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THE INVARIANT CONNECTION OF A $\frac{1}{2}$-PINCHED ANOSOV DIFFEOMORPHISM AND RIGIDITY

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Let f be a C^∞ Anosov diffeomorphism of a compact manifold M, preserving a smooth measure. If f satisfies the $\frac{1}{2}$-pinching assumption defined below, it must preserve a continuous affine connection for which the leaves of the Anosov foliations are totally geodesic, geodesically complete, and flat (its tangential curvature is defined along individual leaves). If this connection, which is the unique f-invariant affine connection on M, is C^r-differentiable, $r \geq 2$, then f is conjugate via a C^{r+2}-affine diffeomorphism to a hyperbolic automorphism of a geodesically complete flat manifold. If f preserves a smooth symplectic form, has C^3 Anosov foliations, and satisfies the 2 : 1-nonresonance condition (an assumption that is weaker than pinching), then f is C^∞ conjugate to a hyperbolic automorphism of a complete flat manifold. (In the symplectic case, the invariant connection is the one previously defined by Kanai in the context of geodesic flows.) If the foliations are C^2 and the holonomy pseudo-groups satisfy a certain growth condition, the same conclusion holds.

1. Introduction

Let M be a compact C^∞ manifold and f a C^∞ diffeomorphism of M. The diffeomorphism is *Anosov* if TM decomposes continuously as a direct sum $TM = E^+ \oplus E^-$ of invariant subbundles E^+ and E^-, so that the following estimate applies: For some (in fact, any) Riemannian metric $\|\cdot\|$, there exist positive constants $C > 1$, $\epsilon < a < A$, such that for all $x \in M$, for all positive integers n, and for all $v \in E^\pm(x)$,

$$\frac{1}{C}\|v\|e^{-nA} \leq \|Tf_x^{\mp n}v\| \leq C\|v\|e^{-na}. \tag{1}$$

The subbundles $E^\pm$ are the tangent bundles of C^0 foliations $\mathcal{E}^\pm$, the *Anosov foliations* of f, whose leaves are smooth (if f is smooth.)

We say that f satisfies the $\frac{1}{2}$*-pinching condition* if $A < 2a$.

In this paper we assume that f leaves invariant a smooth measure λ, which implies, in particular, that it is ergodic.

The following theorem is proved in [**BL**]:

Theorem 1. *Let M be a C^∞ compact manifold equipped with a C^∞ affine connection. Let f be a topologically transitive Anosov diffeomorphism preserving the connection and such that the stable and unstable distributions E^+ and E^- are C^∞. Then f is C^∞ conjugate to a hyperbolic automorphism of an infranilmanifold.*

If f satisfies the $\frac{1}{2}$-pinching condition (more generally, if it satisfies the non-resonance condition defined below), the conclusion of the above theorem holds with the infranilmanifold replaced with a flat one, as can be easily verified.

The main result of the present paper is the following observation:

Theorem 2. *Let M be a C^∞ compact manifold and f a C^∞ Anosov diffeomorphism that satisfies the $\frac{1}{2}$-pinching assumption. Then f preserves a continuous affine connection. This connection is unique among the f-invariant affine connections, it is torsion-free and, with respect to it, the leaves of the stable and unstable Anosov foliations are totally geodesic, complete and flat. (The restriction of this connection to the leaves of the stable and unstable foliations is differentiable, so it makes sense to define its curvature tensor there.) If this connection is C^r-differentiable, $r \geq 2$, then f is C^{r+2}-conjugate to a hyperbolic automorphism of a complete flat manifold.*

In [**Kanai**], M. Kanai defined an affine connection on the unit tangent bundle of a negatively curved Riemannian manifold, invariant under the geodesic flow. Essentially the same construction produces an affine connection that is invariant under symplectic Anosov diffeomorphisms (or Anosov diffeomorphisms preserving any other nondegenerate bilinear form) with differentiable stable and unstable foliations: one uses the Bott connection to define the covariant derivative transversely to the Anosov foliations and the duality obtained from the symplectic form to define a covariant derivative tangentially (here one uses that the Anosov foliations must be Lagrangian foliations). In this symplectic case, C^{r+1}-regularity of the Anosov foliations is equivalent to the C^r regularity of the invariant connection. Before stating the next theorem, we define a nonresonance condition for f that is implied by $\frac{1}{2}$-pinching. Many of the results discussed here require only this weaker assumption.

Definition 3. Let f be a diffeomorphism of a compact manifold M. Define for $v \in TM_x \backslash \{0\}$ the numbers

$$\chi^{\pm}(v) = \limsup_{n \to \pm\infty} \frac{1}{n} \log \|Tf_x^n v\|.$$

We say that f satifies the $2:1$-nonresonance condition if for all $x \in M$ there do not exist vectors $v_1, v_2, v_3 \in TM_x \backslash \{0\}$ such that

$$\chi^+(v_1) = \chi^+(v_2) + \chi^+(v_3).$$

Theorem 4. *Let M be a C^∞ compact manifold and f a C^∞ Anosov diffeomorphism preserving a C^∞ symplectic form, whose Anosov foliations are C^3. Assume moreover that f satifies the $2:1$-nonresonance condition. Then f is C^∞ conjugate to a hyperbolic automorphism of a complete flat manifold.*

It is interesting to note that in [**Llave**] de la Llave constructed, for every r, examples of C^∞ Anosov diffeomorphisms on the torus that are C^r conjugate, but not C^{r+1} conjugate to a linear automorphism of the torus. We note, however, that under the assumption that f be symplectic, C^1 conjugacy implies C^∞ conjugacy.[1]

For systems satisfying the $\frac{1}{2}$-pinching assumption we show that the invariant connection can be defined without recourse to a symplectic structure. In this case it can be shown (as we do later; in fact, avoiding a $2:1$-resonance, rather than $\frac{1}{2}$-pinching, is sufficient for this) that the distributions E^+ and E^- are parallel, so assuming C^r regularity of the connection implies C^{r+1}-regularity of those distributions. For symplectic systems the converse can be easily verified. Such is not always the case, however, for nonsymplectic systems, due again to de la Llave's examples. In fact, if f is C^k (but not C^{k+1}) conjugate to a hyperbolic linear automorphism of a torus, by pulling back the Euclidian connection under the conjugating diffeomorphism we obtain a C^{k-2} invariant connection, which cannot be C^{k-1} (due to a simple argument used in the last section of this paper, which shows that the conjugacy is two units more differentiable than the pulled-back connection). But the Anosov foliations must be C^k since they are preserved by the conjugacy.

