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MULTIPLIERS AND BOURGAIN ALGEBRAS OF $H^\infty + C$ ON THE POLYDISK

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It is well-known that $H^\infty + C$ on the unit circle is a closed subalgebra of $L^\infty(T)$, and Rudin proved the $(H^\infty + C)(T^2)$ is a closed subspace of $L^\infty(T^2)$ but it is not an algebra. The multiplier algebra $\mathcal{M}$ of $(H^\infty+C)(T^2)$ is determined. Using this charaterization, we study Bourgain algebras of type $H^\infty+C$ on the torus T^2 and the polydisk U^2. Both Bourgain algebras of $H^\infty+C$ and $\mathcal{M}$ on the torus coincide with $\mathcal{M}$. We denote by $\tilde{\mathcal{M}}$ the space pf Poisson integral of functions in $\mathcal{M}$ and $C_{T^2}(\bar{U}^2)$ the space of continuous functions on $\bar{U}^2$ which vanish on T^2. It is proved that all higher Bourgain algebras of $H(U^2)+C(\bar{U}^2)$ and $H(U^2) + C_{T^2}(\bar{U}^2)$ are all distinct respectively, but every higher Bourgain algebra of $H(U^2) + C_0(U^2)$ coincides with $H(U^2) + C_0(U^2)$. It is also proved that all higher Bourgain algebras of $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}} + C_0(U^2)$ are all distinct respectively, but every higher Bourgain algebra of $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$ coincides with the first Bourgain algebra of $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$.

1. Introduction.

Let U^2 be the 2-dimensional unit polydisk and let T^2 be the torus. We denote by $H^\infty(U^2)$ the space of bounded holomorphic functions in U^2 and by $H^\infty(T^2)$ the space of radial limits of functions in $H^\infty(U^2)$. Then $H^\infty(T^2)$ is an essential supremum norm closed subalgebra of $L^\infty(T^2)$, the usual Lebesgue space with respect to $d\theta d\psi/(2\pi)^2$ (see [**12**]). Let denote by $C(X)$ the space of continuous functions on a topological space X. The algebra $A(T^2) = H^\infty(T^2) \cap C(T^2)$ or $A(\bar{U}^2) = H^\infty(U^2) \cap C(\bar{U}^2)$ is called the polydisk algebra, where $\bar{U}^2$ is the closed polydisk. In [**13**, Theorem 2.2], Rudin proved that $(H^\infty + C)(T^2) = H^\infty(T^2) + C(T^2) = \{f + g; f \in H^\infty(T^2), g \in C(T^2)\}$ is a closed subspace of $L^\infty(T^2)$ but it is not an algebra. On the unit circle T, it is well known that $(H^\infty + C)(T)$ is a closed subalgebra of $L^\infty(T)$ [**14**]. Let $\mathcal{M}$ be the space of multipliers of $(H^\infty + C)(T^2)$, that is,

$$\mathcal{M} = \{f \in L^\infty(T^2);\ f \cdot (H^\infty + C)(T^2) \subset (H^\infty + C)(T^2)\}.$$

Then $\mathcal{M}$ is a closed subalgebra of $L^\infty(T^2)$. Since constant functions are contained in $(H^\infty+C)(T^2)$, $\mathcal{M} \subset (H^\infty+C)(T^2)$. Let $C^\infty(U^2)$ be the space of bounded continuous functions in U^2 and let $C_0(U^2)$ be the space of functions in $C(\bar{U}^2)$ which vanish on the topological boundary ∂U^2 of U^2. We note that $\partial U^2 \neq T^2$. We denote by $C_{T^2}(\bar{U}^2)$ the space of functions in $C(\bar{U}^2)$ which vanish on the distinguished boundary T^2. Then $C_0(U^2) \subset C_{T^2}(\bar{U}^2) \subset C(\bar{U}^2)$. For a function f in $L^\infty(T^2)$, the Poisson integral is denoted by $\tilde{f}(z,w)$;

$$\tilde{f}(z,w) = \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta},e^{i\psi})P_z(e^{i\theta})P_w(e^{i\psi})\,d\theta d\psi/(2\pi)^2 \qquad \text{for } (z,w) \in U^2,$$

where $P_{re^{it}}(e^{i\theta}) = (1-r^2)/(1-2r\cos(t-\theta)+r^2)$. Then $\tilde{f} \in C^\infty(U^2)$.

In Theorem 2.1, we give a characterization of a function in $\mathcal{M}$, and we prove that $\mathcal{M}$ coincides with the space of functions f in $H^\infty(T^2)$ such that the Poisson integral $\tilde{f}$ can be extended continously on $\bar{U}^2 \setminus T^2$. By this fact, $\mathcal{M}$ becomes one of the interesting subalgebras between $A(T^2)$ and $H^\infty(T^2)$. The purpose of this paper is to investigate the multiplier algebra $\mathcal{M}$. For a closed subalgebra $\mathcal{A}$ with $A(T^2) \subset \mathcal{A} \subset H^\infty(T^2)$, $\mathcal{A} + C(T^2)$ is a closed subspace (see the proof of [**13**, Theorem 2.2]). In Theorem 2.2, we give a necessary and sufficient condition on $\mathcal{A}$ for which $\mathcal{A} + C(T^2)$ becomes an algebra.

In [**4**], Cima and Timoney introduced the concept of Bourgain algebras. Let $\mathcal{X}$ be a commutative Banach algebra with identity and let $\mathcal{Y}$ be a closed subspace of $\mathcal{X}$. A sequence $\{f_n\}_n$ in $\mathcal{Y}$ is called weakly null if $F(f_n) \to 0$ for every bounded linear functional F of $\mathcal{Y}$. For $f \in \mathcal{X}$, put $\|f+\mathcal{Y}\| = \inf\{\|f+g\|\,; g \in \mathcal{Y}\}$. Let

$$\mathcal{Y}_b = \{f \in \mathcal{X};\, \|ff_n + \mathcal{Y}\| \to 0 \ \text{ for every weakly null sequence} \{f_n\}_n \ \text{ in } \mathcal{Y}\}.$$

Cima and Timoney proved that $\mathcal{Y}_b$ is a closed subalgebra of $\mathcal{X}$, and they called $\mathcal{Y}_b$ the Bourgain algebra of $\mathcal{Y}$ relative to $\mathcal{X}$. If $\mathcal{Y}$ is an algebra, then $\mathcal{Y} \subset \mathcal{Y}_b$. We shall write $\mathcal{Y}_{bb}$ for $(\mathcal{Y}_b)_b$. We also write $\mathcal{Y}_{b(n)}$ for $(\mathcal{Y}_{b(n-1)})_b$ and $\mathcal{Y}_{b(0)} = \mathcal{Y}$. We call $\mathcal{Y}_{b(n)}$ the n-th Bourgain algebra of $\mathcal{Y}$.

In [**1**], Cima, Janson and Yale proved that $H^\infty(T)_b = (H^\infty + C)(T)$ relative to $L^\infty(T)$. In [**7**], Gorkin, the first author and Mortini studied the Bourgain algebras of closed algebras B between $H^\infty(T)$ and $L^\infty(T)$, and they proved that $(H^\infty + C)(T)_b = (H^\infty + C)(T)$ and $B_b = B_{bb}$ relative to $L^\infty(T)$. On the torus, we have $H^\infty(T^2)_b = H^\infty(T^2)$ relative to $L^\infty(T^2)$ (see [**9**, Corollary 1]). Since $(H^\infty + C)(T^2)$ is not an algebra, we can not expect that $(H^\infty + C)(T^2) \subset (H^\infty + C)(T^2)_b$. Since $\mathcal{M}$ is the multiplier algebra of $(H^\infty + C)(T^2)$, by the definition of Bourgain algebras we have $\mathcal{M} \subset (H^\infty + C)(T^2)_b$. We shall prove in Theorem 3.1 that

$$(H^\infty + C)(T^2)_b = \mathcal{M} = \mathcal{M}_b.$$

We note that the Bourgain algebra $(H^\infty + C)(\partial B_n)_b$ has not been determined yet, where B_n is the n-dimensional ball, $n \geq 2$ (see [**9**]). These are studies of Bourgain algebras on the boundaries.

When $f \in H^\infty(T^2)$, we have $\tilde{f} \in H^\infty(U^2)$ and f coincides with the radial

In [**2**], Cima, Stroethoff and Yale proved that $H^\infty(U)_b = H^\infty(U) + C(\bar{U}) = (H^\infty(T)_b)^\sim + C_0(U)$ relative to $C^\infty(U)$, where $H^\infty(T)_b$ is the Bourgain algebra relative to $L^\infty(T)$. It is proved in [**11**] that $H^\infty(U)_b = H^\infty(U)_{bb}$. In Section 4, we study Bourgain algebras relative to $C^\infty(U^2)$. We shall prove that $H^\infty(U^2)_b = (H^\infty + C_0)(U^2)_b = (H^\infty + C_0)(U^2)$. Hence $H^\infty(U^2)_b = H^\infty(U^2)_{bb}$. We also prove that

$$(\tilde{\mathcal{M}})_b \;=\; (H^\infty(U^2) + C(\bar{U}^2))_b \;=\; (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b \;=\; \tilde{\mathcal{M}} + C_0(U^2).$$

In Section 5, we study the n-th Bourgain algebras relative to $C^\infty(U^2)$. Since $H^\infty(U^2)_b = H^\infty(U^2)_{bb} = (H^\infty + C_0)(U^2)$, $H^\infty(U^2)_{b(n)} = (H^\infty + C_0)(U^2)_{b(n)} = H^\infty(U^2)_b$. But to our great surprise, the higher Bourgain algebras of $\tilde{\mathcal{M}}$ are all distinct. Actually we prove

$$(\tilde{\mathcal{M}} + C_0(U^2))_{b(n)} \;\neq\; (\tilde{\mathcal{M}} + C_0(U^2))_{b(n+1)} \qquad \text{for every } n \geq 0.$$

Spaces $\tilde{\mathcal{M}} + C_0(U^2)$ and $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$ are similar to each other in their forms, but we can prove that

$$(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)} \;=\; (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b \qquad \text{for every } n \geq 1.$$

In [**9**], the first author gave the conjecture that $\mathcal{A}_b = \mathcal{A}_{bb}$ for every closed subalgebra $\mathcal{A}$ with $A(T^2) \subset \mathcal{A} \subset H^\infty(T^2)$. This conjecture is still open, but $\tilde{\mathcal{M}}$ gives a counterexample for the same kind of problem on the polydisk.

In Section 6, we study Bourgain algebras of the polydisk algebra $A(\bar{U}^2)$ relative to $C^\infty(U^2)$. Some properties are similar to ones of $\tilde{\mathcal{M}}$, but some properties are different from $\tilde{\mathcal{M}}$. The Bourgain algebra of the disk algebra was studied in [**3**].

As a summary, we have the following about Bourgain algebras.

(a) $(H^\infty)_b = (H^\infty)_{bb}$ on the circle, disk, torus, and polydisk;

(b) $(H^\infty + C)_b = (H^\infty + C)_{bb}$ on the circle and the torus;

(c) $(H^\infty(U) + C(\bar{U}))_{b(n)} = (H^\infty(U) + C(\bar{U}))_b$, but $(H^\infty(U^2) + C(\bar{U}^2))_{b(n)} \neq (H^\infty(U^2) + C(\bar{U}^2))_{b(n+1)}$.

(d) $(H^\infty + C_0)(U^2)_b = (H^\infty + C_0)(U^2)$,but

$$(H^\infty(U^2) + C(\bar{U}^2))_b = (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b = \tilde{\mathcal{M}} + C_0(U^2).$$

(e) $\mathcal{M}_{b(n)} = \mathcal{M}$ on the torus, but $(\tilde{\mathcal{M}})_{b(n)} \neq (\tilde{\mathcal{M}})_{b(n+1)}$ on the polydisk.

