 + \mathcal{A}_s^m$, for appropriately chosen m , are replaced by integrals over K'_E and B_s^m . Since $b \in B_s^m$ has the form $d(h)h$ for some $h \in 1 + \mathcal{A}_s^m$ and $d(h)$ is scalar, the integral over B_s^m equals the integral over $1 + \mathcal{A}_s^m$, and thus has the vanishing properties required to prove the lemma. The proof for $i \leq 0$ is obtained the same way as Lemmas 3.8 and 3.9 of [Mu1]. □

The next lemma will be used in the case $j_r = 1$.

Lemma 2.4. *Let $\bar{\psi}$ be a nontrivial character of a finite field \mathbf{F} . Let $\bar{G} = GL_m(\mathbf{F})$ and $\bar{G}' = SL_m(\mathbf{F})$, $m \geq 2$. Suppose that $|\cdot|$ denotes cardinality, and tr is the trace map on the Lie algebra of \bar{G} . Let S , resp. X , be a regular elliptic, resp. arbitrary, element of the Lie algebra of \bar{G} . Then*

$$|\bar{G}|^{-1} \sum_{x \in \bar{G}} \bar{\psi}(\text{tr}(S \text{Ad } x^{-1}(X))) = |\bar{G}'|^{-1} \sum_{x \in \bar{G}'} \bar{\psi}(\text{tr}(S \text{Ad } x^{-1}(X))).$$

Proof. It suffices to show that

$$\sum_{x \in \bar{G}'} \bar{\psi}(\text{tr}(S \text{Ad}(xy)^{-1}(X)))$$

is independent of the choice of $y \in \bar{G}$. $\mathbf{E} = \mathbf{F}[S]$ is a degree m extension of \mathbf{F} . Since the norm map $N_{\mathbf{E}/\mathbf{F}}$ from \mathbf{E}^\times to \mathbf{F}^\times is onto, there exists $\alpha \in \mathbf{E}^\times$ such that $N_{\mathbf{E}/\mathbf{F}}(\alpha) = \det y$. Identifying α with an element of \bar{G} which commutes with S ,

$$\begin{aligned} \bar{\psi}(\operatorname{tr}(S \operatorname{Ad}(xy)^{-1}(X))) &= \bar{\psi}(\operatorname{tr}(\operatorname{Ad} \alpha(S) \operatorname{Ad}(\alpha y^{-1} x^{-1})(X))) \\ &= \bar{\psi}(\operatorname{tr}(S \operatorname{Ad}(\alpha y^{-1} x^{-1})(X))). \end{aligned}$$

Because $\det(\alpha y^{-1}) = N_{\mathbf{E}/\mathbf{F}}(\alpha) \det y^{-1} = 1$, αy^{-1} can be absorbed into the sum over $x \in \bar{G}'$. □

Suppose $\pi_\theta = \operatorname{Ind}_{K_\theta}^G \kappa_\theta$. Let ρ_θ be the character of κ_θ . Define $f_\theta : G \rightarrow C$ by

$$f_\theta(x) = \begin{cases} \rho_\theta(x) & \text{if } x \in K_\theta, \\ 0 & \text{otherwise.} \end{cases}$$

The representation

$$\bar{\pi} = \operatorname{Ind}_{K'_\theta}^{G'}(\kappa_\theta|_{K'_\theta})$$

is a supercuspidal representation of G' and is a component of the restriction of π_θ to G' ([MS]). The restriction of f_θ to G' is a matrix coefficient of $\bar{\pi}$. Define

$$(2.5) \quad S_\theta = X_\theta - \frac{(\operatorname{tr}_{E/F} X_\theta)}{n} I_n,$$

where I_n is the $n \times n$ identity matrix.

Proposition 2.6. *Let $X \in \mathcal{N}$. Then*

$$f_\theta(1)^{-1} \int_{K'_E} f_\theta(k^{-1} \exp X k) dk = \mathcal{I}(S_\theta, X; K'_E).$$

Proof. Because $\operatorname{tr} X = 0$, and X_θ and S_θ differ by a scalar matrix,

$$\mathcal{I}(X_\theta, X; K'_E) = \mathcal{I}(S_\theta, X; K'_E).$$

Thus in the statement of the proposition S_θ can be replaced by X_θ .

The proof of this proposition is a slight modification of the proof of Proposition 3.10 of [Mu1].

The representation κ_θ is a tensor product $(\chi \circ \det) \otimes \kappa_1 \otimes \cdots \otimes \kappa_r$. ρ_s , $1 \leq s \leq r$, denotes the character of κ_s .

As observed in [Mu1], if $X \in \mathcal{N}$, then

$$\exp X \in K_\theta \iff \exp X \in \bar{K}_\theta$$

Thus

$$f_\theta(1)^{-1} \int_{K'_E} f_\theta(k^{-1} \exp Xk) dk = \rho_\theta(1)^{-1} \int_{H_X^0} \rho_\theta(k^{-1} \exp Xk) dk.$$

Case 1: As shown in [Mu1], if $X \in \mathcal{N}$, then

$$\frac{\rho_\theta(\exp X)}{\rho_\theta(1)} = \begin{cases} \psi_0(\text{tr}(X_\theta X)), & \text{if } \exp X \in \widetilde{K}_\theta, \\ 0 & \text{if } \exp X \in \overline{K}_\theta - \widetilde{K}_\theta. \end{cases}$$

Therefore

$$\begin{aligned} f_\theta(1)^{-1} \int_{K'_E} f_\theta(k^{-1} \exp Xk) dk &= \int_{H'_X} \psi_0(\text{tr}(X_\theta \text{Ad } k^{-1}(X))) dk \\ &= \mathcal{I}(X_\theta, X; H'_X) = \mathcal{I}(X_\theta, X; K'_E). \end{aligned}$$

The last equality is Lemma 2.3(1).