The conjugacy claimed in Theorem 1 was obtained with the help of a general result due to M. Gromov [**Gr**] concerning transformation groups of *rigid geometric structures.* (An example of such structure is a (differentiable) affine connection.) Gromov's theorem was used there to show that the pseudo-group of local transformations preserving a C^∞ connection and the (C^∞) distributions E^+ and E^- is a Lie pseudo-group that acts transitively on an open dense f-invariant set. The proof of Theorem 2 does not rely on Gromov's theorem and uses instead a more direct argument that

[1]We thank R. de la Llave for pointing this out.

requires much less differentiability. We show here that the pseudo-group referred to above acts transitively on M by proving that the curvature of the f-invariant connection is parallel and using classical results on affine locally symmetric connections.

It is conjectured that the conclusion of the previous theorem should still hold for systems whose Anosov foliations are only C^2 (the connection being C^1). The place where we use more regularity is at showing that the invariant connection is locally symmetric (there we need to work with the covariant derivative of the curvature tensor). If by other means one shows that the connection is flat, the same conclusion will hold with the optimal degree of smoothness. The next two results illustrate this point.

By allowing the leaves of the Anosov foliations to spread apart only very slowly, as in the definition below, we obtain the theorem that follows next.

Definition 5. Let $\mathcal{F}$ be a C^2 foliation of codimension q. Given a differentiable path $\gamma : [0,1] \to M$ in a leaf of $\mathcal{F}$ and smooth cross-sections Σ_0, Σ_1 of dimension q transversal to $\mathcal{F}$ at $\gamma(0)$ and $\gamma(1)$, we may consider the corresponding element of the holonomy pseudogroup of (germs of) transverse maps. By making the cross-sections sufficiently small, we may describe this element as a C^2 diffeomorphism $\mathcal{H}_\gamma : \Sigma_0 \to \Sigma_1$. Let $\|\cdot\|^{(1)}$ be a C^1 norm defined on such maps as $y \in \Sigma_0 \to (T\mathcal{H}_\gamma)_y$ and assume that

$$K_1 length(\gamma)^{-\delta} \leq \|T\mathcal{H}_\gamma\|^{(1)}_{\gamma(0)} \leq K_2 length(\gamma)^{\delta}$$

for positive constants K_1, K_2, δ that are independent of γ. In this case we say that the foliation $\mathcal{F}$ *spreads more slowly than power* l^δ.

Theorem 6. *Assume that f is a C^∞ Anosov diffeomorphism of a compact manifold M, whose Anosov foliations are C^2 and spread more slowly than power $l^{\frac{2a-A}{A}-\epsilon}$, for some $\epsilon > 0$. Then*

1. *If f is $\frac{1}{2}$-pinched, the f-invariant connection ∇ is flat and f is C^3 conjugate to an automorphism of a complete flat manifold.*
2. *If f is symplectic, the Kanai connection is flat and f is C^∞ conjugate to an automorphism of a complete flat manifold.*

It is interesting to note what happens in dimension 2. In this case, it is well known that if the Anosov foliations of a volume preserving Anosov diffeomorphism of a compact manifold M are C^2 differentiable, they are automatically C^∞ and the diffeomorphism is smoothly conjugate to a linear automorphism of the 2-torus. (For a precise description of the threshold of regularity where rigidity occurs in this case, see [**HK**]. We also refer to that paper for historical and bibliographical information concerning these questions.) We give below a geometric proof of this fact.

Theorem 7. *Let f be a C^∞ volume preserving Anosov diffeomorphism of a compact 2-dimensional manifold whose Anosov foliations are C^2. Then these foliations are C^∞ and f is smoothly conjugate to a linear hyperbolic automorphism of the 2-torus.*

2. Existence of an invariant affine connection

Our first concern is the question of existence of invariant connections under the Anosov diffeomorphism. In this section we show the first part of Theorem 2:

Proposition 8. *Let M be a C^∞ compact manifold and f a C^∞ Anosov diffeomorphism. If f avoids a $2:1$-resonance, it must preserve a unique Borel connection on TM. Suppose now the stronger condition that f satisfies the $\frac{1}{2}$-pinching assumption (in particular, the Anosov distributions $E^\pm$ are C^1, according to* [**Ha**].) *Then f preserves a continuous affine connection. With respect to this connection the Anosov distributions $E^\pm$ are parallel; its restriction to the leaves of the stable and unstable foliations is differentiable, so it makes sense to define its (tangential) curvature tensor. The connection is torsion-free and the leaves of the Anosov foliations are totally geadesic, complete and flat.*

We begin with a few basic comments about invariant connnections.

Let $p : F \to M$ be a C^r vector bundle over M. Let $J_1(F)$ denote the vector bundle over M consisting of first jets of germs of differentiable sections of F. The following short sequence of vector bundles is exact ([**KS**]):

$$0 \to T^*M \otimes F \xrightarrow{i} J_1(F) \xrightarrow{\pi} F \to 0.$$

A connection on F can be described as a splitting of this exact sequence: $\sigma : F \to J_1(F)$, $\pi \circ \sigma = \mathrm{Id}_F$. Such splitting defines a covariant derivative map $\nabla : \Gamma^1(F) \to \Gamma^0(T^*M \otimes F)$, where $\Gamma^r(F)$ denotes the space of C^r-sections of F, as follows: For $X \in \Gamma^1(F)$, and denoting $j_1 X \in \Gamma^0(J_1(F))$ the first jet of X, set: $\nabla X = (\mathrm{Id} - \sigma \circ \pi) j_1 X$.

We now assume that $(\bar{f}, f)$ is a C^r automorphism of $p : F \to M$ (so that $p \circ \bar{f} = f \circ p$), from which we obtain automorphisms of $J_1(F)$ and $T^*M \otimes F$ as follows: For $j_1 X(x) \in J_1(F)_x$, $\bar{f} \cdot j_1 X(x) = j_1(\bar{f} X \circ f^{-1})(f(x))$ and for $\alpha \otimes X \in T^*M_x \otimes F_x$, $\bar{f}(\alpha \otimes X) = \bar{f}\alpha \otimes \bar{f} X$, where $\bar{f}\alpha = \alpha \circ Tf^{-1}|_{TM_{f(x)}}$.

