TANGENTIAL DEFORMATIONS ON THE DUAL OF
NILPOTENT SPECIAL LIE ALGEBRAS

NABIHA BEN AMAR

TANGENTIAL DEFORMATIONS ON THE DUAL OF NILPOTENT SPECIAL LIE ALGEBRAS

N. Ben Amar

The relationship between $\star$ - products (formal deformations) and quantization deformations (non formal deformations) on dual of nilpotent Lie algebras are studied. An explicit, tangential quantization deformation is given on algebras of polynomial functions and C^∞, rapidly decreasing function on the dual of any nilpotent special Lie algebras.

Introduction.

$\star$ - products i.e. associative deformations of usual multiplication of functions have been introduced by F.Bayen, M.Flato, C.Fronsdal, A.Lichnerowicz and D.Sternheimer (see [6]) as a tool for the quantization of a symplectic (or Poisson [12]) manifold. This geometrical notion has been used to give an autonomous phase space formulation of quantum mechanics without operators. In the first definition of $\star$ - products, people consider formal deformations in which $f \star_\hbar g$ is not a function, but rather a formal power series in $\hbar$ with functions as coefficients. But recently M.A. Rieffel introduced the notion of (strict) deformation quantization, a framework in which the convergence question could be handled. He constructed a deformation quantization on the space $\mathcal{S}(\mathfrak{g}^*)$ of the Schwartz (C^∞, rapidly decreasing) functions on the dual of a nilpotent Lie algebra $\mathfrak{g}$ ([16]). This deformation is the convergent version of the vertical part of the $\star$ - product constructed for a general Lie group G on its cotangent bundle T^*G by S.Gutt ([9]) and coincides with the $\star$ - product given by V.Lugo for nilpotent Lie algebras ([13]). Unfortunately, this product is not tangential in general. This means we cannot restrict it to a coadjoint orbit even in general position and so it is useless for the construction of irreducible representations of the corresponding Lie group G. On the other hand, D.Arnal and J.C.Cortet constructed formal tangential $\star$ - products on a dense invariant subset of the dual $\mathfrak{g}^*$ of any nilpotent Lie algebra ([1]). But, in general, this $\star$ - product cannot be extended to the whole dual of $\mathfrak{g}$. Moreover, R.Howe, G.Ratcliff and N.Wildberger constructed symbolic calculus, through the Cayley transform, for OKP Cayley-stable (nilpotent) Lie groups ([10]) and for special nilpotent Lie algebras, D.Arnal, M.Cahen and S.Gutt defined a formal $\star$ - product on $\mathfrak{g}^*$. These two constructions are

while defined on the whole dual and tangential on a dense open subset of $\mathfrak{g}^*$ ([2]).

Our aim here is the construction of a tangential deformation quantization on $\mathfrak{g}^*$ defined on the whole algebras of polynomial and Schwartz functions. Such a construction is realized in the present article in the case of a special nilpotent Lie algebra, using a deformed version of the formulae of M.A.Rieffel. More precisely, for each value of $\hbar$, we consider a new diffeomorphism φ which is a product of exponentials, instead of the usual exponential mapping, to identify G with $\mathfrak{g}$.

The present paper is organized as follows. In the first section, we recall the definition of $\star$ - product, we show that Rieffel's deformation quantization can be viewed as a $\star$ - product, we extend it to the space $\mathcal{P}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$ and give an explicit expression of it. In the second section, we construct a convergent tangential and graded $\star$ - product on $\mathcal{S}(\mathfrak{g}^*) \oplus \mathcal{P}(\mathfrak{g}^*)$ in the case of a special nilpotent Lie algebra $\mathfrak{g}$. This product coincides on $\mathcal{P}(\mathfrak{g}^*)$ with the formal product of [2], but the expressions we give here are totally explicit and have integral form which is much more adapted for use. In the third section, we define the $\star$-exponential map and the adapted Fourier transform and we related it to unitary representations of the corresponding Lie group. Finally, we compare our construction to those of Arnal, Cahen, Gutt and of Howe, Ratcliff, Wildberger.

1. $\star$ - products and deformation quantization.

Definition 1.1. ([16],[17]). Let (W, Λ) be a differentiable Poisson manifold and A an associative and Lie (for Poisson bracket) subalgebra of $C^\infty(W)$. A strict deformation quantization of A in the direction of Λ is a family of associative products, involutions and C^*-norms on A for each $\hbar \in \mathbb{R}$ which are, for $\hbar = 0$, the original product, complex conjugation and supremum norm and such that:

(1)   For every f in A, the function $\hbar \mapsto \|f\|_\hbar$ is continuous.

(2)   For every f, g in A, $\|\frac{f \star_\hbar g - g \star_\hbar f}{i\hbar} - \{f, g\}\|_\hbar$ converges to 0 as $\hbar$ goes to 0.

Let $\mathfrak{g}$ be a real, nilpotent and finite dimensional Lie algebra, $\mathfrak{g}^*$ be its dual vector space and G the connected and simply connected Lie group of the Lie algebra $\mathfrak{g}$. We define the linear Poisson bracket of two C^∞ functions f, $,g$ on $\mathfrak{g}^*$ by:

$$\{f, g\}(\mu) = \langle [df(\mu), dg(\mu)], \mu \rangle,$$

where $df(\mu)$ (respectively $df(\mu)$) is the differential of f (respectively g) at $\mu \in \mathfrak{g}^*$ and $[\ ,\ ]$ denotes the Lie bracket in $\mathfrak{g}$. Let us first recall the example of strict deformation quantization given in this case by Rieffel ([16]): using

the exponential map exp, he identifies $\mathfrak{g}$ with G. Let the corresponding Lie group structure on $\mathfrak{g}$ denoted by $*$. For each $\hbar \in \mathbb{R}$, he defines a Lie group structure on $\mathfrak{g}$ by putting, for all X and Y in $\mathfrak{g}$,

$$X *_\hbar Y = \hbar^{-1}((\hbar X)_*(\hbar Y))$$

(for $\hbar = 0$, $X *_\hbar Y$ means $X + Y$). Let $G_\hbar$ be the Lie group G equipped with the product $*_\hbar$. If f and g are functions in $\mathcal{S}(\mathfrak{g})$, their convolution product for the $G_\hbar$ structure will be denoted by $f *_\hbar g$.

Proposition 1.2. [16]. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\{\ ,\ \}$ be the corresponding linear Poisson bracket on $\mathcal{S}(\mathfrak{g}^*)$. Finally let $\hat{\ }$ (resp. $\check{\ }$) the Fourier transform (resp. the inverse Fourier transform) between $\mathcal{S}(\mathfrak{g}^*)$ and $\mathcal{S}(\mathfrak{g})$. For any $\hbar \in \mathbb{R}$, let us define product, involution and norm on $\mathcal{S}(\mathfrak{g}^*)$ by:*
  (i) $f \star_\hbar g = (\hat{f} *_\hbar \hat{g})^\vee$ $(f, g \in \mathcal{S}(\mathfrak{g}\mathfrak{g}^*))$
  (ii) $f^*(X) = \overline{f}(X)$
  (iii) $\|f\|_\hbar$ *is the norm of f in the group C^*-algebra $C^*(G_\hbar)$.*
Then this structure provides a strict deformation quantization of $\mathcal{S}(\mathfrak{g}^*)$ in the direction of $-(2\pi)^{-1}\{\ ,\ \}$.