(f) $(\tilde{\mathcal{M}} + C_0(U^2))_{b(n)} \neq (\tilde{\mathcal{M}} + C_0(U^2))_{b(n+1)}$, but $(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)} = (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$.

For the sake of simplicity, we discuss on T^2 and U^2. Using the same ideas of this paper, we can get the same results for T^n and $U^n, n > 2$. To determine Bourgain algebras of various spaces $\mathcal{X}$, the key is that if we want to prove $f \notin \mathcal{X}_b$, how to prove it. We need to choose a weakly null sequence $\{f_n\}_n$ in $\mathcal{X}$ such that $\|ff_n + \mathcal{X}\|$ does not converges to zero. How to choose

2. Multipliers of $(H^\infty + C)(T^2)$.

As mentioned in the introduction, $(H^\infty + C)(T^2)$ is a closed subspace of $L^\infty(T^2)$, but is not an algebra. In this section, we study the multiplier algebra $\mathcal{M}$ of $(H^\infty + C)(T^2)$. The following is a trivial fact.

Lemma 2.1. *Let $f \in H^\infty(T^2)$. Then $f \in \mathcal{M}$ if and only if $f \cdot C(T^2) \subset (H^\infty + C)(T^2)$.*

For a function f in $L^\infty(T)$, let $\hat{f}(n) = \int_0^{2\pi} f(e^{it})e^{-int}\, dt/2\pi$, the n-th Fourier coefficient. Also for $f \in L^\infty(T^2)$, let

$$\hat{f}(n,m) = \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta}, e^{i\psi})e^{-i(n\theta+m\psi)}\, d\theta d\psi/(2\pi)^2 \qquad \text{for } (n,m) \in Z \times Z,$$

where Z is the set of integers. Then $f \in H^\infty(T^2)$ if and only if $\hat{f}(n,m) = 0$ for every $(n,m) \notin Z_+ \times Z_+$, where $Z_+ = \{0,1,2,\dots\}$. Let $f \in L^\infty(T^2)$ and $k \in Z$. Then for almost every point $e^{i\theta} \in T$, there exists

$$I_k(f)(e^{i\theta}) = \int_0^{2\pi} f(e^{i\theta}, e^{i\psi})e^{-ik\psi}\, d\psi/2\pi,$$

and $I_k(f) \in L^\infty(T)$. We note that $(I_k(f))\hat{}(n) = \hat{f}(n,k)$. By the same way, let

$$J_k(f)(e^{i\psi}) = \int_0^{2\pi} f(e^{i\theta}, e^{i\psi})e^{-ik\theta}\, d\theta/2\pi \quad \text{a.e. } e^{i\psi} \in T.$$

Then $J_k(f) \in L^\infty(T)$ and $(J_k(f))\hat{}(n) = \hat{f}(k,n)$. For a function $f \in L^\infty(T^2)$, $\|f\|_{T^2}$ means the essential supremum norm on T^2. For functions $f(e^{i\theta})$, $g(e^{i\psi})$ in $L^\infty(T)$, we may identify these functions with $f(e^{i\theta}, e^{i\psi}) = f(e^{i\theta})$ and $g(e^{i\theta}, e^{i\psi}) = g(e^{i\psi})$, respectively. By these identifications, we may consider $L^\infty(T)$ as a subalgebra of $L^\infty(T^2)$. The following lemmas follow from the above definitions.

Lemma 2.2. *For $f \in L^\infty(T^2)$, $\|I_k(f)\|_T \le \|f\|_{T^2}$ and $\|J_k(f)\|_T \le \|f\|_{T^2}$.*

Lemma 2.3. *Let $f \in L^\infty(T^2)$.*
(i) *If $h(e^{i\theta}) \in L^\infty(T)$, then $I_k(h(e^{i\theta})f) = h(e^{i\theta})I_k(f)$ a.e. on T.*
(ii) *If $g(e^{i\psi}) \in L^\infty(T)$, then $J_k(g(e^{i\psi})f) = g(e^{i\psi})J_k(f)$ a.e. on T.*

For continuous functions on T^2, we have

Lemma 2.4. *If $f \in C(T^2)$, then $I_k(f)$ and $J_k(f)$ are contained in $C(T)$.*

Proof. Let $e^{i\theta_n} \to e^{i\theta_0}$ in T. Then $f(e^{i\theta_n}, e^{i\psi}) \to f(e^{i\theta_0}, e^{i\psi})$ uniformly in $e^{i\psi}$ as $n \to \infty$. Hence

$$I_k(f)(e^{i\theta_n}) = \int_0^{2\pi} f(e^{i\theta_n}, e^{i\psi})e^{-ik\psi}\, d\psi/2\pi$$

This implies $I_k(f) \in C(T)$. By the same way, $J_k(f) \in C(T)$. $\square$

Lemma 2.5. *Let $f \in L^\infty(T^2)$. Then $f \in H^\infty(T^2)$ if and only if $I_k(f) = J_k(f) = 0$ a.e. on T for every $k < 0$.*

Proof. We have $\hat{f}(n,m) = (I_m(f))\hat{\ }(n) = (J_n(f))\hat{\ }(m)$. Then $f \in H^\infty(T^2)$ if and only if $(I_k(f))\hat{\ }(n) = (J_k(f))\hat{\ }(n) = 0$ for every $k < 0$ and $n \in Z$. This is equivalent to $I_k(f) = J_k(f) = 0$ a.e. on T for every $k < 0$. $\square$

Let $f \in H^\infty(T^2)$. Then the Poisson integral $\tilde{f}$ is analytic in U^2 and can be represented by the Taylor series as

$$\tilde{f}(z,w) = \sum_{n,k=0}^{\infty} \hat{f}(n,k)\, z^n w^k \qquad \text{for } (z,w) \in U^2.$$

We can rewrite $\tilde{f}$ as the following forms;

$$(A_1) \qquad \tilde{f}(z,w) = \sum_{n=0}^{\infty} f_n(z) w^n = \sum_{n=0}^{\infty} g_n(w) z^n \qquad \text{for } (z,w) \in U^2,$$

where $f_n(z) = \sum_{k=0}^{\infty} \hat{f}(k,n) z^k$ and $g_n(w) = \sum_{k=0}^{\infty} \hat{f}(n,k) w^k$. Then we have

Lemma 2.6. *Let $f \in H^\infty(T^2)$. Then for $n \geq 0$,*
(i) *$f_n, g_n \in H^\infty(U)$, hence there exist boundary functions $f_n(e^{i\theta})$ and $g_n(e^{i\psi})$ in $H^\infty(T)$.*
(ii) *$f_n(e^{i\theta}) = I_n(f)(e^{i\theta})$ and $g_n(e^{i\psi}) = J_n(f)(e^{i\psi})$ a.e. on T.*

Therefore for $f \in H^\infty(T^2)$, we can write f as the following forms for every $k \geq 0$;

$$(A_2) \quad f(e^{i\theta}, e^{i\psi}) = \sum_{n=0}^{k} I_n(f)(e^{i\theta}) e^{in\psi} + e^{i(k+1)\psi}\, F(e^{i\theta}, e^{i\psi}) \quad \text{a.e. on } T^2;$$

$$(A_2) \quad f(e^{i\theta}, e^{i\psi}) = \sum_{n=0}^{k} J_n(f)(e^{i\psi}) e^{in\theta} + e^{i(k+1)\theta}\, G(e^{i\theta}, e^{i\psi}) \quad \text{a.e. on } T^2$$

for some $F, G \in H^\infty(T^2)$. Also we can write f as follows;

$$(A_3) \qquad f(e^{i\theta}, e^{i\psi}) = F_n(e^{i\theta}, e^{i\psi}) + e^{in(\theta+\psi)}\, G_n(e^{i\theta}, e^{i\psi}) \quad \text{a.e. on } T^2,$$

where $F_n, G_n \in H^\infty(T^2)$ and $\hat{F}_n(i,j) = 0$ for $i, j \geq n \geq 0$. We note that

$$F_n = \sum_{k=0}^{n-1} I_k(f)(e^{i\theta}) e^{ik\psi} + \sum_{k=0}^{n-1} J_k(f)(e^{i\psi}) e^{ik\theta} - \sum_{0 \leq s,t < n} \hat{f}(s,t) e^{i(s\theta + t\psi)}.$$

The following lemma is proved in [**13**, p. 105].

Lemma 2.7. *Let* $f(e^{i\theta}) \in H^\infty(T) \setminus A(T)$. *Then* $f(e^{i\theta})e^{-i\psi} \notin (H^\infty + C)(T^2)$.

Now we have the following characterization of $\mathcal{M}$.

Theorem 2.1. $\mathcal{M} = \{f \in H^\infty(T^2);\ I_k(f), J_k(f) \in A(T)$ *for every* $k \geq 0\}$.

Proof. First we shall prove that $\mathcal{M} \subset H^\infty(T^2)$. To prove this, let $f \in L^\infty(T^2) \setminus H^\infty(T^2)$. By Lemma 2.5, $I_k(f) \neq 0$ or $J_k(f) \neq 0$ for some $k < 0$. We assume that $I_k(f) \neq 0$. It is not difficult to find a function $h(e^{i\theta}) \in H^\infty(T)$ such that $h(e^{i\theta})I_k(f) \notin C(T)$ (see Lemma 3.5). By Lemmas 2.2–2.5, we have

$$\left\|h(e^{i\theta})f + (H^\infty + C)(T^2)\right\|_{T^2} \geq \left\|h(e^{i\theta})I_k(f) + C(T)\right\|_T > 0.$$

Hence $f \notin \mathcal{M}$, so that we have $\mathcal{M} \subset H^\infty(T^2)$.

Next we prove that if $f \in \mathcal{M}$ then $I_k(f), J_k(f) \in A(T)$ for $k \geq 0$. To prove this, suppose not. Then we may assume that

$$(1) \qquad I_k(f) \notin A(T) \qquad \text{for some } k \geq 0.$$

Moreover we may assume that

$$(2) \qquad I_j(f) \in A(T) \qquad \text{for } 0 \leq j < k.$$

By the first paragraph, we have $f \in H^\infty(T^2)$. We can write f as the form (A_2)

$$f(e^{i\theta}, e^{i\psi}) = \sum_{j=0}^{k} I_j(f)(e^{i\theta})e^{ij\psi} + e^{i(k+1)\psi} F(e^{i\theta}, e^{i\psi}) \quad \text{a.e. on } T^2,$$

where $F \in H^\infty(T^2)$. Then

$$(3) \quad f(e^{i\theta}, e^{i\psi})e^{-i(k+1)\psi} = I_k(f)(e^{i\theta})e^{-i\psi} + \sum_{j=0}^{k-1} I_j(f)(e^{i\theta})e^{i(j-k-1)\psi} + F(e^{i\theta}, e^{i\psi}) \quad \text{a.e. on } T^2.$$

Since $I_k(f) \in H^\infty(T)$, by (1) we have $I_k(f)(e^{i\theta}) \in H^\infty(T) \setminus A(T)$. Hence by Lemma 2.7, $I_k(f)(e^{i\theta})e^{-i\psi} \notin (H^\infty + C)(T^2)$. By (2), the second term in the right hand side of (3) is contained in $C(T^2)$. Therefore $fe^{-i(k+1)\psi} \notin (H^\infty + C)(T^2)$. Hence $f \notin \mathcal{M}$. This is a desired contradiction.