With these definitions, the diagram

$$\begin{tikzcd} 0 \arrow[r] & T^*M \otimes F \arrow[r, "i"] \arrow[d, "\bar{f}"] & J_1(F) \arrow[r, "\pi"] \arrow[d, "\bar{f}"] & F \arrow[r] \arrow[d, "\bar{f}"] & 0 \\ 0 \arrow[r] & T^*M \otimes F \arrow[r, "i"] & J_1(F) \arrow[r, "\pi"] & F \arrow[r] & 0 \end{tikzcd}$$

commutes.

An $\bar{f}$-invariant C^r-connection $\nabla : \Gamma^{r+1}(F) \to \Gamma^r(T^*M \otimes F)$ can be described as a C^r-splitting σ of the last diagram, for which $\sigma \circ \bar{f}|_F = \bar{f}|_{J_1(F)} \circ \sigma$. Equivalently, it can be described as an $\bar{f}$-equivariant C^r subbundle in $J_1(F)$, complementary to $i(T^*M \otimes F)$.

Let $p : E \to M$ be a smooth vector bundle over M and $(\bar{f}, f)$ a bundle automorphism, so that $\bar{f} \circ p = p \circ f$. In the following theorem (adapted from [**Margulis**, **Mane**, **Wa**]) $\mathcal{B}$ denotes the σ-algebra of Borel subsets of M and $\mathcal{M}(M, f)$ the set of all f-invariant probability measures on $\mathcal{B}$, for a diffeomorphism f of M.

Theorem 9 (Multiplicative ergodic theorem). *Let $E \to M$ be a smooth, real vector bundle of rank q over a compact manifold M and $(\bar{f}, f)$ a bundle automorphism. Then there exist:*

1. *a set $\Lambda \in \mathcal{B}$ such that $\mu(\Lambda) = 1$ for all $\mu \in \mathcal{M}(M, f)$ and $f(\Lambda) = \Lambda$,*
2. *measurable, f-invariant functions $s : \Lambda \to \{1, \cdots, q\}$, $\chi_i : \Lambda \to \mathbb{R}$ $(1 \leq i \leq s)$ satisfying $\chi_1 < \cdots < \chi_s$, and*
3. *a measurable, $\bar{f}$-invariant decomposition $E|_\Lambda = E_1 \oplus \cdots \oplus E_s$ (the Oseledec decomposition for $\bar{f}$) such that for all $v \in E_i(x)\backslash\{0\}$,*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\bar{f}^n v\| = \chi_i(x)$$

exists and the convergence is uniform over the vectors of unit norm in $E_i(x)$.

For all $x \in \Lambda$ and all $v \in E(x)\backslash\{0\}$, the limits

$$\chi^+(\bar{f}, v) \stackrel{\text{def}}{=} \lim_{n \to +\infty} \frac{1}{n} \log \|\bar{f}^n v\|, \quad \chi^-(\bar{f}, v) \stackrel{\text{def}}{=} \lim_{n \to -\infty} \frac{1}{n} \log \|\bar{f}^n v\|$$

also exist. $\chi^+(\bar{f}, v) \geq \chi^-(\bar{f}, v)$ and we have $\chi^+(\bar{f}, v) = \chi^-(\bar{f}, v)$ if and only if $v \in E_i(x)\backslash\{0\}$ for some i.

If L is a smooth, $\bar{f}$-invariant (vector) subbundle of E with projection $\pi : E \to F = E/L$, let $F = F_1 \oplus \cdots \oplus F_{s'}$ be the decomposition of F associated with characteristic exponents ${\chi'}_1 < \cdots < {\chi'}_{s'}$ (s' and ${\chi'}_j$ being measurable functions on Λ). Then, for each j $(1 \leq j \leq s')$ there is $i(j)$, $1 \leq i(j) \leq s$, such that ${\chi'}_j = \chi_{i(j)}$ and $F_j = \pi(E_{i(j)})$ on Λ. Moreover $L|_\Lambda = \bigoplus_{i=1}^{s}(L \cap E_i)$.

Definition 10. We will refer to $\Lambda = \Lambda_E$ as the set of *regular* points of $(\bar{f}, f)$. The measurable decomposition $E = \bigoplus E_i$ is the *Oseledec decomposition* of E and χ_i are the *Lyapunov exponents.*

Lemma 11. *Let $E \to M$ and $(\bar{f}, f)$ be as above and consider the vector bundle*

$$F = \underbrace{E^* \otimes \cdots \otimes E^*}_{r} \otimes \underbrace{E \otimes \cdots \otimes E}_{s}.$$

Then F has a measurable, $\bar{f}$-invariant decomposition $F = \bigoplus_{I,J} E_{I,J}$, where $I = (i_1, \cdots, i_r)$, $J = (j_1, \cdots, j_s)$ and

$$E_{I,J} = E^*_{i_1} \otimes \cdots \otimes E^*_{i_r} \otimes E_{j_1} \otimes \cdots \otimes E_{j_s},$$

so that for every f-invariant Borel probability measure μ, μ-a.e. $x \in M$, and $v \in E_{I,J} \backslash \{0\}$,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\bar{f}^n_x v\| = -\chi_{i_1}(x) - \cdots - \chi_{i_r}(x) + \chi_{j_1}(x) + \cdots + \chi_{i_s}(x).$$

Proof. We first show that the Lyapunov exponents on the dual bundle E^* are the negative of those on E. Denoting by $\langle \cdot, \cdot \rangle$ the pairing between E_i^* and E_i, we have: for any $0 \neq \eta^* \in E_i^*(x)$ there exists $\eta \in E_i(x)$ such that $\langle \eta^*, \eta \rangle \neq 0$ and

$$0 < \text{const.} = |\langle \eta^*, \eta \rangle| = |\langle \bar{f}^n \eta^*, \bar{f}^n \eta \rangle| \leq \|\bar{f}^n \eta^*\| \; \|\bar{f}^n \eta\|.$$

Therefore, for $n > 0$,

$$\frac{1}{n} \ln \|\bar{f}^n \eta^*\| \geq -\frac{1}{n} \ln \|\bar{f}^n \eta\| + \frac{1}{n} \text{const.}$$