On the other hand, we have the notion of $\star$ - product:

Definition 1.3. ([6],[11],[12]). Let W be a differentiable Poisson manifold with a Poisson bracket $\{\ ,\ \}$ and E be the space of formal series in the parameter $\hbar$ with coefficients in $C^\infty(W)$. A $\star$ - product on $C^\infty(W)$ is defined by a bilinear map from $C^\infty(W) \times C^\infty(W)$ into E:

$$(u, v) \mapsto u \star_\hbar v = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} C^r(u, v) \in E$$

where:
  (i) C^r is a bidifferential operator on $C^\infty(W)$ of maximum order r $(r > 1)$ in each argument, null on the constants,
  (ii) $C^0(u, v) = u.v$, $C^1(u, v) = \frac{i}{2}\{u, v\}$,
  (iii) C^r is symmetric (resp. skew symmetric) in (u, v) if r is even (resp. odd),
  (iv) $\displaystyle\sum_{r+s=t} (r!s!)^{-1} C^r(C^s(u, v), w)$

$$= \sum_{r+s=t} (r!s!)^{-1} C^r(u, C^s(u, w))   (t = 1, 2, \dots).$$

Roughly speaking, we can say that $\star$ - product are related to deformations quantization by asymptotic developments. More precisely, in our case:

Proposition 1.4. *Let f,g be in $\mathcal{S}(\mathfrak{g}^*)$ such that $\hat{f}$ and $\hat{g}$ are compactly supported, then $f \star_\hbar g$ admits an expansion in the power series:*

$$f \star_\hbar g = \sum_{k\geq 0} \frac{\hbar^k}{k!} C^k(f,g)$$

in the variable $\hbar$, converging pointwise and in the space $\mathcal{S}'(\mathfrak{g}^*)$ of tempered distributions. This series is a $\star$ - product.Moreover the following properties hold:

1) $\displaystyle\int_{\mathfrak{g}^*} f \star_\hbar g(\mu)\, d\mu = \int_{\mathfrak{g}^*} f(\mu)g(\mu)\, d\mu.$

2) $\overline{g \star_\hbar f} = \overline{f} \star_\hbar \overline{g}.$

Proof. Let $\{E_j, \quad 1 \leq j \leq n\}$ be a basis for $\mathfrak{g}$ and $x_1,\ldots,x_n$ be the corresponding coordinates of X in $\mathfrak{g}$. Let $\mu_1,\ldots,\mu_n$ be the coordinates of μ in $\mathfrak{g}^*$ for the dual basis. One has:

$$f \star_\hbar g(\mu) = (\hat{f} \star_\hbar \hat{g})\check{\ }(\mu)$$
$$= \int_{\mathfrak{g}\times\mathfrak{g}} \hat{f}(X)\hat{g}(Y)e^{2i\pi\langle X *_\hbar Y,\mu\rangle}\, dX\, dY.$$

We set:

$$e^{\langle X *_\hbar Y - X - Y,\mu\rangle} = 1 + \sum_{k\geq 1} \sum_{\substack{1\leq|\alpha|\leq k \\ k+ 1-\alpha\leq|\beta| \leq k}} \hbar^k a_{\alpha\beta k}(\mu) X^\alpha Y^\beta,$$

where:

$$X^\alpha = (x_1)^{\alpha_1}\cdots(x_n)^{\alpha_n}, \quad Y^\beta = (y_1)^{\beta_1}\cdots(y_n)^{\beta_n},$$
$$\alpha = (\alpha_1,\ldots,\alpha_n) \in I\!N^n, \quad \beta = (\beta_1,\ldots,\beta_n) \in I\!N^n.$$

Since $\hat{f}$ and $\hat{g}$ are compactly supported, we consider only X and Y in a given compact set and the above series converges absolutely. Then the following series:

$$f \star_\hbar g(\mu) = fg(\mu) + \sum_{k\leq 1} \sum_{\substack{1\leq\alpha\leq k \\ k+ 1-\alpha\leq\beta \leq k}} \hbar^k \frac{a_{\alpha\beta k}(\mu)}{(2i\pi)^k} D^\alpha f(\mu) D^\beta f(\mu),$$

where D^α means $\dfrac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial\mu_1^{\alpha_1}\cdots\partial\mu_n^{\alpha_n}}$, converges in the space of tempered distributions.

One finds that:

$$C^0(f,g) = fg \quad \text{and}$$
$$C^k(f,g) = \sum_{\substack{1\leq\alpha\leq k \\ k+ 1-\alpha\leq\beta \leq k}} k!\frac{a_{\alpha\beta k}(\mu)}{(2i\pi)^k} D^\alpha f D^\beta f.$$

C^k is, for each positive k, a bidifferential operator on $\mathcal{S}(\mathfrak{g}^*)$ of maximal order k in each argument, vanishing on constants. A quick calculation shows that (see [16]):

$$\{f,g\}(\mu) = -4\pi^2 \int \hat{f}(X)\hat{g}(Y)e^{2i\pi\langle X+Y,\mu\rangle}\langle [X,Y],\mu\rangle \, dX \, dY.$$

Then (ii) of Definition 1.3 holds. Now let:

$$X *_\hbar Y - X - Y = \sum_{r=1}^{\infty} \hbar^r b_r(X,Y),$$

using the Campbell-Hausdorff formula ([8]), it is not difficult to show that b_r is symmetric (resp. skew symmetric) in (X,Y) if r is even (resp. odd). Then we check easily that the same is true for C^r.

Finally, we prove that the series:

$$\sum_{r \geq 0} \frac{\hbar^r}{r!} C^r(f *_\hbar g, u),$$

where $u \in \mathcal{S}(\mathfrak{g}^*)$ and $\hat{u}$ is compactly supported, converges uniformly on compact sets then (iv) is clear.

One has:

$$\int_{\mathfrak{g}^*} f *_\hbar g(\mu) \, d\mu = \int_{\mathfrak{g}^*} (\hat{f} *_\hbar \hat{g})^\vee(\mu) \, d\mu$$

$$= (\hat{f} *_\hbar \hat{g})(0) = \int_{\mathfrak{g}} \hat{f}(X)\hat{g}(-X) \, dX$$

$$= (\hat{f} *_0 \hat{g})(0) = \int_{\mathfrak{g}^*} (f \cdot g)(\mu) \, d\mu.$$

Property (1) is proved. For (2) let us compute $\overline{f} *_\hbar \overline{g}$:

$$\overline{f} *_\hbar \overline{g}(\mu) = \int_{\mathfrak{g}\times\mathfrak{g}} \widehat{\overline{f}}(X)\widehat{\overline{g}}(Y)e^{2i\pi\langle X *_\hbar Y,\mu\rangle} \, dX \, DY$$

$$= \int_{\mathfrak{g}\times\mathfrak{g}} \overline{\hat{f}}(X)\overline{\hat{g}}(Y)e^{2i\pi\langle -X *_\hbar -Y,\mu\rangle} \, dX \, DY$$

$$= \int_{\mathfrak{g}\times\mathfrak{g}} \overline{\hat{f}(X)\hat{g}(Y)e^{2i\pi\langle Y *_\hbar X,\mu\rangle}} \, dX \, DY$$

$$= \overline{g *_\hbar f(\mu)}.$$

$\square$

Let us remark that this $\star$ - product was defined by Lugo in [13]. Let f,g be functions in $\mathcal{S}(\mathfrak{g}^*) \times \mathcal{S}(\mathfrak{g}^*)$ (or in $\mathcal{P}(\mathfrak{g}^*) \times \mathcal{S}(\mathfrak{g}^*)$, or in $\mathcal{P}(\mathfrak{g}^*) \times \mathcal{P}(\mathfrak{g}^*)$),

where $\mathcal{P}(\mathfrak{g}^*)$ denotes the set of polynomial functions on $\mathfrak{g}^*$ and let Φ be a function in $\mathcal{S}(\mathfrak{g}^*)$. Then Lugo put:

$$\langle (g \star f)(\mu), \Phi(\mu) \rangle = \langle \hat{f}(Y), \langle \hat{g}(X), \hat{\Phi}(X * Y) \rangle \rangle.$$