Case 2: $j_r = 1$. The representations κ_s , $1 \leq s \leq r-1$ and κ_r are considered separately.

A certain cuspidal representation of the finite general linear group

$$\left(\text{cal } A_{r-1}^0 \right)^* / 1 + \mathcal{A}_{r-1}^1$$

is used to produce the representation K_r . Lemma 2.4 shows that the Green functions attached to elliptic Cartan subgroups are the same the finite general linear and special linear groups. As shown in Proposition 3.10 of [Mu1], if $x \in \mathcal{N}$ is such that $\exp X \in \overline{K}_\theta$, then

$$\frac{\rho_r(\exp X)}{\rho_r(1)} = \int_{(\text{cal } A_{r-1}^0)^*} \Psi_F(\text{tr}(c_r \text{Ad } h^{-1}(X))) dh.$$

By Lemma 2.4, we may replace $\left(\text{cal } A_{r-1}^0 \right)^*$ with $\left(\text{cal } A_{r-1}^0 \right)^* \cap G'$ in the above integral.

For $1 \leq i \leq r-1$, define

$$K_s = (1 + \mathcal{A}_{r-1}^{i_r}) \cdots (1 + \mathcal{A}_s^{i_s+1}) \quad \text{and} \quad L_s = (1 + \mathcal{A}_{s-1}^{i_s}) \cdots (1 + \mathcal{A}_0^{i_1}).$$

Set $L_0 = \{1\}$. As was shown in [Mu1], if $X \in \mathcal{N}$ is such that $\exp X \in \overline{K}_\theta$,

$$\frac{\rho_s(\exp X)}{\rho_s(1)} = \begin{cases} \int_{1 + \mathcal{A}_{s-1}^{i_s}} \psi_F(\text{tr}(c_s \text{Ad } h^{-1}(X))) dh & \text{if } \exp X \in K_s(1 + \mathcal{A}_{s-1}^{m_s})L_{s-1} \cup (\overline{K}_\theta - K_s L_s), \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as in the proof of Lemma 2.3,

$$\int_{1+\mathcal{A}_{s-1}^{i_s}} \psi_F(\text{tr}(c_s \text{Ad } h^{-1}(X))) \, dh = \int_{B_{s-1}^{i_s}} \psi_F(\text{tr}(c_s \text{Ad } b^{-1}(X))) \, db.$$

Let $X \in \mathcal{N}$. If $X \in \mathcal{A}_0^1$ and $\exp X \in \overline{K}_\theta - \widetilde{K}_\theta$, then $\exp X \in K_s L_s - K_s(1 + \mathcal{A}_{s-1}^{m_{s-1}})L_{s-1}$ for some s , so $\rho_s(\exp X) = 0$. Thus $\rho_\theta(\exp X) = 0$. All remaining $X \in \mathcal{N}$ such that $\exp X \in K_\theta$ satisfy one of the following:

- (i) $X \in \mathcal{A}_0^1$ and $\exp X \in \widetilde{K}_\theta$
- (ii) $X \in \mathcal{A}_0^0 - \mathcal{A}_0^1$ and $\exp X \in \overline{K}_\theta$.

For these X ,

$$\begin{aligned} (2.7) \quad \frac{\rho_\theta(\exp X)}{\rho_\theta(1)} &= \left(\int_{((\mathcal{A}_{r-1}^0)^\times \cap G')} \psi_F(\text{tr}(c_r \text{Ad } h^{-1}(X))) \, dh \right) \\ &\quad \cdot \prod_{s=1}^{r-1} \int_{B_{s-1}^{i_s}} \psi_F(\text{tr}(c_s \text{Ad } b^{-1}(X))) \, db \\ &= \int_{L_\theta} \psi_0(\text{tr}(X_\theta \text{Ad } h^{-1}(X))) \, dh. \end{aligned}$$

To obtain the second equality argue as in [Mu1] (following equation (3.14)). Here

$$L_\theta = ((\mathcal{A}_{r-1}^0)^\times \cap G') \prod_{s=1}^{r-1} B_{s-1}^{i_s}$$

is a subgroup of K'_E . It follows from (2.7) that for $X \in \mathcal{N} \cap \mathcal{A}_0^1$,

$$\begin{aligned} f_\theta(1)^{-1} \int_{K'_E} f_\theta(k^{-1} \exp X k) \, dk &= \rho_\theta(1)^{-1} \int_{H'_X} \rho_\theta(k^{-1} \exp X k) \, dk \\ &= \int_{H'_X} \int_{L_\theta} \psi_0(\text{tr}(X_\theta \text{Ad}(kh)^{-1}(X))) \, dh \, dk \\ &= \mathcal{I}(X_\theta, X; H'_X). \end{aligned}$$

The last equality holds because H'_X is invariant under translation by L_θ . A similar equality holds for $X \in \mathcal{N} \cap (\mathcal{A}_0^0 - \mathcal{A}_0^1)$, except with H_X replaced by $H_{X'}^{0'}$. If $X \in \mathcal{N}$ and $X \notin \mathcal{A}_0^0$, then $f_\theta(k^{-1} \exp X k) = 0$ for all $k \in K'_E$, and $H_{X'}^{0'} = \emptyset$. Apply Lemma 2.3(2) and (3) to complete the proof. \square