Let $x \in \Lambda_E \cap \Lambda_{E^*}$, where Λ_F is the set of regular points defined in the previous theorem. For such points we may take limits and conclude:

$$\lim_{n \to +\infty} \frac{1}{n} \ln \|\bar{f}^n \eta^*\| \geq -\chi_i(x).$$

Similarly,

$$\lim_{n \to -\infty} \frac{1}{n} \ln \|\bar{f}^n \eta^*\| \leq -\chi_i(x).$$

It now follows from Theorem 9 (3) that

$$\lim_{n \to \pm\infty} \frac{1}{n} \ln \|\bar{f}^n \eta^*\| = -\chi_i(x).$$

In a similar way, one shows that given $v_i \in F_i(x) \backslash \{0\}$, $i = 1, 2$, where F_i is a tensor product of subbundles chosen among the E_j and E_j^*, and assuming that $\lim_{n \to \pm\infty} \frac{1}{n} \ln \|\bar{f}^n v_i\| = \nu_i$, then for $w = v_1 \otimes v_2$ we have $\lim_{n \to \pm\infty} \frac{1}{n} \ln \|\bar{f}^n w\| = \nu_1 + \nu_2$. The claim now follows. □

Denote by $\pi_{I,J} : F \to E_{I,J}$ the natural (measurable) $\bar{f}$-invariant projections. It follows from the above remark that:

Lemma 12. *If τ is $\bar{f}$-invariant and $\pi_{I,J}\tau \neq 0$ on a set of positive μ-measure for an f-invariant Borel probability measure μ, then for μ-almost every x in that set,*

$$-\chi_{i_1}(x) - \cdots - \chi_{i_r}(x) + \chi_{j_1}(x) + \cdots + \chi_{i_s}(x) = 0.$$

Proof. Let Ω denote the f-invariant set of positive measure consisting of regular points at which $\tau_x \neq 0$. Then for every $\epsilon > 0$, there exists a set $\Omega_\epsilon \subset \Omega$ and a constant $L \geq 1$ such that $\mu(\Omega \backslash \Omega_\epsilon) < \epsilon$ and $\|\tau_x\| < L$ for all $x \in \Omega_\epsilon$. On the other hand, according to Poincaré's recurrence theorem, for a.e. $x \in \Omega_\epsilon$ we can find a biinfinite sequence $\{n_i : i \in \mathbb{Z}\}$ such that $f^{n_i}(x) \in \Omega_\epsilon$. For any x with this property, let $\xi = v_1 \otimes \cdots \otimes v_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_s$ be in $E^*_{I,J}(x)$ so that $\langle \tau_x, \xi \rangle \neq 0$. Since τ is $\bar{f}$-invariant, we have:

$$\begin{aligned} 0 < |\langle \tau_x, v_1 \otimes \cdots \otimes \alpha_s \rangle| &= |\langle \bar{f}^{-n_i} \tau_{f^{n_i}(x)}, v_1 \otimes \cdots \otimes \alpha_s \rangle| \\ &= |\langle \tau_{f^{n_i}(x)}, \bar{f}^{n_i} v_1 \otimes \cdots \otimes \bar{f}^{n_i} \alpha_s \rangle| \\ &\leq L \prod_{p=1}^{r} \|\bar{f}^{n_i} v_p\| \prod_{q=1}^{s} \|\bar{f}^{n_i} \alpha_q\|. \end{aligned}$$

After taking ln, dividing by n_i and passing to the limit as $i \to \infty$ and $i \to -\infty$, the claim follows. □

Lemma 13. *If f is a diffeomorphism of a compact manifold M preserving a Borel connection ∇ and such that it avoids a $2:1$-resonance, then ∇ is the unique f-invariant affine connection on TM and its torsion vanishes identically. If f is Anosov with C^1 distributions $E^\pm$, then E^+ and E^- are parallel.*

Proof. If ∇' is another f-invariant continuous connection, $\nabla - \nabla'$ is an f-invariant tensor field. If it is not zero, it produces $2:1$-resonance as indicated in Lemma 12. More generally, if τ is an f-invariant measurable tensor field of type (r, s) for $r + s = 3$, then if $\tau \neq 0$ it would produce a $2:1$-resonance, contradicting the assumption. Other tensors of this type are the torsion of ∇ and the covariant derivatives of the (C^1) projections $\Pi^\pm : TM \to E^\pm$. Therefore ∇ is torsion-free and the Anosov subbundles $E^\pm$ are parallel. □

By the uniqueness property obtained in the previous lemma, the covariant derivative of vector fields tangent to one Anosov foliation along a vector tangent to the complementary foliation can be characterized as follows: If $X \in \Gamma^1(E^\pm), Y \in \Gamma^1(E^\mp)$, $\nabla_X Y = \Pi^\mp[X,Y]$ (the Bott connection for the foliations $\mathcal{E}^\pm$).

Before we establish the existence of the invariant connection for a general $\frac{1}{2}$-pinched Anosov diffeomorphism, we recall for later use how the connection can be defined in the presence of an invariant symplectic form ω. The construction is due to Kanai. Define the involution $a : E^+ \oplus E^- \to E^+ \oplus E^-$ given by $v^+ + v^- \to v^+ - v^-$. Then the bilinear form $g(\cdot\ ,\ \cdot) = \omega(\cdot\ ,\ a\ \cdot)$ is symmetric and nondegenerate. In general g is only continuous, but it is clearly differentiable if the subbundles $E^\pm$ are differentiable. If such is the case, one obtains the continuous Levi-Civita connection associated to this pseudo-Riemannian C^1 metric. That the connection defined in this way agrees with the previously constructed one is a consequence of the uniqueness property established in Lemma 13.

Definition 14. We refer to the invariant symplectic connection defined above as the Kanai connection. It is immediate from the definition that the connection is C^r if and only if the foliations are C^{r+1}.