Now we prove that if $f \in H^\infty(T^2)$ and $I_k(f), J_k(f) \in A(T)$ for every $k \geq 0$, then $f \in \mathcal{M}$. To prove $f \in \mathcal{M}$, by Lemma 2.1 it is sufficient to show that $fc \in (H^\infty + C)(T^2)$ for every $c \in C(T^2)$. Let $f = F_n + e^{in(\theta+\psi)}G_n$ be the form given by (A_3) for $n \geq 0$. By our assumption, $F_n \in A(T^2)$ and $G_n \in H^\infty(T^2)$. For $c \in C(T^2)$, there is a sequence $\{c_n\}_n$ in $C(T^2)$ such that

$$(4) \qquad c_n = \sum_{k,j=-n}^{n} a_{n,k,j}\, e^{i(k\theta+j\psi)} \quad \text{and} \quad \|c - c_n\|_{T^2} \to 0 \ (n \to \infty).$$

Then

$$(5) \qquad F_n c_n \in C(T^2) \quad \text{and} \quad G_n c_n e^{in(\theta+\psi)} \in H^\infty(T^2).$$

Now we have

$$\begin{aligned} \|fc + (H^\infty + C)(T^2)\|_{T^2} &= \|f(c - c_n) + fc_n + (H^\infty + C)(T^2)\|_{T^2} \\ &\leq \|f(c - c_n)\|_{T^2} + \left\|(F_n + e^{in(\theta+\psi)}\, G_n)c_n + (H^\infty + C)(T^2)\right\|_{T^2} \\ &= \|f(c - c_n)\|_{T^2} \qquad \text{by (5)} \\ &\to 0 \quad (n \to \infty) \qquad \text{by (4).} \end{aligned}$$

Therefore $fc \in (H^\infty + C)(T^2)$. This completes the proof. □

Corollary 2.1. *$\mathcal{M}$ is a proper subalgebra between $A(T^2)$ and $H^\infty(T^2)$.*

Proof. $A(T^2) \subset \mathcal{M}$ is a trivial fact. Let $f(e^{i\theta}) \in H^\infty(T) \setminus A(T)$. Then by Theorem 2.1, $f(e^{i\theta}) \in H^\infty(T^2) \setminus \mathcal{M}$. Let $\tilde{f}(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$ be the power series of $\tilde{f}$ and let $g(z,w) = \sum_{n=0}^{\infty} a_n (zw)^n$ for $(z,w) \in U^2$. Then $g(e^{i\theta}, e^{i\psi}) \in H^\infty(T^2) \setminus A(T^2)$, and by Theorem 2.1 $g \in \mathcal{M}$. □

Corollary 2.2. *Let $f \in H^\infty(T^2)$. Then $f \in \mathcal{M}$ if and only if the Poisson integral $\tilde{f}$ can be extended continuously on $\bar{U}^2 \setminus T^2$.*

Proof. Let $f \in H^\infty(T^2)$. By (A_1) and Lemma 2.6,

$$(6) \qquad \tilde{f}(z,w) = \sum_{n=0}^{\infty} \tilde{I}_n(f)(z) w^n \qquad \text{for } (z,w) \in U^2,$$

where $\tilde{I}_n(f)$ is the Poisson integral of $I_n(f) \in H^\infty(T)$.

First, suppose that $f \in \mathcal{M}$. By Theorem 2.1, $I_n(f) \in A(T)$. Since $\|I_n(f)\|_T \leq \|f\|_{T^2}$ by Lemma 2.2, $\tilde{f}(z,w)$ is continuous on $\bar{U} \times \{w; |w| \leq r\}$ for every $0 < r < 1$. Hence $\tilde{f}$ can be extended continuously on $\bar{U} \times U$. By the same way as above, $\tilde{f}$ can be extended continuously on $U \times \bar{U}$. Since $\bar{U}^2 \setminus T^2 = (\bar{U} \times U) \cup (U \times \bar{U})$, $\tilde{f}$ can be extended continuously on $\bar{U}^2 \setminus T^2$.

Next, suppose that $\tilde{f}$ is continuous on $\bar{U}^2 \setminus T^2$. By (6), $\tilde{I}_0(f)(z) = \tilde{f}(z,0) \in C(\bar{U})$, hence $I_0(f) \in A(T)$. Since $(\tilde{f}(z,w) - \tilde{I}_0(f)(z))/w$ is also continuous on $\bar{U}^2 \setminus T^2, I_1(f) \in A(T)$. By induction, we can prove $I_n(f) \in A(T)$ for $n \geq 0$. By the same way, we have $J_n(f) \in A(T)$ for $n \geq 0$. By Theorem 2.1, $f \in \mathcal{M}$. □

Let $\mathcal{A}$ be a closed subalgebra with $A(T^2) \subset \mathcal{A} \subset H^\infty(T^2)$. $\mathcal{A}$ is called to be $*$-invariant if $f = F_n + e^{in(\theta+\psi)}\, G_n \in \mathcal{A}$ in the form (A_3), then G_n is contained in $\mathcal{A}$ for every $n \geq 0$ (see [**10**, **14**] for the case of the unit circle T). By Theorem 2.1, we have the following corollary.

Corollary 2.3.
(i) *Let $f \in H^\infty(T^2)$ with $f = F_n + e^{in(\theta+\psi)}\, G_n$ in the form (A_3). Then $f \in \mathcal{M}$ if and only if $F_n \in A(T^2)$ and $G_n \in \mathcal{M}$ for every $n \geq 0$.*
(ii) *$\mathcal{M}$ is $*$-invariant.*

By the proof of [**13**, Theorem 2.2], $\mathcal{A} + C(T^2)$ is a closed subspace of $L^\infty(T^2)$. Now we can characterize $\mathcal{A}$ for which $\mathcal{A} + C(T^2)$ becomes an algebra.

Theorem 2.2. *Let $\mathcal{A}$ be a closed algebra with $A(T^2) \subset \mathcal{A} \subset H^\infty(T^2)$. Then $\mathcal{A} + C(T^2)$ is a closed subalgebra of $L^\infty(T^2)$ if and only if $\mathcal{A} \subset \mathcal{M}$ and $\mathcal{A}$ is $*$-invariant.*

Proof. We already know that $\mathcal{A} + C(T^2)$ is a closed subspace.Hence we need to prove that $\mathcal{A} \cdot C(T^2) \subset \mathcal{A} + C(T^2)$ if and only if $\mathcal{A} \subset \mathcal{M}$ and $\mathcal{A}$ is $*$-invariant.

First, suppose that $\mathcal{A}C(T^2) \subset \mathcal{A} + C(T^2)$. By Lemma 2.1, we have $\mathcal{A} \subset \mathcal{M}$. To prove that $\mathcal{A}$ is $*$- invariant, let $f \in \mathcal{A}$. Since $\mathcal{A} \subset \mathcal{M}$, by Corollary 2.3 (i) we can write as $f = F_n + e^{in(\theta+\psi)}\, G_n, F_n \in A(T^2)$ and $G_n \in \mathcal{M}$ for every $n \geq 0$. Since $A(T^2) \subset \mathcal{A}, f - F_n \in \mathcal{A}$ and

$$G_n = (f - F_n)e^{-in(\theta+\psi)} \in (\mathcal{A} \cdot C(T^2)) \cap H^\infty(T^2) \subset \mathcal{A} + A(T^2) = \mathcal{A}.$$

Hence $\mathcal{A}$ is $*$-invariant.

Next, suppose that $\mathcal{A} \subset \mathcal{M}$ and $\mathcal{A}$ is $*$- invariant. Let $f \in \mathcal{A}$ with $f = F_n + e^{in(\theta+\psi)}\, G_n$ in the form (A_3), and $c \in C(T^2)$. Since $\mathcal{A}$ is $*$-invariant, $G_n \in \mathcal{A}$. Since $f \in \mathcal{M}$, by Corollary 2.3 (i) we have $F_n \in A(T^2)$. As the proof of Theorem 2.1,

$$\|c - c_n\|_{T^2} \to 0 \quad \text{for some } c_n = \sum_{k,j=-n}^{n} a_{n,k,j}\, e^{i(k\theta + j\psi)}.$$

Then $fc_n \in \mathcal{A} + C(T^2)$, and we have

$$\begin{aligned} \|fc + \mathcal{A} + C(T^2)\|_{T^2} &= \|f(c - c_n) + fc_n + \mathcal{A} + C(T^2)\|_{T^2} \\ &\leq \|f(c - c_n)\|_{T^2} \to 0 \ (n \to \infty). \end{aligned}$$

Hence $fc \in \mathcal{A} + C(T^2)$. $\square$

By Theorem 2.2 and Corollary 2.3 (ii), we have

Corollary 2.4. *$\mathcal{M} + C(T^2)$ is a closed subalgebra of $L^\infty(T^2)$.*

3. Bourgain algebras on the torus.

In this section, we study the Bourgain algebras relative to $L^\infty(T^2)$. We shall prove the following theorem.

Theorem 3.1. $\quad (H^\infty + C)(T^2)_b \;=\; (H^\infty + C)(T^2)_{bb} \;=\; \mathcal{M}.$

The following corollary follows the above theorem.

Corollary 3.1.
(i) $\mathcal{M}_{b(n)} \;=\; \mathcal{M}$ *for every* $n \geq 0$.
(ii) $(H^\infty + C)(T^2)_{b(n)} \;=\; \mathcal{M}$ *for every* $n \geq 1$.

To prove our theorem, we need some preliminary observations. For $f \in L^\infty(T^2)$, we have $\left|\tilde{f}(z,w)\right| \leq \|f\|_{T^2}$ and $\tilde{f}(z,w)$ is harmonic in $z \in U$ for each fixed $w \in U$. We denote by $\tilde{f}(e^{i\theta}, w)$ the boundary function of the function $\tilde{f}(z,w)$ in $z \in U$. Then

$$\tilde{f}(e^{i\theta}, w) \;\in\; L^\infty(T), \; \left\|\tilde{f}(e^{i\theta}, w)\right\|_T \;\leq\; \|f\|_{T^2}$$
$$\text{and} \;\; \tilde{f}(z,w) \;=\; \int_0^{2\pi} \tilde{f}(e^{i\theta}, w) P_z(e^{i\theta})\, d\theta/2\pi.$$

By the same way, we have $\tilde{f}(z, e^{i\psi}) \in L^\infty(T)$ for each fixed $z \in U$.