Thus the Lugo product is:

$$g \star f = (\hat{g} \star_1 \hat{f})^{\check{}} = g \star_1 f.$$

This $\star$ - product on $\mathfrak{g}^*$ constitutes also the vertical part of a $\star$ - product constructed, for a general Lie group G, on its cotangent bundle $T^*(G)$ in [9]. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let σ be the linear bijection defined by complete symmetrisation:

$$\sigma : \mathcal{P}(\mathfrak{g}^*) \longrightarrow \mathcal{U}(\mathfrak{g}),$$

$$\sigma(X_{i_1} \cdots X_{i_p}) = \frac{1}{p!} \sum_{s \in \mathfrak{S}_p} X_{i_{s(1)}} \circ \cdots \circ X_{i_{s(p)}},$$

where X_{i_k} belongs to $\mathfrak{g}, \mathfrak{S}_p$ is the permutation group of p elements and $\cdot$ denotes the product in $\mathcal{U}(\mathfrak{g})$. If $\mathcal{P}^l$ is the space of homogeneous polynomials of degree l,

$$\mathcal{U}(\mathfrak{g}) = \bigoplus_{l=0}^{\infty} \sigma(\mathcal{P}^l).$$

If U belongs to $\mathcal{U}(\mathfrak{g})$, we denote by u_l its component in $\sigma(\mathcal{P}^l)$. Finally, for P in $\mathcal{P}^p$ and Q in $\mathcal{P}^q$, we put:

$$P \star'_\hbar Q = \sum_{r=0}^{\infty} (2\hbar)^r \sigma^{-1}[(\sigma(P) \circ \sigma(Q))]_{p+q-r}.$$

S.Gutt proved in [9] that this formula can be extended in a $\star$ - product on $T \star G$ and this product is totally determined by the expression of $X \star'_\hbar P$ where X belongs to $\mathfrak{g}$. Thus, we first compute $E_i \star_\hbar f$ for the Lugo-Rieffel $\star$ - product:

Theorem 1.5. *Let* E_i, $(1 \leq i \leq n)$ *be a basis of* $\mathfrak{g}$ *and* f *in* $\mathcal{S}(\mathfrak{g}^*)$ *or in* $\mathcal{P}(\mathfrak{g}^*)$, *one has:*

$$E_i \star_\hbar f = \sum_{k \geq 0} (-1)^k \frac{B_k}{k!} \frac{\hbar^k}{(2i\pi)^k}$$

$$\cdot E_{m_k} C_{ij_1}^{m_1} C_{m_1 j_2}^{m_2} \cdots C_{m_{k-1} j_k}^{m_k} \partial^k_{j_1 \cdots j_k} f,$$

where B_k is the kth Bernoulli number ([14]) and C_{ij}^k are the structure constants of $\mathfrak{g}$. This expression coincides with the Gutt product $E_i \star'_\hbar P$, thus the $\star$ - product of [9] coincides with the Lugo-Rieffel $\star$ - product.

Proof. Let f be in $\mathcal{S}(\mathfrak{g}^*)$, such that $\hat{f}$ is compactly supported, one has:

$$\langle E_i \star_\hbar f, \Phi \rangle = \langle \hat{f}(Y) \langle \hat{E}_i(x), \hat{\Phi}(X \star_\hbar Y) \rangle \rangle$$

$$= \langle \hat{f}(Y), \frac{1}{2i\pi} \frac{\partial}{\partial x_i} \hat{\Phi}(X \star_\hbar Y)|_{X=0} \rangle.$$

But, by direct computation,

$$\frac{\partial}{\partial x_i}(X \star_\hbar Y)|_{X=0} = \sum_{k\geq 0} \frac{B_k}{k!} \hbar^k (ad\, Y)^k E_i.$$

Therefore:

$$\langle E_i \star_\hbar f, \Phi \rangle = \frac{1}{2i\pi} \left\langle \hat{f}(Y), \hat{\Phi}'(Y) \sum_{k\geq 0} \frac{B_k}{k!} \hbar^k (ad\, Y)^k E_i \right\rangle.$$

Thus:

$$E_i \star_\hbar f = -\frac{1}{2i\pi} \left(\hat{f}'(\cdot) \sum_{k\geq 0} \frac{B_k}{k!} \hbar^k (ad\, \cdot)^k E_i \right)^{\check{}}$$

$$= \sum_{k\geq 0} (-1)^k \frac{B_k}{k!} \frac{\hbar^k}{(2i\pi)^k} E_{m_k} C_{ij_1}^{m_1} C_{m_1 j_2}^{m_2} \cdots C_{m_{k-1} j_k}^{m_k} \partial_{j_1 \cdots j_k}^k f.$$

Now the product $\star_\hbar$ is determined by these expression, since:

$$X_{i_1} \cdots X_{i_k} \star_\hbar f = \frac{1}{k} \sum_{l=1}^{k} (X_{i_l} \star_\hbar X_{i_1} \cdots \check{X}_{i_l} \cdots X_{i_k}) \star_\hbar f$$

$$= \frac{1}{k} \sum_{l=1}^{k} X_{i_l} \star_\hbar [(X_{i_1} \cdots \check{X}_{i_l} \cdots X_{i_k}) \star_\hbar f],$$

where $\check{}$ denotes omission. Thus, $C^r(X_{i_1} \cdots X_{i_k}, f)$ is determined, by induction on k:

$$C^r(X_{i_1} \cdots X_{i_k}, f)$$

$$= \frac{1}{k} \sum_{l=1}^{k} \sum_{s=0}^{r} C^{r-s}(X_{i_l}, C^s(X_{i_1} \cdots \check{X}_{i_l} \cdots X_{i_k}, f)).$$

Finally, as the cochains C^r are bidifferential operators they are totally determined by their values on polynomial functions. $\square$

2. Tangential $\star$ - product on the dual of a special Lie algebra.

Definition 2.1. ([**12**], [**2**]). Let Ω be a G-invariant open set in $\mathfrak{g}^*$ and A be an associative subalgebra of $C^\infty(\Omega)$ stable by the Poisson bracket. A deformation of A is called tangential if for every orbit O of G, contained in Ω and for all pairs of functions u, v in A, such that $u_{|O} = 0, (u \star f)_{|O} = 0$ vanishes for all f in A.

Definition 2.2. ([**5**]). A deformation on $\mathcal{P}(\mathfrak{g}^*)$ or $\mathcal{S}(\mathfrak{g}^*)$ is called covariant, if for all X and Y in $\mathfrak{g}$,

$$\frac{2i\pi}{\hbar}(X \star_\hbar Y - Y \star_\hbar X) = \{X, Y\}.$$

A deformation on $\mathcal{P}(\mathfrak{g}^*)$ is called graded if, for each p, q and r, $\quad C_r(\mathcal{P}^p, \mathcal{P}^q)$ is in $\mathcal{P}^{p+q-r}$.

Let us remark that Lugo-Gutt-Rieffel $\star$ - product is covariant, graded but is not tangential in general. Of course, to build up irreducible representations of the corresponding Lie group G, using $\star$ - product on $\mathfrak{g}^*$, we need restrictions of our $\star$ - product to coadjoint orbits. Thus, we cannot use the Lugo-Gutt-Rieffel $\star$ - product in this manner. In this section, we define a convergent and tangential $\star$ - product for special nilpotent Lie algebras by deformation of the Rieffel formula using, instead of the usual exponential map, a diffeomorphism φ between $\mathfrak{g}$ and G. But let us first recall the definition of a special Lie algebra.