Lemma 15. *Let f be a $\frac{1}{2}$-pinched Anosov diffeomorphism of a compact manifold. Then f preserves a continuous affine connection ∇. The leaves of the stable and unstable foliations $\mathcal{E}^-$ and $\mathcal{E}^+$ are totally geodesic with respect to ∇ and the restriction of ∇ to individual leaves is differentiable.*

Proof. Let $\nabla^{(1)}$ be a smooth affine connection on M. Define $\nabla'_X Y = \Pi^+\nabla^{(1)}_X \Pi^+ Y + \Pi^-\nabla^{(1)}_X \Pi^- Y$, a C^0-connection. Note that the restriction of ∇' to the leaves of $\mathcal{E}^+$ and $\mathcal{E}^-$ is C^1. Define $\Theta(n) = \nabla'^{f^n} - \nabla'$, a C^0 section of $T^*M \otimes T^*M \otimes TM$, $n \in \mathbb{Z}$. Here, we denote $\nabla'^f_X Y = f_*^{-1}\nabla'_{f_*X} f_* Y$. Note that the component of $\Theta(n)$ in $E^{\pm *} \otimes E^{\pm *} \otimes E^\pm$, which we denote $\Theta^\pm(n)$, is differentiable along leaves of $\mathcal{E}^\pm$. Θ satisfies the cocycle property: $\Theta(n+1) = \Theta(1) + f \cdot \Theta(n)$, where $f \cdot \Theta = f_*^{-1}\Theta(f_*\cdot, f_*\cdot)$. We have: $\Theta(0) = 0$, $\Theta(-1) = -f^{-1} \cdot \Theta(1)$, $\Theta(n) = \sum_{i=0}^{n-1} f^i \cdot \Theta(1)$.

To obtain a C^0 connection preserved under f, we need to verify that the cohomology class represented by Θ, over the $\mathbb{Z}$-action defined by f, is trivial; i.e., there exists a continuous section Ψ of $T^*M \otimes T^*M \otimes TM$ such that $\Theta(1) = \Psi - f \cdot \Psi$; in that case $\nabla = \nabla' + \Psi$ will be preserved under f. Also note that it suffices to solve for $\Psi^\pm$ in $\Theta^\pm(1) = \Psi^\pm - f \cdot \Psi^\pm$, where $\Psi^\pm$ is now a C^0 section of $E^{\pm *} \otimes E^{\pm *} \otimes E^\pm$. In fact, if $\nabla^{(+)}$ and $\nabla^{(-)}$ define continuous families of connections on the leaves of $\mathcal{E}^+$ and $\mathcal{E}^-$, we obtain a connection on TM as follows: For C^1 vector fields X and Y, with (C^1) decomposition $X = X^+ + X^-$ and $Y = Y^+ + Y^-$,

$$\nabla_X Y = \nabla^{(+)}_{X^+} Y^+ + \nabla^{(-)}_{X^-} Y^- + \Pi^-[X^+, Y^-] + \Pi^+[X^-, Y^+].$$

If a solution to $\Psi^\pm = \Theta^\pm(1) + f \cdot \Psi^\pm$ exists, it must satisfy

$$\Psi^\pm = \Theta^\pm(1) + f \cdot \Theta^\pm(1) + \cdots + f^{n-1} \cdot \Theta^\pm(1) + f^n \cdot \Psi^\pm$$
$$\Psi^\pm = \Theta^\pm(-1) + f^{-1} \cdot \Theta^\pm(-1) + \cdots + f^{-n+1} \cdot \Theta^\pm(-1) + f^{-n} \cdot \Psi^\pm.$$

We are thus led to define $\Psi^- = \sum_{i=0}^{\infty} f^{+i} \cdot \Theta^-(1)$ (respectively, $\Psi^+ = \sum_{i=0}^{\infty} f^{-i} \cdot \Theta^+(1)$). To see that this series actually defines a continuous section Ψ^-, we set $\eta = \Theta^-(1)$ (a similar discussion will apply to Ψ^+) and note that for any vectors X, Y in $E^-(x)$,

$$\begin{aligned}\|f^n \cdot \eta_x(X,Y)\| &= \left\|Tf^{-n}\eta_{f^n(x)}(Tf_x^n X, Tf_x^n Y)\right\| \\ &\leq Ce^{nA}\left\|\eta_{f^n(x)}(Tf_x^n X, Tf_x^n Y)\right\| \\ &\leq C'e^{nA}\left\|Tf_x^n X\right\| \left\|Tf_x^n Y\right\| \\ &\leq C''e^{-n(2a-A)}\|X\|\ \|Y\|.\end{aligned}$$

Therefore we are justified in writing the uniformly convergent series defined above. Note that each term $f^i \cdot \eta$ of this series is differentiable along the leaves of $\mathcal{E}^-$. We show next that the series consisting of derivatives of $f^i \cdot \eta$ also converges uniformly.

By considering a sufficiently high power of f rather than f itself, we may assume instead of (1) that for any a', A' for which $0 < a' < a < A < A'$,

$$\|v\|e^{-nA'} \leq \left\|Tf_x^{\pm n} v\right\| \leq \|v\|e^{-na'}, \ v \in E^\mp(x). \tag{1}$$

(Note that if ∇' is preserved by f^k, then $\nabla = \frac{1}{k}\sum_{i=0}^{k-1}(f^i)^*\nabla'$ is preserved by f. If $A < 2a$, a' and A' can be chosen so that $A' < 2a'$. Therefore, there is no loss in generality to assume, as we do from now on, that $C = 1$, in (1).)

Let $\mathcal{U} \subset \mathcal{E}^-(x)$ be an open neighborhood of x' in $\mathcal{E}^-(x)$. Consider the neighborhoods $\mathcal{U}, f(\mathcal{U}), \cdots, f^s(\mathcal{U}), \cdots$, where $f^s(\mathcal{U})$ is a neighborhood of $f^s(x')$ in $\mathcal{E}^-(f^s(x))$. We may assume that each open set $f^s(\mathcal{U})$ is contained in a coordinate neighborhood (note that their diameters are decreasing). Denote the coordinate vector fields: $e_i^{(s)} = \frac{\partial}{\partial x_i}|_{f^s(\mathcal{U})}$. We write $Tf_x^n e_i^{(s)} = \sum_j A_{ij}^{(n)}(x) e_j^{(s+n)}$, where $A_{ij}^{(n)}$ are functions defined on $\bigcup_{s=0}^{\infty} f^s(\mathcal{U})$. Let $A_{ij} = A_{ij}^{(1)}$. Then the following relations are satisfied:

$$\begin{aligned}A^{(n)}(x) &= A(x)A(f(x))\cdots A(f^{n-1}(x)) \\ A^{(-1)}(f(x)) &= A^{-1}(x) \\ A^{(-n)}(f^n(x)) &= A^{(-1)}(f^n(x))A^{(-1)}(f^{n-1}(x))\cdots A^{(-1)}(f(x))\end{aligned}$$

and $e^{-A} \leq \|A(x)\| \leq e^{-a}$, $e^a \leq \left\|A^{(-1)}(x)\right\| \leq e^A$ for all x. Also set

$$\eta(e_i^{(s)}, e_j^{(s)}) = \sum_k \eta_{ij}^{(s)k} e_k^{(s)},$$

so that

$$(f^n \cdot \eta)_{ij}^{(0)k} = \sum_{r,s,k} \eta_{r,s}^{(n)k}(f^n(x)) A_{ir}^{(n)}(x).A_{js}^{(n)}(x) A_{kl}^{(-n)}(f^n(x))$$