Lemma 3.1. *Let (z,w) be a point in U^2.*
(i) *If $f \in H^\infty(T^2)$, then $\tilde{f}(e^{i\theta}, w)$, $\tilde{f}(z, e^{i\psi}) \in H^\infty(T)$.*
(ii) *If $f \in C(T^2)$, then $\tilde{f}(e^{i\theta}, w)$, $\tilde{f}(z, e^{i\psi}) \in C(T)$.*
(iii) *If $f \in \mathcal{M}$, then $\tilde{f}(e^{i\theta}, w)$, $\tilde{f}(z, e^{i\psi}) \in A(T)$.*

Proof. (i) and (ii) follow from [**12**, p. 18]. We note that if $f \in C(T^2)$ then $\tilde{f}$ can be extended continuously to $\bar{U}^2$. (iii) follows from Corollary 2.2. $\square$

Let $M(H^\infty(T))$ be the space of nonzero mulitiplicative linear functionals of $H^\infty(T)$. With the weak*-topology, $M(H^\infty(T))$ becomes a compact Hausdorff space and is called the maximal ideal space of $H^\infty(T)$. It is well known that the open unit disk U is a dense open subset of $M(H^\infty(T))$. We identify a function f in $H^\infty(T)$ with its Gelfand transform. Then $H^\infty(T)$ becomes a sup-norm closed subalgebra of $C(M(H^\infty(T)))$, the space of continuous functions on $M(H^\infty(T))$ (see [**8**]). For $\lambda \in T$ and $f \in H^\infty(T)$, let

$$\begin{aligned} M_\lambda(H^\infty(T)) &= \{x \in M(H^\infty(T)); z(x) = \lambda\}; \\ \omega_0(f,\lambda) &= \sup\{|f(\zeta_1) - f(\zeta_2)|; \zeta_1, \zeta_2 \in M_\lambda(H^\infty(T))\}. \end{aligned}$$

By the same way, we can define $M(L^\infty(T))$ and $M_\lambda(L^\infty(T))$. We can consider that $M_\lambda(L^\infty(T)) \subset M_\lambda(H^\infty(T))$. Since every continuous function on T is constant on each $M_\lambda(H^\infty(T))$, for a non-continuous function f in $H^\infty(T)$ we have (for example, see [**9**, Lemma 1])

$$\text{(1)} \quad \sup\{\omega_0(f,\lambda); \lambda \in T\}/2 \le \|f + C(T)\|_T < \sup\{\omega_0(f,\lambda); \lambda \in T\}.$$

Here we see the right hand side strict inequality briefly. Since $f \notin C(T)$, we have $\|f + C(T)\|_T \neq 0$. Since $\|f + C(T)\|_T$ is the quotient norm of $f \in H^\infty(T) \subset L^\infty(T)$ in $L^\infty(T)/C(T)$, there exists an extreme point μ in the unit ball of $(L^\infty(T)/C(T))^*$ such that $\|f + C(T)\|_T = |\mu(f)|$. We can identify μ with the Borel measure on $M(L^\infty(T))$ such that $\int_{M(L^\infty(T))} g\, d\mu = 0$ for every $g \in C(T)$. Since μ is an extreme point, there exists $\lambda \in T$ such that $M_\lambda(L^\infty(T))$ contains the support set of μ. Then

$$\|f + C(T)\|_T = \left|\int_{M_\lambda(L^\infty(T))} f\, d\mu\right| = \left|\int_{M_\lambda(L^\infty(T))} f + c\, d\mu\right|$$

for every constant c. Taking c_0 such that $\|f + c_0\|_{M_\lambda(L^\infty(T))}$ is the smallest, then by the definition of $\omega_0(f,\lambda)$ and $\omega_0(f,\lambda) \neq 0$ we have

$$\|f + c_0\|_{M_\lambda(L^\infty(T))} < \omega_0(f,\lambda).$$

Therefore we get $\|f + C(T)\|_T < \omega_0(f,\lambda)$.

The following lemma is a key to prove Theorem 3.1.

Lemma 3.2. *Suppose that $f \in H^\infty(T^2)$ and $I_k(f) \notin A(T)$ for some $k \ge 0$. Then for each $0 < r < 1$, we have*

$$\sup_{|w|=r} \left\|\tilde{f}(e^{i\theta}, w) + C(T)\right\|_T > r^k \|I_k(f) + C(T)\|_T / 2 \neq 0.$$

Proof. By Lemma 2.6, $I_k(f) \in H^\infty(T)$. Since $I_k(f) \notin A(T)$, $I_k(f) \notin C(T)$ and $\|I_k(f) + C(T)\|_T \neq 0$. Let

$$\Gamma = M(H^\infty(T)) \times \{w;\ |w| \le r\}.$$

Since $I_j(f) \in H^\infty(T)$ for $j \geq 0$, we consider that $I_j(f)$ is a continuous function on $M(H^\infty(T))$. For each $\zeta \in M(H^\infty(T))$ and $w \in U$, we have

$$\left|\sum_{j=n}^{\infty} I_j(f)(\zeta)w^j\right| \leq \|f\|_{T^2} \sum_{j=n}^{\infty} |w|^j \quad \text{by Lemma 2.2}$$
$$\to 0 \ (n \to \infty).$$

This means that the function

$$(2) \qquad F(\zeta, w) = \sum_{j=0}^{\infty} I_j(f)(\zeta)w^j, \quad \zeta \in M(H^\infty(T)) \text{ and } w \in U$$

converges uniformly on Γ. Hence $F(\zeta, w)$ is a continuous function on Γ, and for each fixed ζ in $M(H^\infty(T))$

$$(3) \qquad F(\zeta, w) \text{ is analytic in } w \text{ on } U.$$

We note that $F(z, w) = \tilde{f}(z, w)$ on U^2. Hence for each fixed w_0 with $|w_0| = r$,

$$(4) \quad F(\zeta, w_0),\ \zeta \in M(H^\infty(T)), \text{ is the Gelfand transform of } \tilde{f}(e^{i\theta}, w_0).$$

By our starting assumption, $I_k(f) \notin A(T)$. Hence by (1) there exist points ζ_1 and ζ_2 in $M_\lambda(H^\infty(T))$ for some $\lambda \in T$ such that

$$(5) \qquad |I_k(f)(\zeta_1) - I_k(f)(\zeta_2)| > \|I_k(f) + C(T)\|_T.$$

Recall the elementary fact that for an analytic function $\sum_{n=0}^{\infty} a_n w^n$ on U,

$$(6) \qquad r^k |a_k| \leq \sup_{|w|=r} \left|\sum_{n=0}^{\infty} a_n w^n\right|.$$

Then we have

$$\sup_{|w|=r} |F(\zeta_1, w) - F(\zeta_2, w)| = \sup_{|w|=r} \left|\sum_{j=0}^{\infty} (I_j(f)(\zeta_1) - I_j(f)(\zeta_2))w^j\right| \quad \text{by (2)}$$
$$\geq r^k |I_k(f)(\zeta_1) - I_k(f)(\zeta_2)| \quad \text{by (3) and (6)}$$
$$> r^k \|I_k(f) + C(T)\|_T \quad \text{by (5)}.$$

Hence there exists w_0 with $|w_0| = r$ such that

$$|F(\zeta_1, w_0) - F(\zeta_2, w_0)| > r^k \|I_k(f) + C(T)\|_T.$$

Therefore by (4) we have

$$\omega_0(\tilde{f}(e^{i\theta}, w_0), \lambda) > r^k \|I_k(f) + C(T)\|_T.$$

Hence by (1),

$$\left\|\tilde{f}(e^{i\theta}, w_0) + C(T)\right\|_T > r^k \left\|I_k(f) + C(T)\right\|_T /2.$$

This completes the proof. □

The following is a characterization of a weakly null sequence in the space of continuous functions, and it is a direct corollary of the Lebesgue dominated convergence theorem.

Lemma 3.3. *Let B be a closed subspace of $C(\Omega)$, where Ω is a compact Hausdorff space. Let $\{f_n\}_n$ be a sequence in B. Then $\{f_n\}_n$ is weakly null in B if and only if $\{f_n\}_n$ is norm bounded and $f_n \to 0$ pointwise on Ω.*

The following lemma is given in [**6**] (see also [**9**]).

Lemma 3.4. *Let $\{z_n\}_n$ be a distinct sequence in $\bar{U}$ with $|z_n| \to 1$. Then there is a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ and there is a weakly null sequence $\{h_j\}_j$ in $A(T)$ such that $\tilde{h}_j(z_{n_j}) = 1$ for every j.*

By considering a sequence of peaking functions, we have the following lemma.

Lemma 3.5. *For a function f in $L^\infty(T)$ with $f \neq 0$, there exists a weakly null sequence $\{h_n\}_n$ in $H^\infty(T)$ such that $\|fh_n + C(T)\|_T$ does not converge to 0 as $n \to \infty$.*

Proof. We may consider that $L^\infty(T) = C(M(L^\infty(T)))$ by Gelfand transform. Since $f \neq 0$, there is a point $x_0 \in M_\lambda(L^\infty(T))$ for some $\lambda \in T$ such that $f(x_0) \neq 0$. Since one point set $\{x_0\}$ is not an open and closed subset of $M_\lambda(L^\infty(T))$ (see [**8**, p. 170]), there is a distinct sequence $\{x_n\}_n$ in $M_\lambda(L^\infty(T))$ such that $f(x_n) \to f(x_0)$. By considering a subsequence, we may assume that there is a sequence of disjoint open subsets $\{U_n\}_n$ of $M(L^\infty(T))$ with $x_n \in U_n$. Since x_n is a weak peak point for $H^\infty(T)$ (see [**8**, p. 207]), there is a sequence of functions $\{h_n\}_n$ in $H^\infty(T)$ such that $h_n(x_n) = 1, \|h_n\|_{M(L^\infty(T))} = 1$, and $\|h_n\|_{M(L^\infty(T))\setminus U_n} \to 0$ $(n \to \infty)$. Since $\{U_n\}_n$ is a sequence of disjoint subsets, by Lemma 3.3 $\{h_n\}_n$ is a weakly null sequence in $H^\infty(T)$. Then we have

$$\begin{aligned}
\|fh_n + C(T)\|_T &\geq \omega_0(fh_n, \lambda)/2 \qquad \text{by } (1) \\
&\geq |(fh_n)(x_n) - (fh_n)(x_{n+1})| /2 \\
&\geq (|f(x_n)| - |f(x_{n+1})| \, \|h_n\|_{M(L^\infty(T))\setminus U_n})/2 \\
&\to |f(x_0)| \neq 0.
\end{aligned}$$

□

The following is an elementary fact.

Lemma 3.6.

(i) *Let Ψ be a bounded linear map from a Banach space B_1 to a Banach space B_2. If $\{f_n\}_n$ is a weakly null sequence in B_1, then $\{\Psi(f_n)\}_n$ is weakly null in B_2.*

(ii) *Let B_1 and B_2 be Banach spaces with $B_1 \subset B_2$ and let $\{f_n\}_n \subset B_1$. Then $\{f_n\}_n$is weakly null in B_1 if and only if $\{f_n\}_n$ is weakly null in B_2.*

We use this lemma like as the following ways. Since $H^\infty(T) \subset H^\infty(T^2)$, a weakly null sequence in $H^\infty(T)$ is also weakly null in $H^\infty(T^2)$. Since the Poisson integral is a norm preserving linear map from $L^\infty(T^2)$ to $C^\infty(U^2)$, for a sequence $\{f_n\}_n$ in $L^\infty(T^2)$, $\{f_n\}_n$ is weakly null in $L^\infty(T^2)$ if and only if $\{\tilde{f}_n\}_n$ is weakly null in $C^\infty(U^2)$. Now we can prove Theorem 3.1.

Proof of Theorem 3.1. We devide the proof into two steps.

Step 1. We shall prove $\mathcal{M} = (H^\infty + C)(T^2)_b$. By the definition of the multiplier algebra $\mathcal{M}$ and Bourgain algebras, we have $\mathcal{M} \subset (H^\infty + C)(T^2)_b$. We prove $(H^\infty + C)(T^2)_b \subset \mathcal{M}$. Let $f \in (H^\infty + C)(T^2)_b$.

First we prove $f \in H^\infty(T^2)$. To prove this, suppose not. Then by Lemma 2.5, $I_k(f) \neq 0$ or $J_k(f) \neq 0$ for some $k < 0$. We assume that $I_k(f) \neq 0$ for some $k < 0$. By Lemma 3.5, there is a weakly null sequence $\{h_n(e^{i\theta})\}_n$ in $H^\infty(T)$ and there is $\delta > 0$ such that $\|I_k(f)h_n(e^{i\theta}) + C(T)\|_T > \delta$ for every n. By Lemma 3.6, $\{h_n(e^{i\theta})\}_n$ is weakly null in $(H^\infty + C)(T^2)$. By Lemmas 2.2–2.5, we have

$$\|fh_n(e^{i\theta}) + (H^\infty + C)(T^2)\|_{T^2} \geq \|I_k(f)h_n(e^{i\theta}) + C(T)\|_T > \delta \quad \text{for every } n.$$

Hence $f \notin (H^\infty + C)(T^2)_b$. This contradiction shows that $f \in H^\infty(T^2)$.