3. The character of $\bar{\pi}$ as a Fourier transform.

Let θ be an admissible character of the multiplicative group of a degree n extension E of F . Define

$$G_E = E^\times G' = \{x \in G \mid \det x \in N_{E/F}(E^\times)\}.$$

As in §2, $\bar{\pi}$ denotes the supercuspidal representation of G' defined by

$$\bar{\pi} = \text{Ind}_{K'_\theta}^{G'}(\kappa_\theta \mid K'_\theta).$$

Then ([MS])

$$(3.1) \quad \pi_\theta \mid G' = \bigoplus_{g \in G/G_E} \bar{\pi}^g,$$

where $\bar{\pi}^g(x) = \bar{\pi}(g^{-1}xg)$, $x \in G'$, $g \in G$. Two of the main results, Theorem 3.2 and Corollary 3.5, are proved for the representations $\bar{\pi}^g$, $g \in G/G_E$. As a consequence (Corollary 3.6), (1.1) and (1.2) hold for the irreducible components of $\pi_\theta \mid G'$ whenever there are exactly $|F^\times/N_{E/F}(E^\times)|$ such components.

Given f in $C_c^\infty(\mathfrak{g}')$, the space of locally constant, compactly supported, complex-valued functions on \mathfrak{g}' , let \hat{f} be the function in $C_c^\infty(\mathfrak{g}')$ defined by

$$\hat{f}(X) = \int_{\mathfrak{g}'} \psi_0(\text{tr}(XY))f(Y) dY.$$

The Haar measure dY on \mathfrak{g}' is assumed to be self-dual with respect to $\hat{}$. Given X in \mathfrak{g}' , $\mathcal{O}(X)$ denotes the $\text{Ad } G'$ -orbit of X . If $\mu_{\mathcal{O}(X)}$ is the distribution given by integration over the orbit $\mathcal{O}(X)$, the Fourier transform $\hat{\mu}_{\mathcal{O}(X)}$ is given by $\hat{\mu}_{\mathcal{O}(X)}(f) = \mu_{\mathcal{O}(X)}(\hat{f})$, f in $C_c^\infty(\mathfrak{g}')$. Let \mathfrak{g}'_{reg} be the regular subset of \mathfrak{g}' . Recall ([HC2]) that $\hat{\mu}_{\mathcal{O}(X)}$ can be realized as a locally integrable function (also called $\hat{\mu}_{\mathcal{O}(X)}$) on \mathfrak{g}' which is locally constant on \mathfrak{g}'_{reg} . If a representative of an orbit \mathcal{O} is not specified, the notation $\mu_{\mathcal{O}}$ and $\hat{\mu}_{\mathcal{O}}$ will be used for the corresponding orbital integral and its Fourier transform.

Fix a Haar measure dx on G' . If X is a regular elliptic element in \mathfrak{g}' , the measure on $\mathcal{O}(X)$ is normalized to equal dx . Formal degrees of supercuspidal representations are computed relative to dx . Haar measure on any compact group is normalized so that the total volume of the group equals one.

Let \mathfrak{g}'^* be an open $\text{Ad } G$ -invariant subset of \mathfrak{g}' containing zero such that $\exp : \mathfrak{g}'^* \rightarrow G'$ is defined and $\exp(\text{Ad } x(X)) = x \exp X x^{-1}$ for x in G and X in \mathfrak{g}'^* . Fix an integer $\ell \geq 1$ such that $\mathfrak{g}(\mathfrak{p}^\ell) \subset \mathcal{A}_0^{j_1}$. Choose an integer i large enough that, if $V_\pi = \mathfrak{g}(\mathfrak{p}^i)'$,

- (i) $V_\pi \subset \mathfrak{g}'^*$,

(ii) $i \geq \max \{ \ell, n(\ell + e(F/Q_p))/(p - n + 1) \}$.

Theorem 3.2. *Let S_θ be as in (2.5). Then, if $g \in G$ and $X \in \text{Ad } g(V_\pi) \cap \mathfrak{g}'_{reg}$,*

$$\Theta_{\bar{\pi}^g}(\exp X) = d(\bar{\pi}^g) \widehat{\mu}_{\mathcal{O}(\text{Ad } g(S_\theta))}(X).$$

Proof. By definition of $\bar{\pi}^g$,

$$\Theta_{\bar{\pi}^g}(x) = \Theta_{\bar{\pi}}(g^{-1}xg) \quad x \in G',$$

and $d(\bar{\pi}^g) = d(\bar{\pi})$. Also

$$\widehat{\mu}_{\mathcal{O}(\text{Ad } g(Y))}(X) = \widehat{\mu}_{\mathcal{O}(Y)}(\text{Ad } g^{-1}(X)), \quad X \in \mathfrak{g}'_{reg}, Y \in \mathfrak{g}'.$$

Therefore, it is sufficient to prove the theorem for $g = 1$.