Definition 2.3. (see [**7**] and [**2**]). A nilpotent Lie algebra $\mathfrak{g}$ is called special if it contains an abelian ideal m whose codimension is half of the maximal dimensionality for coadjoint orbits of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a n-dimensional special Lie algebra and

$$0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

be an increasing sequence of ideals $\mathfrak{g}_i$ of $\mathfrak{g}$ with dimension i and such that $\mathfrak{g}_{n-k} = \mathfrak{m}$. The dimension of generic coadjoint orbits is thus $2k$. Let G^i be the corresponding analytic subgroups of G. Let us choose a basis $\{E_i, 1 \leq i \leq n\}$ of $\mathfrak{g}$ such that $E_i \in \mathfrak{g}_i \backslash \mathfrak{g}_{i-1}$. For all i, we consider the diffeomorphism φ_i from $\mathfrak{g}_i$ onto G^i defined by:

$$\varphi_i(Y) = \exp\left(\frac{\hbar}{2}y_i E_i\right) * \exp\left(\frac{\hbar}{2}y_{i-1}E_{i-1}\right) \cdots$$

$$* \exp\left(\frac{\hbar}{2}y_1 E_1\right) * \exp\left(\frac{\hbar}{2}y_1 E_1\right) \cdots * \exp\left(\frac{\hbar}{2}y_i E_i\right),$$

if Y is $\sum y_j E_j$.

For each non zero value of $\hbar$, φ_i is a diffeomorphism, since the product in G^i is a polynomial function, with the form:

$$X \star Y = \sum_{j=1}^{i}(x_j + y_j + P_j(x_{j+1}, \dots, x_i; y_{j+1}, \dots, y_i))E_j,$$

where P_j is a polynomial function in the variables $x_{j+1}, \dots, x_i, y_{j+1}, \dots, y_i$. Thus φ_i is a polynomial and $(\varphi_i)^{-1}$ can be determined by induction on i and is polynomial too. In the following, we will denote by $G^i_\hbar$ the algebra $\mathfrak{g}_i$ equipped with the product $*^i_\hbar$:

$$X *^i_\hbar Y = \varphi_i^{-1}(\varphi_i(X) * \varphi_i(Y))$$

(if $\hbar = 0$, $*^i_\hbar$ is just the usual sum of vectors).

Theorem 2.4. *Let f and g be in $A = \mathcal{S}(\mathfrak{g}^*_i) \oplus \mathcal{P}(\mathfrak{g}^*_i)$ and μ be in $\mathfrak{g}^*_i$. The formula:*

$$f \star^i_\hbar g(\mu) = \int_{\mathfrak{g}_i \times \mathfrak{g}_i} \hat{f}(X)\hat{g}(Y)$$
$$\cdot e^{2i\pi\langle\varphi_i^{-1}(\varphi_i(X)*\varphi_i(Y)),\mu\rangle} \, dX \, dY$$

defines a product $\star^i_\hbar$ on A. This product is a $\star$ - product which is covariant, graded and tangential on a Zariski invariant open subset Ω of $\mathfrak{g}^*$.

Moreover, on $\mathcal{S}(\mathfrak{g}^*_i)$, we define, with this product, an involution and a norm by:

(i) $f^*(X) = \overline{f}(X)$

(ii) $\|f\|_\hbar$ *is the norm of $\hat{f}$ in the group C^*-algebra $C^*(G^i_\hbar)$.*
Then this structure provides a strict deformation quantization of $\mathcal{S}(\mathfrak{g}^*_i)$ in the direction of $-(2\pi)^{-1}\{\ ,\ \}_i$, where $\{\ ,\ \}_i$ is the linear Poisson bracket on $\mathcal{S}(\mathfrak{g}^*_i)$.

Proof. The above formula defines clearly an associative product since:

$$f \star^i_\hbar g = (\hat{f} \star^i_\hbar \hat{g})^{\check{}}$$

where $*^i_\hbar$ denotes the convolution of functions on $G^i_\hbar$.

A similar proof as in the Proposition 1.4 shows that this formula defines a $\star$ - product. To prove the first part of the theorem, it is enough to show that this $\star$ - product coincides on $\mathcal{P}(\mathfrak{g}^*_i)$ with the covariant and generically tangential $\star$ - product defined by induction in [2].

Let us recall briefly here this algebraic construction. We fix the basis $\{E_j\}$ of $\mathfrak{g}_i$, a function on $\mathfrak{g}^*_i$ is a function of variables $\mu_1, \dots, \mu_i$, ∂_j is the jth

partial derivative in these variables. We define $f \star^j g$ as fg if $1 \leq j \leq n - k$, we prove by induction there exists an unique derivation D_j of the algebra $(\mathcal{P}(\mathfrak{g}_{j-1}^*), \star^{j-1})$ such that:

$$D_j = \sum_{k=0}^{\infty} \hbar^{2k} D_{2k}^{(j)}, \quad D_0^{(j)} = \{E_j, \cdot\}, \quad D_{2k}^{(j)} \text{ is a differential operator on } \mathfrak{g}_{j-1}^*,$$

$D_{2k}^{(j)}(\mathcal{P}^p) \subset \mathcal{P}^{p-2k}$ $(0$ if $p - 2k < 0)$, finally, we define:

$$f \star^j g = \sum_{l=0}^{\infty} \left(\frac{\hbar}{4i\pi}\right)^l \sum_{m=0}^{l} \frac{(-1)^m}{m!(l-m)!} (D_j)^m \partial_j^{l-m} f \star^{j-1} (D_j)^{l-m} \partial_j^m g$$

if $j > n - k$.

Let us first prove two lemmas.

Lemma 2.5. *For X in $\mathfrak{g}_{i-1}$ and μ in $\mathfrak{g}_{i-1}^*$, we put:*

$$X^t = \varphi_{i-1}^{-1}(e^{ad\, tE_i} \varphi_{i-1}(X))$$

and for f in $\mathcal{P}(\mathfrak{g}_{i-1}^*)$:

$$f^t(\mu) = \int_{\mathfrak{g}_{i-1}} \hat{f}(X) e^{2i\pi\langle X^t, \mu\rangle} \, dX.$$

Then:

$$D_i f = \frac{d}{dt} f^t|_{t=0} \quad \forall f \in \mathcal{P}(\mathfrak{g}_{i-1}^*).$$

Proof. We show directly that $X \mapsto X^t$ are group and algebra automorphisms:

$$X^t \star_{\hbar}^{i-1} Y^t = (X \star_{\hbar}^{i-1} Y)^t \qquad \text{and}$$

$$f^t \star_{\hbar}^{i-1} g^t = (f \star_{\hbar}^{i-1} g)^t.$$

Then $f \mapsto \frac{d}{dt} f^t(\mu)|_{t=0}$ is a derivation of $(\mathcal{P}(\mathfrak{g}_{i-1}^*), \star_{\hbar}^{i-1})$ and:

$$\frac{d}{dt} f^t(\mu)|_{t=0} = \sum_{s \geq 0} \hbar^s D^s f,$$

D^s is a differential operator on $\mathfrak{g}_{i-1}^*$ with polynomial coefficients.

Now D^{2s+1} vanishes for all s. Indeed, we first show that for each X in $\mathfrak{g}_i$, if x_j is its component on E_j,

$$\varphi_i(X) = \hbar X + \sum_{i_1 \geq 1} \alpha_{i_1, j_1, k'}^m \hbar^{2k'+1} x_{m_1} \cdots x_{m_{2k'}} x_{i_1} E_{j_1}$$

with $m_1 \geq m_2 \geq \ldots \geq m_{2k'} > n - k$, $\quad j_1 < \inf\{i_1 - 2k' + 1, m_{2k'}\}$($m$ is $m_1, \ldots, m_{2k'}$) and $\alpha_{i_1, j_1, k'}^m$ is a real number. For Y, Z in $\mathfrak{g}_{i-1}$, we put:

$$H(Y, Z) = \log((\exp \hbar Y) * (\exp \hbar Z))$$
$$= \sum_{m \geq 1} \hbar^m H_m(Y, Z)$$

by using the Campbell-Hausdorff formula, H_m is a polynomial function of degree m. Now:

$$((\exp \hbar Y) * (\exp \hbar Z))^{-1} = (\exp -\hbar Z) * (\exp -\hbar Y)$$

or

$$H_m(Y, Z) = -H_m(-Z, -Y) = (-1)^{m+1} H_m(Z, Y).$$

For $i = 0$, $\varphi_i(X)$ is $\hbar X$ and by induction, for $i > 0$, if $\vec{X}$ is $X - x_i E_i$,

$$\varphi_i(X) = \exp\left(\frac{\hbar}{2} x_i E_i\right) * \exp\left(\frac{\varphi_{i-1}\left(\vec{X}\right)}{2}\right)$$

$$* \exp\left(\frac{\varphi_{i-1}\left(\vec{X}\right)}{2}\right) * \exp\left(\frac{\hbar}{2} x_i E_i\right).$$