A simple computation shows:

$$\left\| \frac{\partial}{\partial x_u} \eta^{(n)} \circ f^n \right\| \leq K_1 e^{-na}. \tag{2}$$

We also have:

$$\begin{aligned} \left\| \frac{\partial}{\partial x_u} A^{(n)} \right\| &= \left\| \frac{\partial}{\partial x_u} (A\ A \circ f \cdots A \circ f^{n-1}) \right\| \\ &\leq \sum_p \|A\| \cdots \left\| \frac{\partial}{\partial x_u} A \circ f^p \right\| \cdots \|A \circ f^{n-1}\| \\ &\leq e^{-(n-1)a} \sum_p \left\| \frac{\partial}{\partial x_u} A \circ f^p \right\| \\ &\leq K e^{-(n-1)a} \sum_p \left\| A^{(p)} \right\| . \end{aligned}$$

But $\sum_{i=1}^{n-1} \|A^{(p)}\| \leq \sum_{p=0}^{\infty} \|A\|^p \leq \sum_{p=0}^{\infty} e^{-pa} < \infty$, so

$$\left\| \frac{\partial}{\partial x_u} A^{(n)} \right\| \leq K_2 e^{-na}. \tag{3}$$

Also:

$$\begin{aligned} \left\| \frac{\partial}{\partial x_u} A^{(-n)} \circ f^n \right\| &= \left\| \frac{\partial}{\partial x_u} (A^{(-1)} \circ f^n \cdots A^{(-1)} \circ f) \right\| \\ &\leq \sum_{p=1}^{n} \left\| A^{(-1)} \circ f \right\| \cdots \left\| \frac{\partial}{\partial x_u} A^{(-1)} \circ f^p \right\| \cdots \left\| A^{(-1)} \circ f^n \right\| \\ &\leq L e^{(n-1)A} \sum_{p=1}^{n} \left\| A^{(p)} \right\| \\ &\leq L e^{(n-1)A} \sum_{p=1}^{\infty} e^{-pa}. \end{aligned}$$

So

$$\left\| \frac{\partial}{\partial x_u} A^{(-n)} \circ f^n \right\| \leq K_3 e^{nA}. \tag{4}$$

It follows from (3), (4) and (5) that

$$\left\| \frac{\partial}{\partial x_u} f^n \cdot \eta \right\| \leq \text{Const.}\, e^{-n(2a-A)},$$

which establishes the claim. □

Once the restriction of ∇ to individual leaves of $\mathcal{E}^{\pm}$ is differentiable, we may consider its curvature. Using Lemma 12 we conclude that this (tangential) curvature tensor vanishes.

To show that ∇ is complete along the leaves of the stable and unstable foliations, one just has to note the following. Since M is compact, there exists an open set $\mathcal{U} \subset TM$ containing all vectors v of norm $\|v\| \leq a$, for some positive constant a, where the exponential map of ∇ is defined. But as the connection is f-invariant, it follows that $\mathcal{U}$ contains the stable and unstable subbundles. With this, the proof of Proposition 8 is complete.

If ∇ is C^1-differentiable, the foliations $\mathcal{E}^+$ and $\mathcal{E}^-$ are of class C^2, and it makes sense to ask whether ∇ is projectable transversely to these foliations. If such is the case, it would follow immediately that ∇ is flat and the main theorem would hold under the optimal regularity assumption on the connection. The question of projectability of ∇ can be related to growth properties of the holonomy pseudogroup of the Anosov foliations. We make that precise below.

Lemma 16. *Assume that $f : M \to M$ is an Anosov diffeomorphism with $A < 2a$, whose Anosov foliations spread more slowly than power $l^{\frac{2a-A}{A}-\epsilon}$ for some $\epsilon > 0$. Then the f-invariant connection ∇ is projectable tranversely to the Anosov foliations. In particular, the curvature tensor of ∇ vanishes.*

Proof. Given a path γ as in the above definition, define $\alpha_\gamma = \nabla^{\mathcal{H}_\gamma} - \nabla$. Then

$$Tf^n\alpha_\gamma = \alpha_{f^n\circ\gamma}(Tf^n\cdot\ ,\ Tf^n\cdot).$$

Assume the connection is not projectable transversely to the stable foliation. Then there are vectors X and Y such that $\alpha_\gamma(X,Y) \neq 0$. It follows that

$$\begin{aligned} e^{-nA}\,\|\alpha_\gamma(X,Y)\| &\leq \|Tf^n\alpha_\gamma(X,Y)\| \\ &= \|\alpha_{f^n\circ\gamma}(Tf^nX, Tf^nY)\| \\ &\leq \|\alpha_{f^n\circ\gamma}\|\ \ \|Tf^nX\|\ \ \|Tf^nY\| \\ &\leq e^{-2na}\,\|\alpha_{f^n\circ\gamma}\|\ \ \|X\|\ \ \|Y\|\,. \end{aligned}$$

Therefore

$$\|\alpha_{f^n\circ\gamma}\| \geq Ke^{n(2a-A)}. \tag{5}$$

On the other hand

$$\|\alpha_{f^n\circ\gamma}\| \leq Cl(\gamma)^{\frac{2a-A}{A}-\epsilon} \leq C(e^{nA})^{\frac{2a-A}{A}-\epsilon} = Ce^{n(2a-A-\epsilon A)}.$$

The contradiction establishes that α_γ should vanish. □

3. Establishing that ∇ is locally symmetric

In this section we show:

Proposition 17. *Let f be a C^∞ Anosov diffeomorphism of a compact manifold M that preserves a C^r affine connection ∇, $r \geq 2$. Assume that f avoids 2 : 1-resonances. Denoting by $\tilde{M}$ the universal cover of M and $\tilde{E}^{\pm}$ the Anosov distributions pulled-back to $\tilde{M}$, then the group G of diffeomorphisms of $\tilde{M}$ that preserve $\tilde{E}^+$ and $\tilde{E}^-$ is a Lie group which acts transitively on $\tilde{M}$ and the action is C^r.*

The conclusion of the above proposition is the starting point of [**BL**], where Theorem 1 is proved. Since we are assuming the nonresonance condition, the infranilmanifods obtained in that theorem can only be flat. Therefore, once we establish the proposition, it will follow from [**BL**]:

Proposition 18. *Assume the same conditions of Proposition 17. Then f is conjugate via an affine C^r diffeomorphism to a hyperbolic automorphism of a complete flat manifold.*

We proceed as follows: In this section we show that under the conditions of the proposition, $\nabla R \equiv 0$. It then follows from classical results in differential geometry [**KN**] that the peudogroup of local affine diffeomorphisms preserving $E^{\pm}$ is a Lie pseudogroup that acts transitively on M. Due to [**BFL**, Lemma 3.4.4] (see also [**FK**, Proposition 3]), the invariant connection is complete. Therefore the local affine diffeomorphisms induce diffeomorphisms of $\tilde{M}$, establishing the proposition.

Lemma 19. *Assume τ is a f-invariant C^1 tensor field and ∇ an f-invariant affine connection where f is an Anosov diffeomorphism that avoids 2 : 1-resonances and has C^1-differentiable Anosov foliations. Also assume that f leaves invariant a smooth measure λ. Then $\nabla\tau \equiv 0$.*

Proof. Without loss of generality we assume that

$$\tau \in \Gamma^1(E^{\epsilon_1 *} \otimes \cdots \otimes E^{\epsilon_r *} \otimes E^{\delta_1} \otimes \cdots \otimes E^{\delta_s}).$$

Assume $\nabla\tau \neq 0$. Then, by ergodicity, there exists a set of full measure $\mathcal{A}$ where $\nabla\tau \neq 0$ and $\tau \neq 0$. For every point in $\mathcal{A}$ we can find

$$v_1 \otimes \cdots \otimes v_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_s \in E^{\epsilon_1 *}_{i_1}(x) \otimes \cdots \otimes E^{\epsilon_r *}_{i_r}(x) \otimes E^{\delta_1}_{j_1}(x) \otimes \cdots \otimes E^{\delta_s}_{j_s}(x)$$

and $v_0 \in E^{\epsilon_0}_{i_0}(x)$ such that $\langle \nabla_{v_0}\tau, v_1 \otimes \cdots \otimes \alpha_s \rangle \neq 0$. It follows from Lemma 12 that

$$\epsilon_0\chi_{i_0}(x) + \cdots + \epsilon_r\chi_{i_r}(x) - \delta_1\chi_{j_1}(x) - \cdots - \delta_s\chi_{j_s}(x) = 0. \tag{1}$$

Let $\tilde{v}_1, \cdots, \tilde{\alpha}_s$ be C^1 extensions of $v_1, \cdots, \alpha_s$ such that $\tilde{v}_i \in E^{\epsilon_i}, \cdots, \tilde{\alpha}_j \in E^{\delta_j *}$. We can write:

$$\begin{aligned}\langle \nabla_{v_0}\tau, v_1 \otimes \cdots \otimes \alpha_s\rangle &= v_0\langle \tau, \tilde{v}_1 \otimes \cdots \otimes \tilde{\alpha}_s\rangle \\ &\quad -\langle \tau, \nabla_{v_0}\tilde{v}_1 \otimes \cdots \otimes \alpha_s\rangle - \cdots - \langle \tau, v_1 \otimes \cdots \otimes \nabla_{v_0}\tilde{\alpha}_s\rangle.\end{aligned}$$

We claim that $\langle \tau, \nabla_{v_0}\tilde{v}_1 \otimes \cdots \otimes \alpha_s\rangle, \cdots, \langle \tau, v_1 \otimes \cdots \otimes \nabla_{v_0}\tilde{\alpha}_s\rangle$ vanish. Suppose that is not so, say:

$$\langle \tau, v_1 \otimes \cdots \otimes \nabla_{v_0}\tilde{v}_k \otimes \cdots \otimes \alpha_s\rangle \neq 0.$$

It follows, again by Lemma 12,

$$\begin{aligned}\epsilon_0\chi_{i_0}(x) + \quad &\cdots + \epsilon_r\chi_{i_r}(x) - \delta_1\chi_{j_1}(x) \\ - \quad &\cdots - \delta_s\chi_{j_s}(x) - \epsilon_0\chi_{i_0}(x) - \epsilon_k\chi_{i_k}(x) + \epsilon_k\chi'(x) = 0.\end{aligned}$$

Together with the previous equation (7) it yields

$$\epsilon_k\chi' - \epsilon_k\chi_{i_k} = \epsilon_0\chi_{i_0}.$$

Similarly, if

$$\langle \tau, v_1 \otimes \cdots \otimes \nabla_{v_0}\tilde{\alpha}_k \otimes \cdots \otimes \alpha_s\rangle \neq 0.$$

We obtain

$$-\delta_k\chi' + \delta_k\chi_{i_k} = \epsilon_0\chi_{i_0}.$$

Therefore, due to the nonresonance condition,

$$\langle \nabla_{v_0}\tau, v_1 \otimes \cdots \otimes \alpha_s\rangle = v_0\langle \tau, \tilde{v}_1 \otimes \cdots \otimes \tilde{\alpha}_s\rangle$$

at x for almost every $x \in M$ and arbitrary extensions of $v_1, \cdots, \alpha_s$.