Let K_0 be any open compact subgroup of G' . As shown in Lemma 4.1(1) of [Mu1], Harish-Chandra's integral formula for $\widehat{\mu}_{\mathcal{O}(S_\theta)}(X)$ ([HC2], Lemma 19) can be rewritten as:

(3.3)

$$\begin{aligned} \widehat{\mu}_{\mathcal{O}(S_\theta)}(X) &= \int_{G'} \int_{K_0} \left[\int_{K'_E} \psi_0(\text{tr}(S_\theta \text{Ad}(kxh)^{-1}(X))) dh \right] dk dx \\ &= \int_{G'} \int_{K_0} \mathcal{I}(S_\theta, \text{Ad}(kx)^{-1}(X); K'_E) dk dx, \quad X \in \mathfrak{g}'_{reg}. \end{aligned}$$

Since $f_\theta|_{K'_\theta}$ is a matrix coefficient of $\bar{\pi}$, Harish-Chandra's integral formula for $\Theta_{\bar{\pi}}$, ([HC1, p. 60]), can be rewritten as ([Mu1], Lemma 4.1(2)):

$$(3.4) \quad \Theta_{\bar{\pi}}(\exp X) = \frac{d(\bar{\pi})}{f_\theta(1)} \int_{G'} \int_{K_0} \left[\int_{K'_E} f_\theta((kxh)^{-1}(\exp X)kxh) dh \right] dk dx, \\ X \in \mathfrak{g}'^* \cap \mathfrak{g}'_{reg}.$$

Fix $x \in G'$ and $k \in K'_E$. Then there exist $Y \in \mathcal{N}$ and $Z \in V_\pi$ such that $\text{Ad}(kx)^{-1}(X) = Y + Z$. This follows from (see Lemma 4.2 of [Mu1])

$$\text{Ad } x^{-1}(\mathfrak{g}(\mathfrak{p}^t)') \subset \mathcal{N} + \mathfrak{g}(\mathfrak{p}^t)', \quad x \in G', t \geq 1.$$

As shown in the proof of Theorem 4.3 of [Mu1],

$$f_\theta(h^{-1} \exp(Y + Z)h) = f_\theta(h^{-1}(\exp Y)h), \quad h \in K'_E.$$

It follows from $\text{tr } \mathcal{A}_0^1 \subset \mathfrak{p}_F$ and $\varpi_F \mathcal{A}_0^t = \mathcal{A}_0^{t+e}$, $e = e(E/F)$, that $\text{tr } \mathcal{A}_0^m \subset \mathfrak{p}_F^{[(m-1)/e]+1}$. As a consequence of $\varpi_F X_\theta \in \mathcal{A}_0^{-j_1+1}$ and $Z \in \mathcal{A}_0^{j_1}$, we have

$X_\theta Z \in \varpi_F^{-1} \mathcal{A}_0^1$, $\text{tr}_{E/F} X_\theta = \text{tr} X_\theta \in \mathfrak{p}_F^{[-j_1/e]+1}$ and $\text{tr} Z \in \mathfrak{p}_F^{[(j_1-1)/e]+1}$. Therefore,

$$\psi_0(\text{tr}(S_\theta Z)) = \psi_0(\text{tr}(X_\theta Z)) \psi_0(\text{tr}_{E/F} X_\theta \text{tr} Z)^{-1} = 1.$$

Thus

$$\psi_0(\text{tr}(S_\theta(Y + Z))) = \psi_0(\text{tr}(S_\theta Y)).$$

We can now apply Proposition 2.6 to the inner integrals in (3.3) and (3.4), completing the proof. \square

Let $(\mathcal{N})'$ be the set of nilpotent $\text{Ad } G'$ -orbits in \mathfrak{g}' . Suppose π is an admissible representation of G' of finite length. If $\mathcal{O} \in (\mathcal{N})'$, $c_{\mathcal{O}}(\pi)$ denotes the coefficient of $\widehat{\mu}_{\mathcal{O}}$ in Harish-Chandra's local character expansion of π at the identity ([HC2]):

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \in (\mathcal{N})'} c_{\mathcal{O}}(\pi) \widehat{\mu}_{\mathcal{O}}(X),$$

for $X \in \mathfrak{g}'_{reg}$ sufficiently close to zero. For $\mathcal{O} \in (\mathcal{N})'$, let $\Gamma_{\mathcal{O}} : \mathfrak{g}'_{reg} \rightarrow R$ be the Shalika germ corresponding to \mathcal{O} ([HC2]).

Corollary 3.5. *Let $g \in G$. Then*

$$c_{\mathcal{O}}(\overline{\pi}^g) = d(\overline{\pi}^g) \Gamma_{\mathcal{O}}(\text{Ad } g(S_\theta)), \quad \mathcal{O} \in (\mathcal{N})'.$$

Proof. As follows from Lemma 21 of [HC2], there exists an open neighbourhood V of zero in \mathfrak{g}' such that:

$$\widehat{\mu}_{\mathcal{O}(\text{Ad } g(S_\theta))}(X) = \sum_{\mathcal{O} \in (\mathcal{N})'} \Gamma_{\mathcal{O}}(\text{Ad } g(S_\theta)) \widehat{\mu}_{\mathcal{O}}(X), \quad X \in V \cap \mathfrak{g}'_{reg}.$$

The corollary is now a consequence of Theorem 3.2 and the linear independence of the functions $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N})'$ ([HC2]). \square

An irreducible supercuspidal representation of G' is a component of $\pi_\theta | G'$, for some admissible character θ of E^\times , where E is a degree n extension of F ([MS]). Each π_θ decomposes with multiplicity one upon restriction to G' ([T]). An L-packet of supercuspidal representations of G' consists of the irreducible components of the restriction of an irreducible supercuspidal representation of G to G' ([GK]).