Next we prove $f \in \mathcal{M}$. To prove this, suppose not. By Theorem 2.1, we may assume that $I_k(f) \notin A(T)$ for some $k \geq 0$. By Lemma 3.2, there is a sequence $\{w_n\}_n$ in U such that $|w_n| \to 1, |w_n|^k > 2/3$, and

$$\left\|\tilde{f}(e^{i\theta}, w_n) + C(T)\right\|_T > \|I_k(f) + C(T)\|_T / 3 \neq 0 \qquad \text{for every } n. \tag{7}$$

By Lemma 3.4, by considering a subsequence of $\{w_n\}_n$ we may assume that there is a weakly null sequence $\{h_n(e^{i\psi})\}_n$ in $A(T)$ such that $\tilde{h}_n(w_n) = 1$ for every n. Let $b_n(w) = (w - w_n)/(1 - \bar{w}_n w)$ be the Blaschke factor, and let

$$g_n(e^{i\psi}) = b_n(e^{i\psi})^{-1}\, h_n(e^{i\psi}).$$

Then $g_n(e^{i\psi}) \in C(T^2)$, and by Lemma 3.6 $\{g_n\}_n$ is weakly null in $(H^\infty +$

$C)(T^2)$. Here for every $h \in H^\infty(T^2)$ and $c' \in C(T^2)$, put $c_n = c'\, b_n$, we have

$$\begin{aligned}
\|fg_n + h + c'\|_{T^2} &= \|fh_n(e^{i\psi}) + b_n(e^{i\psi})h + c_n\|_{T^2} \\
&\geq \sup_{|z|<1} \left|\tilde{f}(z, w_n)\tilde{h}_n(w_n) + b_n(w_n)\tilde{h}(z, w_n) + \tilde{c}_n(z, w_n)\right| \\
&= \sup_{|z|<1} \left|\tilde{f}(z, w_n) + \tilde{c}_n(z, w_n)\right| \\
&= \left\|\tilde{f}(e^{i\theta}, w_n) + \tilde{c}_n(e^{i\theta}, w_n)\right\|_T \\
&\geq \left\|\tilde{f}(e^{i\theta}, w_n) + C(T)\right\|_T \qquad \text{by Lemma 3.1 (ii)} \\
&> \|I_k(f) + C(T)\|_T/3 \neq 0 \qquad \text{by (7).}
\end{aligned}$$

Therefore $\|fg_n + (H^\infty + C)(T^2)\|_{T^2}$ does not converge to 0, hence $f \notin (H^\infty + C)(T^2)_b$. This contradiction shows $f \in \mathcal{M}$. Thus we get $(H^\infty + C)(T^2)_b = \mathcal{M}$.

Step 2. We prove $\mathcal{M}_b = \mathcal{M}$. Since $\mathcal{M}$ is an algebra, $\mathcal{M} \subset \mathcal{M}_b$. To prove $\mathcal{M}_b \subset \mathcal{M}$, let $f \in \mathcal{M}_b$. By [**9**, Theorem 4], we have $\mathcal{M}_b \subset H^\infty(T^2)$. Hence $f \in H^\infty(T^2)$. We shall prove $f \in \mathcal{M}$ by the same way as Step 1, but this case is more easier. To prove $f \in \mathcal{M}$, suppose not. By Theorem 2.1, we may assume that there is $k \geq 0$ such that $I_k(f) \notin A(T)$. By Lemma 3.2, there is a sequence $\{w_n\}_n$ in U such that $|w_n| \to 1$ and

$$\left\|\tilde{f}(e^{i\theta}, w_n) + C(T)\right\|_T > \|I_k(f) + C(T)\|_T/3 \neq 0. \tag{8}$$

By Lemma 3.4, we may assume that there is a weakly null sequence $\{h_n(e^{i\psi})\}_n$ in $A(T)$ such that $\tilde{h}_n(w_n) = 1$ for every n. By Lemma 3.6, $\{h_n(e^{i\psi})\}_n$ is weakly null in $\mathcal{M}$. Then for $h \in \mathcal{M}$ we have

$$\begin{aligned}
\|fh_n(e^{i\psi}) + h\|_{T^2} &\geq \sup_{|z|<1} \left|\tilde{f}(z, w_n)\tilde{h}_n(w_n) + \tilde{h}(z, w_n)\right| \\
&\geq \left\|\tilde{f}(e^{i\theta}, w_n) + A(T)\right\|_T \qquad \text{by Lemma 3.1 (iii)} \\
&\geq \left\|\tilde{f}(e^{i\theta}, w_n) + C(T)\right\|_T \\
&> \|I_k(f) + C(T)\|_T/3 \qquad \text{by (8).}
\end{aligned}$$

Hence $\|fh_n(e^{i\psi}) + \mathcal{M}\|_{T^2}$ does not converge to 0, and $f \notin \mathcal{M}_b$. This is a contradiction. Therefore $f \in \mathcal{M}$. This completes the proof. □

By Corollary 2.4, $\mathcal{M} + C(T^2)$ is a closed subalgebra of $L^\infty(T^2)$. We have the following problem.

Problem 3.1. Is $(\mathcal{M} + C(T^2))_b = \mathcal{M} + C(T^2)$ true ?

4. Bourgain algebras on the polydisk.

In this and next sections, we shall study the Bourgain algebras relative to $C^\infty(U^2)$. Recall the results on the open unit disk U. By [**2**, **11**], we have

$$H^\infty(U)_b = H^\infty(U)_{bb} = H^\infty(U) + C(\bar{U}) = (H^\infty(T)_b)^{\tilde{}} + C_0(U).$$

First we shall prove the following theorem of Bourgain algebras relative to $C^\infty(U^2)$.

Theorem 4.1. *Let $\mathcal{A}$ be the closed subspace such that $A(T^2) \subset \mathcal{A} \subset H^\infty(T^2)$. Then $(\tilde{\mathcal{A}})_b = (\mathcal{A}_b)^{\tilde{}} + C_0(U^2)$ relative to $C^\infty(U^2)$, where $\mathcal{A}_b$ is the Bourgain algebra relative to $L^\infty(T^2)$.*

Let $f \in C^\infty(U^2)$ and $\zeta \in T^2$. If f has the radial limit $\lim_{r\to 1} f(r\zeta)$, we denote it by $f^*(\zeta)$.

Lemma 4.1. *Let B be a closed subspace of $L^\infty(T^2)$. Let V be a closed subspace of $C^\infty(U^2)$ such that every f in V has the radial limit f^* and $f^* = 0$ a.e. on T^2. Then $\tilde{B} + V$ is a closed subspace of $C^\infty(U^2)$.*

Proof. Let $\{\tilde{f}_n + g_n\}_n$ be a Cauchy sequence in $\tilde{B} + V$. By considering the radial limits, $\{f_n\}_n$ is a Cauchy sequence in B. Hence $\{\tilde{f}_n\}_n$ and $\{g_n\}_n$ are Cauchy sequences in $C^\infty(U^2)$. Hence we have our assertion. □

We use the following lemma frequently.

Lemma 4.2 (see the proof of [2, Theorem 8]). *Let B be a closed subspace with $A(T^2) \subset B \subset L^\infty(T^2)$. Let V be a closed subspace of $C^\infty(U^2)$ such that every f in V has the radial limit f^* and $f^* = 0$ a.e. on T^2. If $g \in (\tilde{B} + V)_b$ relative to $C^\infty(U^2)$, then there exists the radial limit g^* a.e. on T^2, and $g^* \in B_b$ relative to $L^\infty(T^2)$.*

Proposition 4.1. *Let B be a closed subspace such that $A(T^2) \subset B \subset L^\infty(T^2)$ relative to $C^\infty(U^2)$. Let $f \in (\tilde{B})_b$. If $f^* = 0$ a.e. on T^2, then $f \in C_0(U^2)$.*

Proof. To prove $f \in C_0(U^2)$, suppose not. Then there is $\delta > 0$ and there exists a sequence $\{(z_n, w_n)\}_n$ in U^2 such that (z_n, w_n) converges to a point in ∂U^2 and

$$(1) \qquad |f(z_n, w_n)| > \delta \qquad \text{for every } n.$$

We may assume that $|z_n| \to 1$. By lemma 3.4 and considering a subsequence of $\{z_n\}_n$, we may assume that there is a weakly null sequence $\{p_n(e^{i\theta})\}_n$ in $A(T)$ such that

$$(2) \qquad \tilde{p}_n(z_n) = 1 \qquad forevery\ n.$$

Since $p_n(e^{i\theta}) \in A(T) \subset A(T^2) \subset B$, by Lemma 3.6 $\{\tilde{p}_n(z)\}_n$ is weakly null in $\tilde{B}$. Since $f \in (\tilde{B})_b$, there exists $\tilde{q}_n$ in $\tilde{B}$ such that

$$\|f\tilde{p}_n + \tilde{q}_n\|_{U^2} \to 0.$$

Since the radial limit of f vanishes a.e. on T^2, $\|\tilde{q}_n\|_{U^2} = \|q_n\|_{T^2} \to 0$. Hence

$$\begin{aligned}\limsup_{n\to\infty} \|f\tilde{p}_n + \tilde{q}_n\|_{U^2} &= \limsup_{n\to\infty} \|f\tilde{p}_n\|_{U^2} \\ &\geq \limsup_{n\to\infty} |f(z_n, w_n)| \qquad \text{by (2)} \\ &\geq \delta \qquad \text{by (1)}.\end{aligned}$$

This is a contradiction. Therefore $f \in C_0(U^2)$. □

Proof of Theorem 4.1. In [**9**, Theorem 4], the first author showed that $\mathcal{A}_b \subset H^\infty(T^2)$ for every closed subalgebra $\mathcal{A}$ with $A(T^2) \subset \mathcal{A} \subset H^\infty(T^2)$. Its proof works for a closed subspace $\mathcal{A}$. Hence we have $\mathcal{A}_b \subset H^\infty(T^2)$. Since a weakly null sequence in $\tilde{\mathcal{A}}$ is sup-norm bounded and converges to 0 uniformly on each compact subset of U^2, we have $C_0(U^2) \subset (\tilde{\mathcal{A}})_b$. To show $(\mathcal{A}_b)^\sim \subset (\tilde{\mathcal{A}})_b$, let $g \in \mathcal{A}_b$ and let $\{\tilde{g}_n\}_n$ be a weakly null sequence in $\tilde{\mathcal{A}}$. Then $\{g_n\}_n$ is weakly null in $\mathcal{A}$ and there exists h_n in $\mathcal{A}$ such that $\|gg_n + h_n\|_{T^2} \to 0$. Since $\mathcal{A}_b \subset H^\infty(T^2)$, $\left\|\tilde{g}\tilde{g}_n + \tilde{h}_n\right\|_{U^2} = \|gg_n + h_n\|_{T^2}$. Hence $\tilde{g} \in (\tilde{\mathcal{A}})_b$ and $(\mathcal{A}_b)^\sim + C_0(U^2) \subset (\tilde{\mathcal{A}})_b$.

To prove the converse inclusion, let $f \in (\tilde{\mathcal{A}})_b$. Then by Lemma 4.2 (in this case $W = \{0\}$), there exists the radial limit f^* of f and $f^* \in \mathcal{A}_b$. By the first paragraph, $(f^*)^\sim \in (\tilde{\mathcal{A}})_b$, hence $f - (f^*)^\sim \in (\tilde{\mathcal{A}})_b$. Since $f - (f^*)^\sim = 0$ a.e. on T^2, by Proposition 4.1 we have $f - (f^*)^\sim \in C_0(U^2)$. Therefore $f \in (\mathcal{A}_b)^\sim + C_0(U^2)$, so that $(\tilde{\mathcal{A}})_b \subset (\mathcal{A}_b)^\sim + C_0(U^2)$. □

Corollary 4.1.
(i) $(\tilde{\mathcal{M}})_b = \tilde{\mathcal{M}} + C_0(U^2)$.
(ii) $H^\infty(U^2)_b = (H^\infty + C_0)(U^2)$.