Then by induction on i, $\exp\left(\frac{\hbar}{2} x_i E_i\right) * \exp\left(\frac{\varphi_{i-1}\left(\vec{X}\right)}{2}\right)$ can be written as

$\exp\left(\sum_{m \geq 1} \hbar^m K_m(X)\right)$ where K_m is a polynomial function with degree m

and

$$\varphi_i(X) = \exp\left(\sum_{m \geq 1} \hbar^m K_m(X)\right)$$

$$* \exp\left(\sum_{m \geq 1} (-1)^{m+1} \hbar^m K_m(X)\right)$$

has the announced form. Now, $\varphi_{i-1}(\cdot)$ is an odd polynomial function in $\hbar$, $(d\varphi_{i-1}(\cdot))^{-1}$ too.

On the other hand, if f is in $\mathcal{P}(\mathfrak{g}_{i-1}^*)$ one has:

$$\frac{d}{dt} f^t(\mu) \mid_{t=0} = 2i\pi \int_{\mathfrak{g}_{i-1}} \hat{f}(X)$$

$$\cdot \langle d\varphi_{i-1}^{-1}(\varphi_{i-1}(X)) \quad ad\, E_i(\varphi_{i-1}(X)), \mu \rangle e^{2i\pi \langle X, \mu \rangle} \, dX$$

and $D^{2s+1}f$ vanishes for all s.

More precisely,

$$D^0 f(\mu) = 2i\pi \int_{\mathfrak{g}_{i-1}} \hat{f}(X) \sum C_{ij}^q x_j \mu_q e^{2i\pi\langle X,\mu\rangle} \, dX$$

(the coefficients C_{ij}^q are the structure constants of $\mathfrak{g}$) and

$$D^{2l'} f = \int_{\mathfrak{g}_{i-1}} \hat{f}(X) \sum_{l=1}^{i-1} \left[\sum_{q=1}^{l-1} \left(\sum_{k'+k''+1=l'} \right. \right.$$

$$\left. a_{ql}^{k''} C_{ij_1}^q \alpha_{i_1,j_1,k'}^m x_{m_1} \cdots x_{m_{2k'}} x_{i_1} + a_{ql}^{l'-1} \sum C_{ij}^q x_j \right)$$

$$+ \sum C_{ij_1}^l \alpha_{i_1,j_1,l'}^m x_{m_1} \cdots x_{m_{2l'}} x_{i_1} \Bigg] \mu_l e^{2i\pi\langle X,\mu\rangle} \, dX$$

where $a_{ql}^{k''} = \beta_{ql}^j x_{j_1} \cdots x_{j_{2k''+2}}$, $j = (j_1, \ldots, j_{2k''+2})$ and β_{ql}^j is a real number.

Then $D^0 = \{E_i, \cdot\}$ and $D^{2l'}$ sends $\mathcal{P}^p$ on $\mathcal{P}^{p-2l'}$, vanishes if $p < 2l'$. Thus:

$$D_i f = \frac{d}{dt} f^t|_{t=0}, \quad \forall f \in \mathcal{P}(\mathfrak{g}_{i-1}^*).$$

$\square$

Lemma 2.6. *Let f be in $\mathcal{P}(\mathfrak{g}_{i-1}^*) \oplus \mathcal{S}(\mathfrak{g}_{i-1}^*)$, then:*

$$E_i \star_\hbar^i f = E_i f + \frac{\hbar}{4i\pi} \frac{d}{dt} f^t|_{t=0}$$

and for all $j < i$

$$E_j \star_\hbar^i f = E_j \star_\hbar^{i-1} f.$$

That means the $\star$ - product $\star_\hbar^i$ coincides with the $\star$ - product defined in [2].

Proof. We keep the notations of the preceeding proof. For X, Y in $\mathfrak{g}_i$,

$$\varphi_i(X) * \varphi_i(Y) = \exp\left(\frac{\hbar}{2}(x_i + y_i)E_i\right) * \exp\left(-\frac{\hbar}{2}y_i E_i\right)$$

$$* \exp(\varphi_{i-1}(\vec{X})) * \exp\left(\frac{\hbar}{2}y_i E_i\right)$$

$$* \exp\left(\frac{\hbar}{2}x_i E_i\right) * \exp(\varphi_{i-1}(\vec{Y}))$$

$$* \exp\left(-\frac{\hbar}{2}x_i E_i\right) * \exp\left(\frac{\hbar}{2}(x_i + y_i)E_i\right).$$

But

$$\exp\left(-\frac{\hbar}{2}y_i E_i\right) * \exp(\varphi_{i-1}((\vec{X}))) * \exp\left(\frac{\hbar}{2}y_i E_i\right)$$

$$= \exp(e^{-ad\,\frac{\hbar}{2}y_i E_i}(\varphi_{i-1}(\vec{X}))),$$

thus:

$$X \star_\hbar^i Y = \varphi_{i-1}^{-1}[(\exp(e^{-ad\,\frac{\hbar}{2}y_i E_i}(\varphi_{i-1}(\vec{X}))))$$

$$*(\exp(e^{ad\,\frac{\hbar}{2}x_i E_i}(\varphi_{i-1}(\vec{Y}))))] + (x_i + y_i)E_i,$$

$$\frac{\partial}{\partial x_i}(X \star_\hbar^i Y)|_{X=0} = E_i + (d\varphi_{i-1}^{-1})(\varphi_{i-1}(\vec{Y}))ad\,\frac{\hbar}{2}E_i(\varphi_{i-1}(\vec{Y}))$$

and for all $j < i$:

$$\frac{\partial}{\partial x_i}(X \star_\hbar^i Y)|_{X=0}$$

$$= d\varphi_{i-1}^{-1}(\varphi_{i-1}(\vec{Y}))(\exp(e^{-ad\,\frac{\hbar}{2}y_i E_i}(\hbar E_j) * \varphi_{i-1}(\vec{Y}))).$$

Then as in Theorem 1.5, we prove that if f is in $\mathcal{S}(\mathfrak{g}_i^*) \oplus \mathcal{P}(\mathfrak{g}_i^*)$:

$$E_i \star_\hbar^i f(\mu) = E_i f(\mu) + \frac{\hbar}{2}\int_{\mathfrak{g}_i} \hat{f}(Y)\langle d\varphi_{i-1}^{-1}(\varphi_{i-1}(\vec{Y}))$$

$$ad\,E_i(\varphi_{i-1}(\vec{Y})),\mu\rangle e^{2i\pi\langle Y,\mu\rangle}\,dY$$

and for all $j < i$,

$$E_j \star_\hbar^i f(\mu) = \int_{\mathfrak{g}_i} \hat{f}(Y)\langle d\varphi_{i-1}^{-1}(\varphi_{i-1}(\vec{Y}))$$

$$((e^{-ad\,\frac{\hbar}{2}x_i E_i}(\hbar E_j) * \varphi_{i-1}(\vec{Y}))),\mu\rangle$$

$$e^{2i\pi\langle Y,\mu\rangle}\,dY.$$

Especially, if f belongs to $\mathcal{P}(\mathfrak{g}_{i-1}^*) \oplus \mathcal{S}(\mathfrak{g}_{i-1}^*)$,

$$E_i \star_\hbar^i f = E_i f + \frac{\hbar}{4i\pi}\frac{d}{dt}f^t|_{t=0}$$

and for all $j < i$,

$$E_j \star_\hbar^i f(\mu) = \int_{\mathfrak{g}_i} \hat{f}(Y)\langle d\varphi_{i-1}^{-1}(\varphi_{i-1}(\vec{Y}))$$

$$((\hbar E_j) * \varphi_{i-1}(\vec{Y})),\mu\rangle e^{2i\pi\langle Y,\mu\rangle}\,dY.$$

But

$$\frac{\partial}{\partial x_j}(X \star_\hbar^{i-1} Y)|_{X=0} = d\varphi_{i-1}^{-1}(\varphi_{i-1}(\vec{Y}))((\hbar E_j) * \varphi_{i-1}(\vec{Y})).$$

Thus

$$E_j \star_\hbar^i f = E_j \star_\hbar^{i-1} f.$$

Then our $\star$ - product $\star_\hbar^i$ coincides with the $\star$ - product of [2] on polynomial functions and since the coefficient of $\hbar^r$ of this product is a bidifferential operator, it is totally determined by its value on polynomial functions. The first part of the Theorem 2.4 is then proved.