Suppose θ is such that $j_r = 1$. Since ϕ_r is a character of E^\times which is trivial on $1 + \mathfrak{p}_E$, ϕ_r may be viewed as a character $\overline{\phi}_r$ of \mathbf{E}^\times , where \mathbf{E} is the residue class field of E . Let N_1 be the kernel of the norm map from \mathbf{E}^\times to \mathbf{E}_{r-1}^\times . As in [MS], we define $\overline{\phi}_r | N_1$ to be *regular* if the number of distinct

conjugates of $\bar{\phi}_r | N_1$ under the action of the Galois group of \mathbf{E} over \mathbf{E}_{r-1} is equal to $[\mathbf{E} : \mathbf{E}_{r-1}]$.

Corollary 3.6. *Let π be an irreducible supercuspidal representation of G' . Choose θ such that π is a component of $\pi_\theta | G'$. Suppose one of the following conditions holds:*

- (i) $j_r > 1$,
- (ii) $j_r = 1$ and $\bar{\phi}_r | N_1$ is regular.

Then there exists a regular elliptic $X_\pi \in \mathfrak{g}'$ such that

- (1) $\Theta_\pi \circ \exp = d(\pi) \hat{\mu}_{\mathcal{O}(X_\pi)}$ on some open neighbourhood of zero intersected with $\mathfrak{g}'^* \cap \mathfrak{g}'_{reg}$,
- (2) $c_{\mathcal{O}}(\pi) = d(\pi) \Gamma_{\mathcal{O}}(X_\pi)$, $\mathcal{O} \in (\mathcal{N})'$,
- (3) *The L-packet of π is $\{\pi^g | g \in G/G_E\}$. (1) and (2) hold for π^g with $X_{\pi^g} = \text{Ad } g(X_\pi)$.*

Proof. As proved in [MS], conditions (i) and (ii) are necessary and sufficient for each of the representations $\bar{\pi}^g$, $g \in G/G_E$, to be irreducible. In that case (see (3.1)), the representations $\bar{\pi}^g$ are the members of the L-packet of π , and (1), (2), and (3) are restatements of Theorem 3.2 and Corollary 3.5. \square

Remark 3.7. Moy and Sally showed that if n and $q-1$ are relatively prime, then, whenever $j_r = 1$, $\bar{\phi}_r | N_1$ is regular ([MS], Cor. 3.15). Therefore, (1) and (2) hold for all irreducible supercuspidal representations of G' when n and $q-1$ are relatively prime.

Two elements X_1 and X_2 of \mathfrak{g}' are *stably conjugate* if there exists g in G such that $X_2 = \text{Ad } g(X_1)$. The *stable orbit* $\mathcal{O}_{st}(X)$ of X in \mathfrak{g}' consists of the set of stable conjugates of X . Given θ , since $E = F[S_\theta]$ and $S_\theta \in \mathfrak{g}'_{reg}$,

$$\mathcal{O}_{st}(S_\theta) = \cup_{g \in G/G_E} \mathcal{O}(\text{Ad } g(S_\theta)).$$

To the L-packet of supercuspidal representations of G' consisting of the components of $\pi_\theta | G'$, we associate the stable orbit $\mathcal{O}_{st}(S_\theta)$. Of course, the choice of θ is not unique. However, as discussed in §4 of [MS], any two choices for θ must satisfy certain conjugacy conditions. Corollary 3.6 deals with those L-packets which contain $|F^\times/N_{E/F}(E^\times)| = |G/G_E|$ representations. In this case, the representations in the L-packet correspond to the $\text{Ad } G'$ -orbits in the associated stable orbit via Corollary 3.6(3). If an L-packet contains more than $|F^\times/N_{E/F}(E^\times)|$ representations, we do not have such a correspondence. The elements of the L-packet are the irreducible components of the representations $\bar{\pi}^g$, g in G/G_E . This case is discussed in more detail in the next section.

4. The case $\bar{\pi}$ reducible.

Let π be an irreducible supercuspidal representation of G' . Choose an admissible character θ such that π is a component of $\pi_\theta | G'$. Let E be the associated degree n extension of F . Define

$$G(\pi) = \{ g \in G \mid \pi^g \sim \pi \}.$$

Here, \sim denotes equivalence of representations. Set $\pi'_\theta = \pi_\theta | G'$. By [T]

$$\pi'_\theta = \bigoplus_{g \in G/G(\pi)} \pi^g.$$

In this section, we assume that the L-packet of π contains more than $|F^\times/N_{E/F}(E^\times)|$ representations. That is,

$$(4.1) \quad |G/G(\pi)| > |F^\times/N_{E/F}(E^\times)|.$$

This is equivalent to the representation $\bar{\pi}$ being reducible ([MS]). The purpose of this section is to prove that $\Theta_\pi \circ \text{exp}$ is not a multiple of the Fourier transform of a semisimple orbit on any neighbourhood of zero (Theorem 4.5). In order for (4.1) to hold, it is necessary that n and $q - 1$ have a nontrivial common divisor (see Remark 3.7).

Let $X \in \mathfrak{g}'$. We assume that the measures on the orbits in the stable orbit $\mathcal{O}_{st}(X)$ of X are normalized so that

$$\mu_{\mathcal{O}(X)}(f^g) = \mu_{\text{Ad } g^{-1} \cdot \mathcal{O}(X)}(f), \quad f \in C_c^\infty(\mathfrak{g}'), \quad g \in G.$$

Here $f^g(X) = f(\text{Ad } g^{-1}(X))$, $X \in \mathfrak{g}'$.