Let $\rho : N = E^{\epsilon_1 *} \otimes \cdots \otimes E^{\delta_s} \to \mathbb{R}$ be the C^1 function defined by pairing with τ. Locally, $N \simeq M \times V$ where V is a vector space and $\rho : M \times V \to \mathbb{R}$ is linear in the second argument. If $X \in TN_{(x,\xi)} = TM_x \oplus TV_\xi$ we can write $X = \bar{v}_0 + \eta$. Let $X = \frac{d}{dt}|_{t=0}\tilde{\xi} \circ \gamma$ where $\gamma'(0) = v_0$ and $\tilde{\xi} = \tilde{v}_1 \otimes \cdots \otimes \tilde{\alpha}_s$. Note that X projects onto v_0. We have

$$\begin{aligned}T\rho_{(x,\xi)}(\bar{v}_0 + \eta) &= \partial_1\rho_{(x,\xi)}\bar{v}_0 + \partial_2\rho_{(x,\xi)}\eta \\ &= \partial_1\rho_{(x,\xi)}\bar{v}_0 + \rho(x,\eta) \ \in \mathbb{R}.\end{aligned}$$

But at x, $\rho(x,\cdot) \neq 0$. Therefore there exists $\eta \in V$ such that $T\rho_{(x,\xi)}(\bar{v}_0 + \eta) = 0$. This contradicts $\langle \nabla_{v_0}\tau, v_1 \otimes \cdots \otimes \alpha_s\rangle \neq 0$. Therefore the lemma.

□

4. Bootstrap of regularity in the symplectic case

In this section we show the following:

Proposition 20. *Let f be a C^∞ Anosov diffeomorphism of a compact C^∞ manifold M. Suppose that f leaves invariant a flat torsion-free C^r connection ∇ ($r \geq 1$). Also assume that the tangent distribution to the Anosov foliations are differentiable and parallel with respect to ∇. Then these distributions are C^{r+2}. If f is a symplectic Anosov diffeomorphism with C^2 foliations and the Kanai connection ∇ is flat, then the Anosov foliations and the connection are C^∞.*

Proof. Let M be a C^∞ manifold equipped with a C^r affine connection ∇, $r \geq 1$. Assume that ∇ is symmetric and flat, that is its curvature and torsion tensors vanish identically. For each $x \in M$ and any basis $\alpha = \{e_1, \cdots, e_n\}$ of TM_x consider the local parallel fields $X_1, \cdots, X_n$ that agree with $e_1, \cdots, e_n$ at x. In a C^∞ coordinate system we may write $X_i = \sum_j a_{ij} \frac{\partial}{\partial x_j}$. The equation describing parallel transport is a system of ordinary differential equations with C^r coefficients given by the Christoffel symbols. By the theorem on C^r dependence of solutions of O.D.E.s on parameters, X_i are C^r. In fact, as $\nabla X_i \equiv 0$, the coefficients a_{ij} of X_i should satisfy $\partial_k a_{ij} = -\sum_l a_{il}\Gamma^j_{kl}$ for all k, i, j. Since the right hand side is C^r, the functions a_{ij} must be C^{r+1}, so that the vector fields are C^{r+1}. As ∇ is symmetric ($T \equiv 0$), these vector fields commute, and so do their local flows $\phi_t^{X_1}, \cdots, \phi_t^{X_n}$. For $x \in M$, define

$$\Phi(t_1, \cdots, t_n) = \phi_{t_1}^{X_1} \circ \cdots \circ \phi_{t_n}^{X_n}(x),$$

for $t_1, \cdots, t_n$ sufficiently small. Φ defines a local C^{r+1} affine diffeomorphism between an open set $\mathcal{V} \subset \mathbb{R}^n$ containing 0 ($\mathbb{R}^n$ is equipped with the Euclidian affine structure) and an open set $\mathcal{U} \subset M$ such that $\Phi(0) = x$.

We claim that Φ (which is the exponential map of ∇ at x) is C^{r+2}. To see that, consider the locally defined pulled-back connection ∇^Φ on $\mathcal{V}$. With respect to the (normal) coordinate vector fields $\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}$, the Christoffel symbols of ∇^Φ vanish, so that

$$0 = \nabla^\Phi_{\frac{\partial}{\partial t_i}} \frac{\partial}{\partial t_j} = \Phi_*^{-1} \nabla_{\Phi_* \frac{\partial}{\partial t_i}} \Phi_* \frac{\partial}{\partial t_j}.$$

Denoting by Φ^l_i, Φ^l_{ij} the partial derivatives of the lth component of Φ, it follows:

$$\Phi^l_{ij} = -\sum_{k,r} \left(\Gamma^l_{kr} \circ \Phi\right) \Phi^k_i \Phi^r_j.$$

The right hand side of this equation is C^r, so that Φ is C^{r+2}.

If E^+ and E^- are two complementary subbundles parallel under ∇, then by choosing the basis $\{e_1, \cdots, e_n\}$ at x so that $\{e_1, \cdots, e_k\}$ spans $E^+(x)$ and $\{e_{k+1}, \cdots, e_n\}$ spans $E^-(x)$, we obtain that E^+ and E^- are locally the tangent bundles of coordinate planes associated to the C^{r+2} normal coordinates defined before. Therefore the foliations $\mathcal{E}^+$ and $\mathcal{E}^-$ that integrate E^+ and E^- are C^{r+2}.

If ∇ is the Kanai connection associated to a symplectic Anosov diffeomorphism, having C^{r+2} Anosov foliations implies that the connection is C^{r+1}. Note that we started with a C^r connection and concluded that it is C^{r+1}. Since r was arbitrary, this yields that ∇ is C^∞. Since $E^\pm$ are parallel, it also follows that the distributions are C^∞. Therefore, the conjugacy is also smooth. □

Theorems 2 and 4 follow now from Propositions 20, 18 and 8. Theorem 6 follows from Lemma 16 and Proposition 20.

Proof of Theorem 7. Under that regularity assumption, the curvature tensor R of the invariant connection ∇ is continuous (in fact, to carry out this proof it seems to be sufficient that the curvature be measurable. Such is the case if the foliations have measurable transversal second derivative). On the other hand, we can write $R = h\omega \otimes I$, where ω is the invariant volume form and h is a continuous f-invariant function. But f is ergodic, so h is constant and R must coincide with a constant multiple of the volume form. Therefore R is smooth and we can consider its covariant derivative ∇R. It follows from Lemma 12 that such tensor vanishes, so that by classifying the possible homogeneous structures that arise (as done in [**FK**]) one obtains that $R \equiv 0$. The result now follows from Proposition 20. □

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