Proof. By Theorems 3.1 and 4.1, we have (i). Since $H^\infty(T^2)_b = H^\infty(T^2)$ [**9**], we have (ii). □

To determine the Bourgain algebra of a subspace of $C^\infty(U^2)$ which contains $C_0(U^2)$, we need the following two lemmas to find a weakly null sequence.

Lemma 4.3 (see the proof of Lemma 1 [**7**]). *Let B be a Banach space and K is a positive number. Let $\{f_n\}_n$ be a sequence in B such that* $\left\|\sum_{n=1}^{k} a_n f_n\right\| <$

K for every k and complex numbers a_n with $|a_n| = 1$. Then f_n is a weakly null sequence in B.

Lemma 4.4. *Let $\{z_n\}_n$ be a sequence in U with $|z_n| \to 1$. Then there is a weakly null sequence $\{h_k\}_k$ in $H^\infty(U)$ such that both $\{z_n; |h_k(z_n)| \geq 1\}$ and $\{z_n; h_k(z_n) = 0\}$ are infinite sets for each k.*

Proof. This is essentially proved in [**7**, Theorem 2]. By considering a subsequence, we may assume that $\{z_n\}_n$ is an interpolating sequence, that is, for each bounded sequence of complex numbers $\{a_n\}_n$ there is a function h in $H^\infty(U)$ such that $h_n(z_n) = a_n$ for every n. By [**5**, p. 294], there is a sequence $\{f_n\}_n$ in $H^\infty(U)$ such that

$$f_n(z_n) = 1 \quad \text{and} \quad f_n(z_k) = 0 \qquad \text{if } n \neq k;$$

$$\sum_{n=1}^{\infty} |f_n(z)| < K \ (z \in U) \qquad \text{for some constant } K.$$

We divide the set of positive integers into infinite disjoint subsets $\bigcup_{k=1}^{\infty} \{n_{k,j}\}_{j=1}^{\infty}$. Let

$$h_k(z) = \sum_{j=1}^{\infty} f_{n_{k,j}}(z) \qquad \text{for each } k.$$

Then $h_k \in H^\infty(U)$ and

$$\sum_{k=1}^{\infty} |h_k(z)| \leq \sum_{n=1}^{\infty} |f_n(z)| < K \qquad \text{for } z \in U.$$

Hence by Lemma 4.3, $\{h_k\}_k$ is a weakly null sequence in $H^\infty(U)$ and

$$\{z_n; |h_k(z_n)| \geq 1\} = \{z_{n_{k,j}}\}_{j=1}^{\infty} \quad \text{and} \quad \{z_n; h_k(z_n) = 0\} = \{z_n\}_n \backslash \{z_{n_{k,j}}\}_{j=1}^{\infty}.$$

□

Recall that $C_{T^2}(\bar{U}^2)$ is the space of functions f in $C(\bar{U}^2)$ which vanish on T^2. Let B be a closed subspace of $L^\infty(T^2)$ and let V be a closed subspace of $C_{T^2}(\bar{U}^2)$. Then by Lemma 4.1, $\tilde{B} + V$ is a closed subspace of $C^\infty(U^2)$. We study the Bourgain algebra of $\tilde{B} + V$ relative to $C^\infty(U^2)$.

Proposition 4.2. *Let B be a closed subspace such that $H^\infty(T^2) \subset B \subset L^\infty(T^2)$. Let V be a closed subspace of $C_{T^2}(\bar{U}^2)$. Let $f \in (\tilde{B} + V)_b$. If $f^* = 0$ a.e. on T^2, then $f \in C_0(U^2)$.*

Proof. Let $f \in (\tilde{B} + V)_b$. Suppose that $f^* = 0$ a.e. on T^2. To prove $f \in C_0(U^2)$, suppose not. Then there is $\delta > 0$, and there is a sequence $\{(z_n, w_n)\}_n$ in U^2 such that $\{(z_n, w_n)\}_n$ converges to a point λ_0 in ∂U^2 and

$$|f(z_n, w_n)| > \delta \quad \text{for every } n. \tag{3}$$

We may assume that $|z_n| \to 1$. Here we use Lemma 4.4. Then there is a weakly null sequence $\{h_k(e^{i\theta})\}_k$ in $H^\infty(T)$ such that both $\left\{z_n; \left|\tilde{h}_k(z_n)\right| \geq 1\right\}$ and $\{z_n; \tilde{h}_k(z_n) = 0\}$ are infinite sets for each k. By Lemma 3.6, we may consider that $\{\tilde{h}_k(z)\}_k$ is a weakly null sequence in $\tilde{B}+V$. Since $f \in (\tilde{B}+V)_b$, there exist $g_k \in B$ and $c_k \in V$ such that

$$\left\|f\tilde{h}_k + \tilde{g}_k + c_k\right\|_{U^2} \to 0.$$

Since $c_k^* = 0$ on T^2 and $f^* = 0$ a.e. on T^2, by taking radial limits we have $\|\tilde{g}_k\|_{U^2} = \|g_k\|_{T^2} \to 0$. Hence

$$\left\|f\tilde{h}_k + c_k\right\|_{U^2} \to 0. \tag{4}$$

Since $c_k \in C_{T^2}(\bar{U}^2)$ and $(z_n, w_n) \to \lambda_0$, $c_k(z_n, w_n) \to c_k(\lambda_0)$ $(n \to \infty)$ for each k. Since $\{z_n; \tilde{h}_k(z_n) = 0\}$ is an infinite set, we have

$$\begin{aligned}\left\|f\tilde{h}_k + c_k\right\|_{U^2} &\geq \limsup_{n\to\infty} |f(z_n, w_n)\tilde{h}_k(z_n) + c_k(z_n, w_n)| \\ &\geq |c_k(\lambda_0)|.\end{aligned}$$

Therefore by (4), $c_k(\lambda_0) \to 0$ $(k \to \infty)$. Since $\{z_n; |\tilde{h}_k(z_n)| \geq 1\}$ is an infinite set, we have

$$\begin{aligned}\left\|f\tilde{h}_k + c_k\right\|_{U^2} &\geq \limsup_{n\to\infty} |f(z_n, w_n)\tilde{h}_k(z_n) + c_k(z_n, w_n)| \\ &\geq \delta - |c_k(\lambda_0)| \quad \text{by (3)}.\end{aligned}$$

Since $c_k(\lambda_0) \to 0$, we have

$$\liminf_{k\to\infty} \left\|f\tilde{h}_k + c_k\right\|_{U^2} \geq \delta.$$

This contradicts (4). Consequently, we have $f \in C_0(U^2)$. □

Corollary 4.2.
(i) $(H^\infty + C_0)(U^2)_b = (H^\infty + C_0)(U^2)$.
(ii) $H^\infty(U^2)_{b(n)} = (H^\infty + C_0)(U^2)_{b(n)} = (H^\infty + C_0)(U^2)$ *for every* $n \geq 1$.

Proof. Since $(H^\infty+C_0)(U^2)$ is an algebra, $(H^\infty+C_0)(U^2) \subset (H^\infty+C_0)(U^2)_b$. To prove the converse inclusion, let $f \in (H^\infty + C_0)(U^2)_b$. By Lemma 4.2, $f^* \in H^\infty(T^2)_b = H^\infty(T^2)$. Therefore $f - (f^*)\tilde{} \in (H^\infty + C_0)(U^2)_b$. By Proposition 4.2, $f - (f^*)\tilde{} \in C_0(U^2)$. Hence $f \in H^\infty(T^2)\tilde{} + C_0(U^2) =$

$(H^\infty + C_0)(U^2)$. Consequently we get (i). By Corollary 4.1 and (i), we have (ii). □

We note that the proof of Proposition 4.2 actually proved the following proposition.

Proposition 4.3. *Let B be a closed subspace such that $H^\infty(T^2) \subset B \subset L^\infty(T^2)$. Let V be a closed subspace of $C_{T^2}(\bar{U}^2)$. Let f be a function in $C^\infty(U^2)$ such that there exists the radial limit f^* and $f^* = 0$ a.e. on T^2. If $\left\| f\tilde{h}_k + \tilde{B} + V \right\|_{U^2} \to \infty$ for every weakly null sequence $\{h_k\}_k$ in $H^\infty(T^2)$, then $f \in C_0(U^2)$.*

Since $H^\infty(U^2) + C(\bar{U}^2) = (H^\infty + C)(T^2)^{\tilde{}} + C_{T^2}(\bar{U}^2)$, by Lemma 4.1 $H^\infty(U^2) + C(\bar{U}^2)$ is a closed subspace of $C^\infty(U^2)$ but it is not an algebra. We denote by $\mathcal{M}_1$ the multiplier algebra of $H^\infty(U^2) + C(\bar{U}^2)$, that is,

$$\mathcal{M}_1 = \{f \in C^\infty(U^2);\ f \cdot (H^\infty(U^2) + C(\bar{U}^2)) \subset H^\infty(U^2) + C(\bar{U}^2)\}.$$

Since constants are contained in $H^\infty(U^2) + C(\bar{U}^2)$, $\mathcal{M}_1 \subset H^\infty(U^2) + C(\bar{U}^2)$. Now we have a following characterization of $\mathcal{M}_1$.

Theorem 4.2. $\mathcal{M}_1 = \tilde{\mathcal{M}} + C_0(U^2) = (H^\infty(U^2) + C(\bar{U}^2))_b$.

Proof. It is tirivial that $C_0(U^2) \subset \mathcal{M}_1$. To show $\tilde{\mathcal{M}} \subset \mathcal{M}_1$, let $f \in \mathcal{M}$. Since $C(\bar{U}^2) = C(T^2)^{\tilde{}} + C_0(U^2)$, it is sufficient to show that $\tilde{f} \cdot C(T^2)^{\tilde{}} \subset H^\infty(U^2) + C(\bar{U}^2)$. By the definiton of the Poisson integral, it is not difficult to see that for $c \in C(T^2)$,

$$(5) \qquad (\tilde{f}\tilde{c} - (fc)^{\tilde{}})(z_n, w_n) \to 0 \quad \text{as } |z_n| \to 1 \text{ and } |w_n| \to 1.$$

Since $f \in \mathcal{M}$, by Corollary 2.4 we have $fc \in \mathcal{M} + C(T^2)$. Hence by Corollary 2.2, $\tilde{f}\tilde{c}$ and $(fc)^{\tilde{}}$ can be extended continuously on $\bar{U}^2 \setminus T^2$. Then by (5), $\tilde{f}\tilde{c} - (fc)^{\tilde{}} \in C(\bar{U}^2)$. Therefore

$$\tilde{f}\tilde{c} = (fc)^{\tilde{}} + (\tilde{f}\tilde{c} - (fc)^{\tilde{}}) \in \tilde{\mathcal{M}} + C(T^2)^{\tilde{}} + C(\bar{U}^2) \subset H^\infty(U^2) + C(\bar{U}^2).$$

Thus $\tilde{f} \in \mathcal{M}_1$, so that $\tilde{\mathcal{M}} + C_0(U^2) \subset \mathcal{M}_1$. By the definition of Bourgain algebras, the multiplier algebra is contained in the Bourgain algebra. Therefore we have

$$\tilde{\mathcal{M}} + C_0(U^2) \subset \mathcal{M}_1 \subset (H^\infty(U^2) + C(\bar{U}^2))_b.$$

Next we prove $(H^\infty(U^2) + C(\bar{U}^2))_b \subset \tilde{\mathcal{M}} + C_0(U^2)$. Let $g \in (H^\infty(U^2) + C(\bar{U}^2))_b$. We have $H^\infty(U^2) + C(\bar{U}^2) = (H^\infty + C)(T^2)^{\tilde{}} + C_{T^2}(\bar{U}^2)$. By

Lemma 4.2, g has the radial limit and $g^* \in (H^\infty + C)(T^2)_b$. By Theorem 3.1, $g^* \in \mathcal{M}$. By the first paragraph, we have $g - (g^*)\tilde{} \in (H^\infty(U^2) + C(\bar{U}^2))_b$. Hence by Proposition 4.2, $g - (g^*)\tilde{} \in C_0(U^2)$, so that $(H^\infty(U^2) + C(\bar{U}^2))_b \subset \tilde{\mathcal{M}} + C_0(U^2)$. This completes the proof. □

By Lemma 4.1, $H^\infty(U^2) + C_{T^2}(\bar{U}^2)$ is also a closed subspace of $C^\infty(U^2)$ but it is not an algebra. We denote by $\mathcal{M}_2$ the multiplier algebra of $H^\infty(U^2) + C_{T^2}(\bar{U}^2)$, that is,

$$\mathcal{M}_2 = \{f \in C^\infty(U^2);\ f \cdot (H^\infty(U^2) + C_{T^2}(\bar{U}^2)) \subset H^\infty(U^2) + C_{T^2}(\bar{U}^2)\}.$$

Then we have the following theorem.