Lemma 4.2. $c_{\mathcal{O}}(\pi^g) = c_{\text{Ad } g^{-1} \cdot \mathcal{O}}(\pi)$, $\mathcal{O} \in (\mathcal{N})'$, $g \in G$.

Proof. The above compatibility conditions on the measures on \mathcal{O} and $\text{Ad } g \cdot \mathcal{O}$, $\mathcal{O} \in (\mathcal{N})'$, imply that

$$\widehat{\mu}_{\text{Ad } g \cdot \mathcal{O}}(X) = \widehat{\mu}_{\mathcal{O}}(\text{Ad } g^{-1}(X)), \quad X \in \mathfrak{g}'_{reg}.$$

The lemma follows from a comparison of the local character expansions of π and π^g and the linear independence of the functions $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N})'$, on neighbourhoods of zero intersected with \mathfrak{g}'_{reg} . \square

Given $\mathcal{O} \in (\mathcal{N})'$, let \mathcal{O}_{st} be the stable orbit containing \mathcal{O} . \mathcal{O}_{st} is an $\text{Ad } G$ -orbit in \mathfrak{g}' . Define a measure $\mu_{\mathcal{O}_{st}}$ on \mathcal{O}_{st} by:

$$\mu_{\mathcal{O}_{st}} = \sum_{\tilde{\mathcal{O}} \subset \mathcal{O}_{st}} \mu_{\tilde{\mathcal{O}}}.$$

Lemma 4.2 holds for any smooth admissible representation of G' of finite length. Therefore, since $(\pi'_\theta)^g \sim \pi'_\theta$ for all g in G , the coefficients $c_{\mathcal{O}}(\pi'_\theta)$ coincide for all orbits \mathcal{O} contained in a stable orbit \mathcal{O}_{st} . Let $c_{\mathcal{O}_{st}}(\pi'_\theta)$ denote their common value. Then

$$\theta_{\pi_\theta}(\exp X) = \sum_{\mathcal{O}_{st} \subset \mathcal{N}} c_{\mathcal{O}_{st}}(\pi'_\theta) \widehat{\mu}_{\mathcal{O}_{st}}(X),$$

for X in \mathfrak{g}'_{reg} sufficiently close to zero.

Lemma 4.3. *Choose $g \in G$ such that π is a component of $\overline{\pi}^g$. Let $\mathcal{O} \in (\mathcal{N})'$.*

(1) *If $\text{Ad } g \cdot \mathcal{O} = \mathcal{O}$ for all $g \in G_E$, then*

$$c_{\mathcal{O}}(\pi) = d(\pi) \Gamma_{\mathcal{O}}(\text{Ad } g(S_\theta)).$$

(2) *If $\text{Ad } g \cdot \mathcal{O} = \mathcal{O}$ for all $g \in G$, that is, $\mathcal{O} = \mathcal{O}_{st}$, then*

$$c_{\mathcal{O}}(\pi) = d(\pi) d(\pi'_\theta)^{-1} c_{\mathcal{O}_{st}}(\pi'_\theta).$$

Proof. (1) Since π'_θ decomposes with multiplicity one, $\overline{\pi}^g$ also decomposes with multiplicity one. Thus ([T])

$$\overline{\pi}^g = \bigoplus_{x \in G_E/G(\pi)} \pi^x.$$

Applying Corollary 3.5 and Lemma 4.2,

$$\begin{aligned} c_{\mathcal{O}}(\overline{\pi}^g) &= d(\overline{\pi}^g) \Gamma_{\mathcal{O}}(\text{Ad } g(S_\theta)) = \sum_{x \in G_E/G(\pi)} c_{\mathcal{O}}(\pi^x) \\ &= \sum_{x \in G_E/G(\pi)} c_{\text{Ad } x^{-1} \cdot \mathcal{O}}(\pi) = |G_E/G(\pi)| c_{\mathcal{O}}(\pi) \\ &= d(\pi)^{-1} d(\overline{\pi}^g) c_{\mathcal{O}}(\pi), \end{aligned}$$

to obtain (1).

(2) Assume $\mathcal{O} = \mathcal{O}_{st}$. By linear independence of the Fourier transforms of nilpotent orbits, and Lemma 4.2,

$$\begin{aligned} c_{\mathcal{O}_{st}}(\pi'_\theta) &= \sum_{g \in G/G(\pi)} c_{\mathcal{O}}(\pi^g) = \sum_{g \in G/G(\pi)} c_{\text{Ad } g^{-1} \cdot \mathcal{O}}(\pi) \\ &= |G/G(\pi)| c_{\mathcal{O}}(\pi) = d(\pi'_\theta) d(\pi)^{-1} c_{\mathcal{O}}(\pi). \end{aligned}$$

□

Remark. As (4.1) was not used in the proof, Lemma 4.3 holds for all irreducible supercuspidal representations of G' . In general there exist $\mathcal{O} \in (\mathcal{N})'$ which are stable under $\text{Ad } G_E$, but not under $\text{Ad } G$.

Let $(\mathcal{N}_{reg})'$ denote the set of regular (maximal dimension) nilpotent $\text{Ad } G'$ -orbits in \mathfrak{g}' . Define $w(\pi)$ to be the number of orbits \mathcal{O} in $(\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi)$ is nonzero.