Now as φ_n and φ_n^{-1} are polynomial, we can write:

$$X \star_\hbar^n Y = X + Y + \hbar M(\hbar, X, Y)$$

where M is a $\mathfrak{g}$-valued polynomial function and, if we denote the coordinate of $M(\hbar, X, Y)$ on E_l by $m_l(\hbar, X, Y)$, then m_l is a polynomial function only of the variables $x_{l+1}, \ldots, x_n, y_{l+1}, \ldots, y_n$ and $\hbar$. We prove the second part of our theorem by repeating exactly the argument of [16].   □

Definition 2.7. ([6], [12]). Two $\star$ - products (respectively tangential $\star$ - products) $\star$ and $\star'$ on a Poisson manifold W are equivalent (respectively tangentially equivalent) if there exists a formal series:

$$T = Id + \sum_{r=0}^{\infty} \hbar^r T^r$$

where Id denotes the identity and the T^r's are differential (respectively differential and tangential) operators vanishing on constants such that for each u and v in $C^\infty(W)$,

$$T(u \star v) = T(u) \star' T(v).$$

Proposition 2.8. *Let $\mathfrak{g}$ be a special nilpotent Lie algebra and $\{\mathfrak{g}_i\}$ and $\{\mathfrak{g}_i'\}$ two increasing sequence of ideals of $\mathfrak{g}$ such that the dimension of $\mathfrak{g}_i$ and $\mathfrak{g}_i'$ is i and:*

$$\mathfrak{g}_{n-k} = \mathfrak{g}_{n-k}' = \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] = 0.$$

We define on $\mathfrak{g}^*$, as in the preceeding theorem, two $\star$ - product: $\star$ by using $\{\mathfrak{g}_i\}$ and $\star'$ by using $\{\mathfrak{g}_i'\}$.

Then $\star$ and $\star'$ are tangentially equivalent on $\Omega \cap \Omega'$, subset of the union of orbits of maximal dimension and they are equivalent to the Lugo-Gutt-Rieffel $\star$ - product.

Proof. Let φ (resp. φ') be the map corresponding to the $\mathfrak{g}_i$ (resp. $\mathfrak{g}_i'$). For f in $\mathcal{P}(\mathfrak{g}^*)$ or in $\mathcal{S}(\mathfrak{g}^*)$, we put:

$$T(f) = (\hat{f} \circ \varphi^{-1} \circ \varphi')^\vee.$$

Then, for all f and g in $\mathcal{S}(\mathfrak{g}^*) \oplus \mathcal{P}(\mathfrak{g}^*)$,

$$Tf \star' Tg = T(f \star g)$$

and $T - Id$ is expandable in a series of tangential operators. Indeed if f in $\mathcal{S}(\mathfrak{g}^*) \oplus \mathcal{P}(\mathfrak{g}^*)$ is such that $\hat{f}$ is compactly supported, we can write:

$$Tf(\mu) = \int_{\mathfrak{g}} \hat{f}(X) e^{2i\pi \langle \varphi'^{-1} \circ \varphi(X), \mu \rangle} \, dX$$

$$= f + \sum \hbar^{2k'} \gamma_{k',i}^{j,l} \frac{\partial^{2k'+l} f}{\partial x_{j_1} \cdots \partial x_{j_{2k'+l}}} \mu_{i_1} \cdots \mu_{i_l}$$

$$= f + \sum_{r=1}^{\infty} \hbar^{2r} T^r f,$$

here $\gamma_{k',i}^{j,l}$ are real numbers.

On the other hand, if P is an invariant polynomial function, P is in $\mathcal{P}(\mathfrak{m}^*)$ (see [19] for instance), by definition, for such a P, the P^t defined in Lemma 2.5 for φ or φ' are just P, thus, for each f:

$$P \star f = P \star' f = Pf.$$

Moreover we can choose a local system of coordinates in $\Omega \cap \Omega'$ of the form (λ_l, p_j, q_j) such that the $\lambda_j(\mu)$ are polynomial invariant functions (transverse to the orbits) and $p_j(\mu), q_j(\mu)$ are rational and define a canonical chart for the orbit through μ. Such a chart is described in [1] or [19]. Now since our T^r contains at least a derivative in x_{j_1} with j_1 larger than $n-k$, they vanish on invariant P, thus on λ_l. If we exprim them in the chart (λ, p, q), they cannot have any derivative in the variables λ, since:

$$T(\lambda f) = T(\lambda \star f) = (T\lambda) \star' (Tf) = \lambda \star' (Tf) = \lambda(Tf).$$

But this means T is tangential.

Finally, we repeat the preceeding argument with the transform:

$$T(f) = (\hat{f} \circ \varphi^{-1} \circ \exp)\check{}$$

to find equivalence between our $\star$ - product and the Lugo-Gutt-Rieffel product.   □

3. Adapted Fourier transform.

Ley us define now the adapted Fourier transform. We follow the method of [4] (or [3]), defining first the $\star$ exponential mapping. Our $\star$ - product being covariant, the map:

$$\rho : X \mapsto \frac{2i\pi}{\hbar} X_\star$$

is a representation of $\mathfrak{g}$ on the space $S(\mathfrak{g}^*)$. By definition, the $\star$ exponential mapping is the representation for G whose differential is ρ.

Definition 3.1. Let u be in $S(\mathfrak{g}^*)$ and X in $\mathfrak{g}$. We define $E(\exp X) \star u$ as $U(1)$ if $U(t)$ is the solution of:

$$\frac{d}{dt}U(t) = (2i\pi X \star U)(t) \quad \text{with} \quad U(0) = u.$$

If f is a function of $S(G)$ (G is identified with $\mathfrak{g}$ by exp or φ), we put:

$$\langle \mathcal{E}(f), u \rangle = \int_G \int_{\mathfrak{g}^*} f(x)(E(x) \star u)(\mu) \, dx \, d\mu.$$

Thanks to an argument of [4], this formula defines a tempered distribution $\mathcal{E}(f)$ on $S(\mathfrak{g}^*)$. We shall write it as:

$$\mathcal{E}(f)(\mu) = \int_G f(x)E(x)(\mu) \, dx.$$

Proposition 3.2. *Let f be in $S(G)$, then:*

$$\mathcal{E}(f)(\mu) = \int_\mathfrak{g} (f \circ \varphi)(X)e^{2i\pi\langle X,\mu \rangle} \, dX$$

where φ is the map φ_n defined in the second section.