Theorem 4.3. $\mathcal{M}_2 = \tilde{\mathcal{M}} + C_0(U^2) = (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b$.

The difficulty of the proof is to show $(H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b \subset \tilde{\mathcal{M}} + C_0(U^2)$. To prove this, we need some lemmas. First we have the notation which will be used in Section 5.

Definition 4.1. For a function f in $C^\infty(U^2)$ and $\zeta \in \partial U^2$, let

$$\omega(f, \zeta) = \sup \left\{ \limsup_{n \to \infty} |f(\zeta_n) - f(\xi_n)|;\ \zeta_n, \xi_n \in U^2, \zeta_n, \xi_n \to \zeta \right\}.$$

Then $\omega(f, \zeta)$ is an upper semicontinuous function in $\zeta \in \partial U^2$, and $\omega(f, \zeta) = 0$ if and only if f can be extended continuously at $\zeta \in \partial U^2$. The following lemma follows from the above definition.

Lemma 4.5. *Let $f, g \in C^\infty(U^2)$. Then*
(i) $\|f\|_{U^2} \geq \omega(f, \zeta)/2$ *for every* $\zeta \in \partial U^2$;
(ii) $\omega(f + g, \zeta) \geq \omega(f, \zeta) - \omega(g, \zeta)$ *for every* $\zeta \in \partial U^2$;
(iii) *if f can be extended continuously at ζ, then* $\omega(fg, \zeta) = |f(\zeta)|\ \omega(g, \zeta)$.

The following lemma is an application of Lemma 3.2. Recall the definition of $\omega_0(f, \lambda)$ for $f \in H^\infty(T)$ and $\lambda \in T$ (see the paragraph above Lemma 3.2). We note that

$$\omega_0(f, \lambda) = \sup \left\{ \limsup_{j \to \infty} |\tilde{f}(x_j) - \tilde{f}(z_j)|;\ x_j, z_j \in U, x_j, z_j \to \lambda \right\}.$$

Lemma 4.6. *Let $f \in H^\infty(T^2)$ and $f \notin \mathcal{M}$. Then there exists $\delta > 0$ such that $\{\zeta \in \partial U^2 \setminus T^2; \omega(\tilde{f}, \zeta) > \delta\}$ is an infinite set.*

Proof. Since $f \in H^\infty(T^2) \setminus \mathcal{M}$, by Theorem 2.1 we may assume that $I_k(f) \notin A(T)$ for some $k \geq 0$. Since $I_k(f) \in H^\infty(T)$ by Lemma 2.6,

$$\|I_k(f) + C(T)\|_T > 0.$$

Let $\delta = \|I_k(f) + C(T)\|_T /3$. By Lemma 3.2, there is a distinct sequence $\{w_n\}_n$ in U such that $|w_n| \to 1$ and

$$\left\|\tilde{f}(e^{i\theta}, w_n) + C(T)\right\|_T > \delta \qquad \text{for every } n,$$

where we consider that $\tilde{f}(e^{i\theta}, w_n)$ is a function in $e^{i\theta} \in T$. By (1) in Section 3,

$$\|h + C(T)\|_T \leq \sup\{\omega_0(h, \lambda); \lambda \in T\} \quad \text{for } h \in H^\infty(T).$$

Hence for each n there exists a point $e^{i\theta_n}$ in T such that

$$\omega_0(\tilde{f}(e^{i\theta}, w_n), e^{i\theta_n}) > \delta.$$

Since

$$\omega_0(\tilde{f}(e^{i\theta}, w_n), e^{i\theta_n}) = \sup\left\{\limsup_{j\to\infty} |\tilde{f}(x_j, w_n) - \tilde{f}(z_j, w_n)|;\ x_j, z_j \in U, x_j, z_j \to e^{i\theta_n}\right\},$$

we have

$$\omega(\tilde{f}, (e^{i\theta_n}, w_n)) \geq \omega_0(\tilde{f}(e^{i\theta}, w_n), e^{i\theta_n}).$$

Put $\zeta_n = (e^{i\theta_n}, w_n)$. Then $\{\zeta_n\}_n$ is a distinct sequence in $\partial U^2 \setminus T^2$ and $\omega(\tilde{f}, \zeta_n) > \delta$ for every n. □

Proof of Theorem 4.3. It is trivial that $C_0(U^2) \subset \mathcal{M}_2$. By Corollary 2.2, we have $\tilde{\mathcal{M}} \cdot C_{T^2}(\bar{U}^2) \subset C_{T^2}(\bar{U}^2)$. Hence $\tilde{\mathcal{M}} \subset \mathcal{M}_2$ and

$$\tilde{\mathcal{M}} + C_0(U^2) \subset \mathcal{M}_2 \subset (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b.$$

We shall prove that $(H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b \subset \tilde{\mathcal{M}} + C_0(U^2)$. To show this, let $g \in (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b$. By Lemma 4.2, $g^* \in H^\infty(T^2)_b = H^\infty(T^2)$. Let $\{h_k\}_k$ be a weakly null sequence in $H^\infty(T^2)$. Since $g \in (H^\infty(U^2)+C_{T^2}(\bar{U}^2))_b$, there exists $q_k \in H^\infty(U^2)$ and $b_k \in C_{T^2}(\bar{U}^2)$ such that

$$\left\|g\tilde{h}_k + q_k + b_k\right\|_{U^2} \to 0. \tag{6}$$

By considering the radial limits, $\|g^*h_k + q_k^*\|_{T^2} \to 0$. Since $g^*, h_k \in H^\infty(T^2)$,

$$\left\|(g^*)^\sim\, \tilde{h}_k + q_k\right\|_{U^2} = \|g^*h_k + q_k^*\|_{T^2} \to 0.$$

Then by (6), we have

$$\left\|(g - (g^*)^\sim)\tilde{h}_k + b_k\right\|_{U^2} \to 0.$$

Hence

$$\left\|(g-(g^*)^{\tilde{}})\tilde{h}_k + H^\infty(U^2) + C_{T^2}(\bar{U}^2)\right\|_{U^2} \to 0.$$

Therefore by Proposition 4.3, we have

$$g-(g^*)^{\tilde{}} \in C_0(U^2). \tag{7}$$

Next, to show $(g^*)^{\tilde{}} \in (H^\infty(U^2)+C_{T^2}(\bar{U}^2))_b$, let $\{f_n+c_n\}_n$ be a weakly null sequence in $H^\infty(U^2)+C_{T^2}(\bar{U}^2)$. Since $g \in (H^\infty(U^2)+C_{T^2}(\bar{U}^2))_b$,

$$\|g(f_n+c_n)+H^\infty(U^2)+C_{T^2}(\bar{U}^2)\|_{U^2} \to 0.$$

By (7), $(g-(g^*)^{\tilde{}})(f_n+c_n) \in C_0(U^2) \subset C_{T^2}(\bar{U}^2)$. Hence

$$\begin{aligned}
&\left\|(g^*)^{\tilde{}}(f_n+c_n)+H^\infty(U^2)+C_{T^2}(\bar{U}^2)\right\|_{U^2}\\
&= \left\|g(f_n+c_n)-(g-(g^*)^{\tilde{}})(f_n+c_n)+H^\infty(U^2)+C_{T^2}(\bar{U}^2)\right\|_{U^2}\\
&= \|g(f_n+c_n)+H^\infty(U^2)+C_{T^2}(\bar{U}^2)\|_{U^2}\\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

This implies that

$$(g^*)^{\tilde{}} \in (H^\infty(U^2)+C_{T^2}(\bar{U}^2))_b. \tag{8}$$

Now we show $g^* \in \mathcal{M}$. To show this, suppose not. Then $g^* \in H^\infty(T^2)\setminus\mathcal{M}$. By Lemma 4.6, there exist $\delta>0$ and a distinct sequence $\{\zeta_n\}_n$ in $\partial U^2 \setminus T^2$ such that $\zeta_n \to \zeta_0 \in \partial U^2$, $\zeta_n \neq \zeta_0$, and

$$\omega((g^*)^{\tilde{}},\zeta_n) > \delta \quad \text{for every } n. \tag{9}$$

Take a sequence of disjoint open subsets $\{V_n\}_n$ of $\bar{U}^2$ such that $V_n \cap T^2 = \emptyset$ and $\zeta_n \in V_n$. Take functions d_n in $C(\bar{U}^2)$ such that

$$d_n(\zeta_n) = 1,\ 0 \le d_n \le 1 \text{ on } \bar{U}^2, \text{ and } d_n = 0 \text{ on } \bar{U}^2\setminus V_n. \tag{10}$$

Then by Lemma 3.3, $\{d_n\}_n$ is a weakly null sequence in $C_{T^2}(\bar{U}^2)$. By (8), there are sequences $\{g_n\}_n$ in $H^\infty(U^2)$ and $\{p_n\}_n$ in $C_{T^2}(\bar{U}^2)$ such that

$$\left\|(g^*)^{\tilde{}}d_n+g_n+p_n\right\|_{U^2} \to 0 \quad \text{as } n\to\infty.$$

By considering the radial limits, $\|g_n\|_{U^2} = \|g_n^*\|_{T^2} \to 0$. Hence

$$\left\|(g^*)^{\tilde{}}d_n+p_n\right\|_{U^2} \to 0 \quad \text{as } n\to\infty. \tag{11}$$

By Lemma 4.5, we have

$$\begin{aligned}\left\|(g^*)^{\sim} d_n + p_n\right\|_{U^2} &\geq \sup_{\zeta\in\partial U^2} \{\omega((g^*)^{\sim} d_n, \zeta)/2\} \\ &= \sup_{\zeta\in\partial U^2} \{|d_n(\zeta)|\omega((g^*)^{\sim}, \zeta)/2\} \\ &\geq \sup_{n} \{|d_n(\zeta_n)|\omega((g^*)^{\sim}, \zeta_n)/2\} \\ &> \delta \quad \text{ by (9) and (10).}\end{aligned}$$

This contradicts (11). Hence $g^* \in \mathcal{M}$. Consequently by (7), we have $g \in \tilde{\mathcal{M}} + C_0(U^2)$, so that $(H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b \subset \tilde{\mathcal{M}} + C_0(U^2)$. This completes the proof. □

By Corollaries 4.1 and Theroems 4.2 and 4.3, we have the following.

Corollary 4.3. $(\tilde{\mathcal{M}})_b = (H^\infty(U^2) + C(\bar{U}^2))_b = (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b$.