Lemma 4.4. *The L -packet of π contains $w(\pi)^{-1}|F^\times/(F^\times)^n|$ representations.*

Proof. Up to a positive constant depending on the normalization of the measure on $\mathcal{O} \in (\mathcal{N}_{reg})'$, $c_{\mathcal{O}}(\pi)$ equals the multiplicity with which some Whittaker model occurs in π ([**Ro**]). As shown in Remark 2.9 of [**T**], for each $\mathcal{O} \in (\mathcal{N}_{reg})'$, there exists exactly one $g \in G/G(\pi)$ such that $c_{\mathcal{O}}(\pi^g) \neq 0$. The determinant map factors to an isomorphism between $G/F^\times G'$ and $F^\times/(F^\times)^n$ and $(\mathcal{N}_{reg})'$ is the disjoint union of the orbits $\text{Ad } g \cdot \mathcal{O}$, $g \in G/F^\times G'$ ([**Re**]). Thus

$$\sum_{g \in G/G(\pi)} w(\pi^g) = |F^\times/(F^\times)^n|.$$

By Lemma 4.2, $w(\pi^g) = w(\pi)$. Therefore

$$w(\pi)|G/G(\pi)| = |F^\times/(F^\times)^n|.$$

□

Theorem 4.5. *Assume that (4.1) holds. $d(\pi)^{-1}\Theta_\pi \circ \exp|V \cap \mathfrak{g}'_{reg}$ is not of the form $\widehat{\mu}_{\mathcal{O}(X)}|V \cap \mathfrak{g}'_{reg}$, for any $X \in \mathfrak{g}'_{reg}$ and open neighbourhood V of zero in \mathfrak{g}' .*

Proof. Suppose that $\Theta_\pi \circ \exp$ and $\lambda \widehat{\mu}_{\mathcal{O}(X)}$ coincide on $V \cap \mathfrak{g}'_{reg}$ for some constant λ and neighbourhood V , where $X \in \mathfrak{g}'_{reg}$. Then

$$c_{\mathcal{O}}(\pi) = \lambda \Gamma_{\mathcal{O}}(X), \quad \mathcal{O} \in (\mathcal{N})'.$$

(To see this, argue as in the proof of Corollary 3.5.) Since $c_{\{0\}}(\pi) = d(\pi)\Gamma_{\{0\}}(X) \neq 0$, ([**HC2**]), $\lambda = d(\pi)$. Also, X is elliptic, because $\Gamma_{\{0\}}$ vanishes off the regular elliptic set ([**HC2**]). Let L be the degree n extension of F such that L^\times is isomorphic to the stabilizer of X in G .

By Theorem 6.3(i) of [**Re**], if $\mathcal{O} \in (\mathcal{N}_{reg})'$ and $g \in G$,

$$\Gamma_{\text{Ad } g \cdot \mathcal{O}}(X) = \begin{cases} \Gamma_{\mathcal{O}}(X), & \text{if } \det g \in N_{L/F}(L^\times), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\det g \in N_{L/F}(L^\times)$. By Lemma 4.2,

$$c_{\text{Ad } g \cdot \mathcal{O}}(\pi) = d(\pi^g) \Gamma_{\text{Ad } g \cdot \mathcal{O}}(X) = d(\pi) \Gamma_{\mathcal{O}}(X) = c_{\mathcal{O}}(\pi), \quad \mathcal{O} \in (\mathcal{N}_{reg})'.$$

Since there exists an $\mathcal{O} \in (\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi) \neq 0$ ([**T**]), $w(\pi) = |N_{L/F}(L^\times)/(F^\times)^n|$. Thus, given the relation between $w(\pi)$, $|G/G(\pi)|$ and $|F^\times/(F^\times)^n|$ described in the proof of Lemma 4.4,

$$|G/G(\pi)| = |F^\times/N_{L/F}(L^\times)|.$$

For g such that $\det g \in N_{L/F}(L^\times)$, the relation

$$\begin{aligned} c_{\mathcal{O}}(\pi^g) &= c_{\text{Ad } g^{-1} \cdot \mathcal{O}}(\pi) = d(\pi^g) \Gamma_{\text{Ad } g^{-1} \cdot \mathcal{O}}(X) = d(\pi) \Gamma_{\mathcal{O}}(X) \\ &= c_{\mathcal{O}}(\pi), \quad \mathcal{O} \in (\mathcal{N}_{reg})', \end{aligned}$$

together with the fact that there is exactly one $g \in G/G(\pi)$ such that $c_{\mathcal{O}}(\pi^g)$ is nonzero ([**T**]), implies that $g \in G(\pi)$. We can now conclude that

$$G(\pi) = \{g \in G \mid \det g \in N_{L/F}(L^\times)\}.$$

Choose $\mathcal{O} \in (\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi) \neq 0$. Fix $x \in G$ such that π is a component of $\bar{\pi}^x$. Then

$$\bar{\pi}^x = \bigoplus_{g \in G_E/G(\pi)} \pi^g.$$

$$c_{\mathcal{O}}(\bar{\pi}^x) = \sum_{g \in G_E/G(\pi)} c_{\mathcal{O}}(\pi^g) = \sum_{g \in G_E/G(\pi)} d(\pi^g) \Gamma_{\text{Ad } g^{-1} \cdot \mathcal{O}}(X) = d(\pi) \Gamma_{\mathcal{O}}(X),$$

the final equality resulting from $\Gamma_{\text{Ad } g^{-1} \cdot \mathcal{O}}(X) = 0$ whenever $g \in G_E - G(\pi)$ (because $\det g \notin N_{L/F}(L^\times)$). By Corollary 3.5,

$$c_{\mathcal{O}}(\bar{\pi}^x) = d(\bar{\pi}^x) \Gamma_{\mathcal{O}}(\text{Ad } x(S_\theta)).$$