Proof. Let us put:

$$\mathcal{E}'(f)(\mu) = \int_\mathfrak{g} (f \circ \varphi)(X)e^{2i\pi\langle X,\mu \rangle} \, dX.$$

A simple computation shows that $\mathcal{E}'(S(G))$ is $S(\mathfrak{g}^*)$ and:

$$\mathcal{E}'(f *_\hbar g) = \mathcal{E}'(f) \star_\hbar \mathcal{E}'(g)$$

where $*_\hbar$ denotes the usual convolution of functions on $S(G)$. Now $\mathcal{E}'$ can be extended in an unitary transformation from $L^2(G)$ onto $L^2(\mathfrak{g}^*)$ and if L_x is the left regular representation of G on $L^2(G)$, for f in $S(G)$,

$$2i\pi X \star_\hbar \mathcal{E}'(f) = \mathcal{E}'(dL_X f).$$

Thus

$$\int_{\mathfrak{g}^*} (E(x) \star_\hbar \mathcal{E}'(f))(\mu) \, d\mu = (L_x f)(0) = f(x^{-1})$$

or for g in $\mathcal{S}(G)$,

$$\langle \mathcal{E}(g),' (f) \rangle = \int_G g(x) f(x^{-1}) \, dx$$
$$= (g, \mathcal{E}'^{-1} \overline{\mathcal{E}'(f)}) = \langle \mathcal{E}'(g), \mathcal{E}'(f) \rangle$$

($(,)$ is the scalar product in $L^2(G)$). Then by definition:

$$\mathcal{E}(g) = \mathcal{E}'(g).$$

$\square$

The adapted Fourier transform is related to the unitary representation associated to an orbit O in $\mathfrak{g}^*$. Let us first recall the construction of such a representation. We choose an element μ_0 in O and a maximal subordinate subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ i.e. a subalgebra such that:

$$\langle \mu_0, [\mathfrak{h}, \mathfrak{h}] \rangle = 0$$

and maximal with that property. Since the exponential map is a diffeomorphism between $\mathfrak{g}$ and G, the analytic subgroup corresponding to $\mathfrak{h}$ is $H = \exp(\mathfrak{h})$ and the formula:

$$\chi_{\mu_0}(\exp X) = e^{i\mu_0(X)} \quad (X \in \mathfrak{h})$$

defines a character of H. We induce this one-dimensional representation from H to G and obtain the unitary representation associated to O:

$$\Pi^O = \mathrm{ind}_{H \uparrow G} \chi_{\mu_0}.$$

If O is isomorphic to $\mathbb{R}^{2k}$ and equipped with the chart (p_j, q_j) of [1] or [19] then Π^O can be realized in $L^2(\mathbb{R}^k)$ (with variables t (see [1] or [15])).

Since $\mathfrak{g}$ is special, we can (and we do) choose $\mathfrak{h} = \mathfrak{m}$, thus the point μ with coordinate (p_j, q_j) is:

$$\mu = \exp -q_k E_n \cdot [\exp -q_{k-1} E_{n-1} \cdot (\cdots (\mu_0) \cdots) + p_{k-1} E_{n-1}^*] + p_k \cdot E_n^*$$

if $\{E_j^*\}$ is the dual basis of $\{E_j\}$. Now there exists an unitary transformation S from $L^2(O)$ onto the space of Hilbert Schmidt operators on $L^2(\mathbb{R}^k)$ defined by:

$$Su(s, t) = F_p u \left(t - s, \frac{s+t}{2} \right)$$

where

$$F_p u(p, q) = \int_{\mathbb{R}^k} u(x, q) e^{-2i\pi xp} \, dx \quad (s, t, p, q \in \mathbb{R}^k).$$

Theorem 3.3. *Let us suppose Π^O explicitly realized on $L^2(\mathbb{R}^k)$ as in [1]. Let f be in $\mathcal{S}(G)$ and g in $L^2(O)$, then:*

(1) $\Pi^O(f)g(t) = \int_O K(t,\tau)g(\tau)\,d\tau$

   where:

$$K(s,t) = \int_m f(\exp(-t_k E_n) * \exp(-t_{k-1}E_{n-1}) \cdots$$
$$* \exp(-t_1 E_{n-k+1}) * m * \exp(s_1 E_{n-k+1})$$
$$\cdots * \exp(s_k E_n))e^{2i\pi\langle m,\mu_0\rangle}\,dm,$$

   here $t = (t_1,\ldots,t_k)$ and $s = (s_1,\ldots,s_k)$.

(2)   *K and $\mathcal{E}(f)$ are related by $\mathcal{E}(f) = S^{-1}(K)$.*

Proof. Pukanszky gives in [15], p. 115 the expression of the kernel of $\Pi^O(f)$ for the unitary representation Π^O associated to an orbit O. In our special case this expression can be written as in the first part of the theorem.

   For the part (2), let G_1 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_1$ spanned by $E_1,\ldots,E_{n-1}$. We define the representation Π_1^t of G_1 by:

$$\Pi_1 = \mathrm{ind}_{H\uparrow G_1}\chi_{\mu_0},$$
$$\Pi_1^t(g_1) = \Pi_1(\exp\,tE_n\,g_1\exp\,-tE_n)\quad (t\in\mathbb{R}).$$

Let now F_1 be a function in $\mathcal{S}(G_1)$, by induction, we suppose that $\Pi_1^t(F_1)$ can be written:

$$(\Pi_1^t(F_1)f)(t_{(1)}) = \int_{\mathbb{R}^{k-1}} K_1^t(t_{(1)},s_{(1)})f(s_{(1)})\,ds_{(1)},$$

here $x_{(1)}$ is for $(x_1,\ldots,x_{k-1})$, and if μ_1^t is $\exp-tE_n\cdot\mu_1(p_1,q_1)$, with:

$$K_1^t(t_{(1)},s_{(1)}) = F_{p_{(1)}}u_1^t\left(t_{(1)} - s_{(1)},\frac{t_{(1)}+s_{(1)}}{2}\right),$$

$$u_1^t(p_{(1)},q_{(1)}) = \int_{\mathfrak{g}_1} F_1\circ\varphi_1(X_1)e^{2i\pi\langle X_1,\mu_1^t\rangle}\,dX_1.$$

Then if F is in $\mathcal{S}(G)$, we obtain:

$$(\Pi^O(F)f)(t_{(1)},t_{(k)}) = \int_{\mathbb{R}}\int_{\mathfrak{g}_1} F\circ\varphi(X_1 + yE_n)$$
$$(\Pi^O(\varphi(X_1 + yE_n)f)(t_{(1)},t_k))\,dX_1\,dy$$
$$= \int_{\mathbb{R}}\int_{\mathfrak{g}_1} F\circ\varphi(X_1 + yE_n)$$
$$\left(\Pi_1^{t_k-\frac{y}{2}}(\varphi_1(X_1))f\right)(t_{(1)},t_k - y)\,dX_1\,dy$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}^{k-1}} K_1^{t_k-\frac{y}{2}}(t_{(1)},s_{(1)},y)$$
$$f(s_{(1)},t_k - y)\,ds_{(1)}\,dy.$$

Where:

$$K_1^q(t_{(1)}, s_{(1)}, t - s) = (F_{p_{(1)}} u_1^q) \left(t_{(1)} - s_{(1)}, \frac{t_{(1)} + s_{(1)}}{2}, t - s \right)$$

and:

$$u_1^q(p_{(1)}, q_{(1)}, x) = \int_{\mathfrak{g}_1} (F \circ \varphi)(X_1 + x E_n)$$
$$\cdot e^{2i\pi\langle X_1, \exp - q E_n \star \mu(p_{(1)}, q_{(1)}) \rangle} \, dX_1.$$

Thus if u is the function on O defined by:

$$u(p, q) = \mathcal{E}(f)(\mu(p, q))$$
$$= \int_{\mathfrak{g}} F \circ \varphi(X) e^{2i\pi\langle X, \mu(p, q) \rangle} \, dX,$$

then:

$$(\Pi^O(F)f)(t) = \int_{\mathbb{R}^k} K(t, s) f(s) \, ds,$$

with:

$$K(t, s) = K_1^{\frac{t_k + s_k}{2}}(t_{(1)}, s_{(1)}, t_k - s_k)$$
$$= (F_{p_k} F_{p_{(1)}} u) \left(t - s, \frac{t + s}{2} \right).$$

□

4. Final remarks.

Our goal was the construction of deformations on algebras of functions or distributions on $\mathfrak{g}^*$, for $\mathfrak{g}$ nilpotent. Many authors considered such deformations, all of them being related to the convolution on the corresponding connected and simply connected Lie group G through an 'adapted Fourier transform' of the form:

$$(\mathcal{E}f)(\mu) = \int_{\mathfrak{g}} f(\exp X) e^{2i\pi a(\mu, X)} \, dX$$
$$\text{or} \quad \int_{\mathfrak{g}} f(\psi(X)) e^{2i\pi\langle \mu, X \rangle} \, dX.$$

Let us compare now our construction with some preceeding ones, starting with that point of view of the Fourier transform. If $\mathfrak{g}$ is any nilpotent Lie algebra, we knew two deformations:

(1)   the deformation of Lugo-Gutt-Rieffel where $a(\mu, X)$ is simply $\langle \mu, X \rangle$ (or ψ is exp). $\mathcal{E}$ is well defined everywhere (for instance if f is a Schwarz function on G) but it is well known that this deformation is not (generically) tangential.