Since $d(\bar{\pi}^x) = |G_E/G(\pi)| d(\pi)$,

$$\Gamma_{\mathcal{O}}(X) = |N_{E/F}(E^\times)/N_{L/F}(L^\times)| \Gamma_{\mathcal{O}}(\text{Ad } x(S_\theta)).$$

Repka ([**Re**]) computed $\Gamma_{\mathcal{O}}$ on the regular set in G' . Lifting the Shalika germs from the group to the Lie algebra, and substituting the values of $\Gamma_{\mathcal{O}}(X)$ and $\Gamma_{\mathcal{O}}(\text{Ad } x(S_\theta))$, we obtain

$$\begin{aligned} (4.6) \quad &|N_{L/F}(O_L^\times)/(O_F^\times)^n| (q^{n/e_L} - 1) q^{n/(2e_L)} |\eta(X)|^{-1/2} \\ &= |N_{E/F}(E^\times)/N_{L/F}(L^\times)| |N_{E/F}(O_E^\times)/(O_F^\times)^n| \\ &\quad \cdot (q^{n/e_E} - 1) q^{n/(2e_E)} |\eta(\text{Ad } x(S_\theta))|^{-1/2}. \end{aligned}$$

Here $\eta : \mathfrak{g}_{reg} \rightarrow C$ is the discriminant function ([**HC2**]), and e_L and e_E are the ramification degrees of L and E over F , respectively.

$N_{L/F}(L^\times)$ is a subset of $N_{E/F}(E^\times)$ ($G(\pi) \subset G_E$). Since $N_{L/F}(L^\times)$ contains an element of valuation n/e_L and the valuation of any element of $N_{E/F}(E^\times)$ is a multiple of n/e_E , e_L is a divisor of e_E . As a consequence,

$$N_{E/F}(O_E^\times) = (O_F^\times)^{e_E} \subset (O_F^\times)^{e_L} = N_{L/F}(O_L^\times) \subset N_{E/F}(O_E^\times),$$

so $(O_F^\times)^{e_L} = (O_F^\times)^{e_E}$. Thus

$$|N_{E/F}(E^\times)/N_{L/F}(L^\times)| = e_E/e_L.$$

Therefore (4.6) becomes

$$e_L (q^{n/e_L} - 1) q^{n/(2e_L)} |\eta(X)|^{1/2} = e_E (q^{n/e_E} - 1) q^{n/(2e_E)} |\eta(\text{Ad } x(S_\theta))|^{1/2}.$$

$q^{n/(2e_L)} |\eta(X)|^{1/2}$ and $q^{n/(2e_E)} |\eta(\text{Ad } x(S_\theta))|^{1/2}$ are powers of q . Because q is a power of p , $p > n$, and e_L and e_E divide n , e_L and e_E are relatively prime to q . Therefore (4.6) implies

$$e_L (q^{n/e_L} - 1) = e_E (q^{n/e_E} - 1).$$

That is,

$$e_E/e_L = (q^{n/e_E} - 1)^{-1} (q^{n/e_L} - 1) = 1 + q^{n/e_E} + \dots + q^{n(e_E - e_L)/(e_E e_L)} > n,$$

which is impossible. \square

Remarks. Suppose $n = \ell$ is prime (not necessarily dividing $q - 1$). Let π be any irreducible supercuspidal representation of G' .

(1) If $\mathcal{O} \in (\mathcal{N}') - (\mathcal{N}_{reg})'$, then $\text{Ad } g \cdot \mathcal{O} = \mathcal{O}$ for every $g \in G$ ([Re]). Thus, by Lemma 4.3,

$$c_{\mathcal{O}}(\pi) = d(\pi) \Gamma_{\mathcal{O}}(S_\theta) = d(\pi) d(\pi'_\theta)^{-1} c_{\mathcal{O},s}(\pi'_\theta).$$

Lemma 4.3(2) was first observed by Assem([As]) in the case where π'_θ has ℓ irreducible components.

(2) The elements of an L-packet containing ℓ representations correspond to G/G_E (Corollary 3.6), and, if π belongs to the L-packet,

$$c_{\mathcal{O}}(\pi) = d(\pi) \Gamma_{\mathcal{O}}(\text{Ad } g(S_\theta)), \quad \mathcal{O} \in (\mathcal{N}_{reg})',$$

where g is a representative of the corresponding coset. If ℓ divides $q - 1$, there exist L-packets containing ℓ^2 representations ([MS]). As noted in ([As]), the elements of such an L-packet correspond to the orbits in $(\mathcal{N}_{reg})'$, each π being identified with the unique $\mathcal{O} \in (\mathcal{N}_{reg})'$ such that $c_{\mathcal{O}}(\pi)$ is nonzero (up to a

constant depending on normalization of $\mu_{\mathcal{O}}$ this nonzero coefficient equals one ([Ro]).

(3) Modulo determination of the values of Shalika germs, (1) and (2) combine to give the values of the coefficients $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in (\mathcal{N})'$, for supercuspidal representations of $SL_{\ell}(F)$.

(4) The functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N})'$, were computed by Assem ([As]). Thus, whenever the coefficients $c_{\mathcal{O}}(\pi)$ are known, substitution of Assem's formulas into the local character expansion of π yields a formula for the character Θ_{π} on a neighbourhood of the identity element.

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