(2)   the deformation of Arnal-Cortet where $a(\mu, X)$ is a function polynomial in the variable X and rational in the variable μ. The function $\mathcal{E}f$ is thus defined only on a dense invariant subset of $\mathfrak{g}^*$, the corresponding deformation is tangential but we cannot surrounded the singularities as the example of $\mathfrak{g}_{5,4}$ (see [2]) shows.

Now, we are obliged to consider only some classes of nilpotent Lie algebras $\mathfrak{g}$, hoping to find generically tangential, globaly defined deformation on $\mathfrak{g}^*$ for such $\mathfrak{g}$.

For instance, let us define the 'filform' Lie algebra $\mathfrak{f}$ to be the algebra with basis Y, X_0, $X_1, \ldots$, X_n and non trivial brackets:

$$[Y, X_i] = X_{i-1}, \quad i = 1, \ldots, n.$$

For n larger than 2, the Lugo-Gutt-Rieffel deformation on $\mathfrak{f}$ is distinct of the Arnal-Cortet deformation (and the same holds for the corresponding Fourier transforms). The Arnal-Cortet deformation is in fact globally defined for each $\mathfrak{f}$, on the other hand $\mathfrak{f}$ is a special Lie algebra and our deformation on $\mathfrak{f}^*$ is distinct of the Arnal-Cortet deformation, for n larger than 2.

Let us now mention the work of Howe, Ratcliff and Wildberger [10]. They consider a globally defined, generically tangential deformation on the dual $\mathfrak{g}^*$ of the Lie algebra of a OKR Cayley-stable Lie group G (see [10] for the definitions). The fundamental example of such $\mathfrak{g}$ is the Lie algebra $\mathfrak{u}_m$ of matrices of the form $\left| \begin{smallmatrix} A & B \\ 0 & -A^t \end{smallmatrix} \right|$ where A, B are $m \times m$ matrices with A upper triangular and B symmetric. For a OKR Cayley stable G, they define a Cayley transform c from G to $\mathfrak{g}$ and put:

$$(\mathcal{E}f)(\mu) = \int_G f(\exp X) e^{-2i\langle \mu, c(\exp X) \rangle} \, dX$$

$$= K \int_{\mathfrak{g}} f\left(c^{-1}\left(-\frac{X}{2}\right) \right) e^{i\langle X, \mu \rangle} \, dX,$$

where K is a constant. Finally, their well defined and tangential deformation can be defined through this Fourier transformation.

Let us compair our construction with the result of [10]. First, each $\mathfrak{u}_m$ is a special Lie algebra and we can apply our construction to them. For $m = 1, 2$, the two Fourier transforms coincide since then:

$$c^{-1}\left(-\frac{X}{2}\right) = 1 + X + \frac{X^2}{2} + \cdots + \frac{X^n}{2^{n-1}} + \cdots = \varphi(X).$$

But a direct computation shows they are distinct for m larger than 2.

Now, if G is any OKR-Cayley stable group and Π^O a generic unitary irreducible representation of G, in [10], the authors use an isomorphism Φ between $G/ker\ \Pi^O$ and some U_m such that $c_{U_m} \circ \Phi = d\Phi \circ c_G$ to define their symbolic calculus. Therefore, $G/ker\ \Pi^O$ is special and we can compare the two constructions. In fact, if m is larger than 2, the two formulae are distinct.

Finally, if $\mathfrak{f}$ is the filiform Lie algebra of dimensionality larger than 4, $\mathfrak{f}$ is special but not OKR, thus, in some sense, our construction is more general than the construction of [10].

Acknowledgements.

References

[1]   D.Arnal, and J.C.Cortet, ⋆ - *product in the method of orbits for nilpotent groups*, J. Geom. Phys., **2** (1985), 83-116.

[2]   D.Arnal, M.Cahen, and S.Gutt, *Deformation on coadjoint orbits*, J. Geom. Phys., **3** (1986), 327-351.

[3]   D.Arnal, ⋆ - *product and representations of nilpotent groups*, Pacific J. Math., **114** (2) (1984), 285-308.

[4]   D.Arnal, and J.C.Cortet, *Nilpotent Fourier transform and applications*, Lett. math. phys., **9** (1985), 25-34.

[5]   D.Arnal, J.C.Cortet, P.Molin, and G.Pinczon, *Covariance and geometrical invariance in ⋆-quantization*, J. Math. Phys., **24** (2) (1983), 276-283.

[6]   F.Bayen, M.Flato, C.Fronsdal, A.Lichnerowicz, and D.Sternheimer, *Deformation theory and quantization, I, II*, Ann. of Phys., **111** (1978), 61-151.

[7]   L.Corwin, and F.P.Greenleaf, *Fourier transforms of smooth functions on certain nilpotent groups*, J. Funct. Anal., **37** (1980), 203-217.

[8]   R.Godement, *Introduction a la theorie des groupes de Lie*, Tome 2, Publ. math. Univ. Paris VII.

[9]   S.Gutt, *An explicit ⋆ - product on the cotangent bundle of a Lie group*, Lett. Math. Phys., **7** (1983), 249-258.

[10] R.Howe, G.Ratcliff, and N.Wildberger, *Symbol mappings for certain nilpotent groups*, Springer Lect. Notes in Math., Vol 1077.

[11] A.Lichnerowicz, *Deformations and quantization*, Geom. and Phys., (1983), 103-116.

[12] A.Lichnerowicz, *Variété de Poisson et feuilletage*, Ann. Fac. Sci. Toulouse, (1982), 195-262.

[13] V.Lugo, *An associative algebra of functions on the orbits of nilpotent Lie groups*, Lett. Math. Phys., **5** (1981), 509-516.

[14]   W.Magnus, F.Oberbettinger, and R.P.Soni, *Formulas and theorems for special functions of mathematical physics*, Springer-Verlag Berlin, Heidelberg, New York (1966).

[15]   L.Pukanszky, *Lecons sur les representations des groupes*, Dunod Paris, (1967).

[16]   M.A.Rieffel, *Lie group convolution algebras as deformation quantizations of linear Poisson structures*, Amer. J. Math., **112** (1990), 657-686.

[17]   M.A.Rieffel, *Deformation quantization of Heisenberg manifolds*, Comm. Math. Phys., **122** (1989), 531-562.

[18]   M.A.Rieffel, *Continuous fields of C^*-algebras coming from group cocycles and actions*, Math. Ann., **283** (1989), 631-643.

[19]   M.Vergne, *La structure de Poisson sur l'agebre symetrique d'une algebre de Lie nilpotente*, Bull. Soc. Math. France, **100** (1972), 301-335.

SCIENCE FACULTY
3038 SFAX
TUNISIA

CONTENTS

PACIFIC JOURNAL OF MATHEMATICS