5. Higher Bourgain algebras on the polydisk.

In Section 4, we proved that

$$H^\infty(U^2)_{b(n)} = (H^\infty + C_0)(U^2)_{b(n)} = (H^\infty + C_0)(U^2)$$

for every $n \geq 1$. In this section, we study the n-th Bourgain algebras $(\tilde{\mathcal{M}})_{b(n)}$, $(\tilde{\mathcal{M}} + C_0(U^2))_{b(n)}$, and $(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)}$ relative to $C^\infty(U^2)$. In Theorem 5.1, we shall prove that

$$(\tilde{\mathcal{M}})_{b(n)} \neq (\tilde{\mathcal{M}})_{b(n+1)} \qquad \text{for every } n \geq 0.$$

Since $(\tilde{\mathcal{M}})_{b(1)} = \tilde{\mathcal{M}} + C_0(U^2)$ by Corollary 4.1 (i), the above fact means that all the higher Bourgain algebras of $\tilde{\mathcal{M}} + C_0(U^2)$ are all distinct. But the situation is not the same if we start from $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$. In Theorem 5.2, we shall prove that

$$(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)} = (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b \qquad \text{for every } n \geq 1.$$

Spaces $\tilde{\mathcal{M}} + C_0(U^2)$ and $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$ are similar to each other, but the properties of these Bourgain algebras are completely different. Notations are little bit complicated, but the essential idea is simple and like the following. Consider Bourgain algebras of $C_0(U)$ and $C(\bar{U})$ relative to $C^\infty(U)$. It is rather easy to show that $C(\bar{U})_{b(n)} = C(\bar{U})_b$ and $C_0(U)_{b(n)} \neq C_0(U)_{b(n+1)}$. When determing $(\tilde{\mathcal{M}} + C_0(U^2))_{b(n)}$, it appears the phenomenon on $\partial U^2 \setminus$

T^2 like as $C_0(U)_{b(n)}$. When determing $(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)}$, it appears the phenomenon on $\partial U^2 \setminus T^2$ like as $C(\bar{U})_{b(n)}$.

To describe $(\tilde{\mathcal{M}} + C_0(U^2))_{b(n)}$ explicitly, we need some notations. Let

$$\tilde{T} = (T \times 0) \cup (0 \times T).$$

Then $\tilde{T}$ is a closed subset of ∂U^2. By induction, we define the families of closed subsets of $\tilde{T}$. If A is a subset of $\tilde{T}$, we denote by A^d, the derived set of A, the set of cluster points of A. We write $A^{d(n)}$ for $(A^{d(n-1)})^d$ and $A^{d(0)} = A$. Let Λ_1 be the family of finite subsets of $\tilde{T}$. Consider that the empty set is contained in Λ_1. Let Λ_2 be the family of closed subsets E of $\tilde{T}$ such that $E^d \in \Lambda_1$. Assume that the family Λ_n is defined. Then Λ_{n+1} is the family of closed subsets F of $\tilde{T}$ such that $F^d \in \Lambda_n$, that is, $F^{d(n)} \in \Lambda_1$. By our definition, every subset in Λ_n is a countable closed set, $\Lambda_n \subset \Lambda_{n+1}$, and

$$\Lambda_n \neq \Lambda_{n+1} \qquad \text{for every } n.$$

For $\xi \in \tilde{T}$ and $0 < r \leq 1$, we put

$$D(\xi, r) = \begin{cases} \{(e^{i\theta}, w);\ |w| \leq r\} \text{ if } \xi = (e^{i\theta}, 0), \\ \{(z, e^{i\psi});\ |z| \leq r\} \ \text{ if } \xi = (0, e^{i\psi}). \end{cases}$$

Then $D(\xi, r) \subset \partial U^2$ and $D(\xi, r) \cap T^2 = \emptyset$ for every $\xi \in \tilde{T}$ and $0 < r < 1$. Now we have the following notation.

Definition 5.1. For $f \in C^\infty(U^2)$ and $\zeta \in \partial U^2$, let

$$\bar{f}(\zeta) = \sup \left\{ \limsup_{n \to \infty} |f(\zeta_n)|;\ \zeta_n \in U^2, \zeta_n \to \zeta \right\}.$$

Then $\bar{f}$ is an upper semicontinuous function on ∂U^2. To describe $(\tilde{\mathcal{M}} + C_0(U^2))_{b(n)}$, we introduce new spaces $C_n(U^2)$.

Definition 5.2. We denote by $C_n(U^2), n \geq 1$, the space of functions f in $C^\infty(U^2)$ such that for each $\delta > 0$, there exists $r, 0 < r < 1$, and there exists $E \in \Lambda_n$ depending on δ such that $\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\} \subset \cup \{D(\xi, r); \xi \in E\}$.

By the upper semicontinuity of $\bar{f}$, we can see that the above definiton is equivalent to the following one.

Definition 5.2'. We denote by $C_n(U^2), n \geq 1$, the space of functions f in $C^\infty(U^2)$ such that for each $\delta > 0$, there exists $E \in \Lambda_n$ such that $\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\} \subset \cup \{D(\xi, 1); \xi \in E\}$ and $\bar{f} = 0$ on T^2.

For, if $f \in C_n(U^2)$ in the sense of Definition 5.2, then it is easy to prove $f \in C_n(U^2)$ in the sense of Definition 5.2'. To see the converse, suppose that $g \in C_n(U^2)$ in the sense of Definition 5.2'. Then for each $\delta > 0$, there exists

$E \in \Lambda_n$ such that $\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\} \subset \cup \{D(\xi, 1); \xi \in E\}$ and $\bar{f} = 0$ on T^2. Since $\bar{f}$ is upper semicontinuous, $\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\}$ is a compact subset of $\partial U^2 \setminus T^2$. Hence for each $\xi \in E$, there exists r_ξ with $0 < r_\xi < 1$ such that

$$\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\} \cap (D(\xi, 1) \setminus D(\xi, r_\xi)) = \emptyset.$$

Therefore

$$\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\} \subset \cup \{D(\xi, r_\xi); \xi \in E\}.$$

We may assume moreover that

$$\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\} \cap (D(\xi, 1) \setminus D(\xi, r)) \neq \emptyset$$

for every r with $0 < r < r_\xi$. We need to prove that $\sup\{r_\xi; \xi \in E\} < 1$. To show this, suppose not. Then there is a sequence $\{\xi_n\}_n$ in E such that $r_{\xi_n} \to 1$. Then there exists ζ_n in ∂U^2 such that

$$\bar{f}(\zeta_n) \geq \delta \quad \text{and} \quad \zeta_n \in D(\xi, r_\xi) \setminus D(\xi, (1 + r_\xi)/2).$$

Let ζ_0 be a cluster point of $\{\zeta_n\}_n$. Then we have

$$\zeta_0 \in T^2 \quad \text{and} \quad \bar{f}(\zeta_0) \geq \delta.$$

This is the desired contradiction. Hence Definitions 5.2 and 5.2' are equivalent.

By our definition, $\bar{f} = 0$ on T^2 for $f \in C_n(U^2)$. Since each element E in Λ_n is a closed subset of $\tilde{T}$, $\cup\{D(\xi, r); \xi \in E\}$ is a closed subset of ∂U^2. For a closed subset F of ∂U^2, it is not difficult to find a function f in $C^\infty(U^2)$ such that $\bar{f} = \chi_F$, the characteristic function for F on ∂U^2. Since $\Lambda_n \neq \Lambda_{n+1}$, we have

$$C_n(U^2) \neq C_{n+1}(U^2) \qquad \text{for every } n \geq 0.$$

Now our theorem is the following.

Theorem 5.1. $(\tilde{\mathcal{M}} + C_0(U^2))_{b(n)} = \tilde{\mathcal{M}} + C_n(U^2)$ *for every* $n \geq 0$.

Corollary 5.1.

(i) $(\tilde{\mathcal{M}})_{b(n)} = (H^\infty(U^2) + C(\bar{U}^2))_{b(n)} = (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_{b(n)} = \tilde{\mathcal{M}} + C_{n-1}(U^2)$ *for every* $n \geq 1$.

(ii) $(\tilde{\mathcal{M}})_{b(n)} \neq (\tilde{\mathcal{M}})_{b(n+1)}$ *for every* $n \geq 0$.

Proof. By Corollary 4.1, Theorems 4.2 and 4.3, $(\tilde{\mathcal{M}})_b = (H^\infty(U^2)+C(\bar{U}^2))_b = (H^\infty(U^2) + C_{T^2}(\bar{U}^2))_b = \tilde{\mathcal{M}} + C_0(U^2)$. Then by Theorem 5.1, we get (i) and (ii). □

To prove our theorem, we need some lemmas. The following lemma follows from Definition 5.1.

Lemma 5.1. *Let $f, g \in C^\infty(U^2)$ and $\zeta \in \partial U^2$. Then*
(i) *if g can be extended continuously at ζ, then $(fg)^{-}(\zeta) = |g(\zeta)|\ \bar{f}(\zeta)$;*
(ii) $(fg)^{-} \le \bar{f}\bar{g}$ *on* ∂U^2;
(iii) $\|f + C_0(U^2)\|_{U^2} = \|\bar{f}\|_{\partial U^2}$;
(iv) $(f+g)^{-}(\zeta) \ge \bar{f}(\zeta) - \bar{g}(\zeta)$.

Lemma 5.2. *Let $f \in C^\infty(U^2)$ and $0 < r < 1$. For $\delta > 0$, the set F of points ξ in $\tilde{T}$ such that $D(\xi, r) \cap \{\zeta \in \partial U^2; \bar{f}(\zeta) \ge \delta\} \ne \emptyset$ is a closed subset of $\tilde{T}$.*

Proof. Let $\{\xi_n\}_n$ be a sequence in F such that $\xi_n \to \xi_0$ for some $\xi_0 \in \tilde{T}$. We may assume that $\xi_n = (e^{i\theta_n}, 0)$, $\xi_0 = (e^{i\theta_0}, 0)$, and $\theta_n \to \theta_0$. By our assumption, there is a point w_n in U with $|w_n| \le r$ such that

$$\bar{f}(e^{i\theta_n}, w_n) \ \ge\ \delta.$$

Let w_0 be a cluster point of $\{w_n\}_n$. Then $|w_0| \le r$, and by the upper semicontinuity of $\bar{f}$,

$$\bar{f}(e^{i\theta_0}, w_0) \ \ge\ \delta.$$

Since $(e^{i\theta_0}, w_0) \in D(\xi_0, r)$, we have $\xi_0 \in F$. Hence F is a closed subset of $\tilde{T}$. □

Lemma 5.3. *Let B be a closed subspace with $\mathcal{M} \subset B \subset L^\infty(T^2)$. Let V be a closed subspace of $C^\infty(U^2)$ such that $\bar{f} = 0$ on T^2 for every $f \in V$. If $G \in (\tilde{B} + V)_b$ and the radial limit $G^* = 0$ a.e. on T^2, then $\bar{G} = 0$ on T^2.*

Proof. We note that by Lemma 4.1, $\tilde{B} + V$ is a closed subspace of $C^\infty(U^2)$. To prove $\bar{G} = 0$ on T^2, suppose not. Then $\bar{G}(\zeta) \ne 0$ for some $\zeta \in T^2$. Hence there exists a sequence $\zeta_n = (z_n, w_n)$ in U^2 such that $|z_n| \to 1, |w_n| \to 1$ and

$$(1) \qquad |G(z_n, w_n)| \ >\ |\bar{G}(\zeta)|/2 \qquad \text{for every } n.$$

Here we have $|z_n w_n| < 1$ and $|z_n w_n| \to 1$. By Lemma 4.4, there is a weakly null sequence $\{g_k\}_k$ in $H^\infty(U)$ such that

$$(2) \qquad \{z_n w_n;\ |g_k(z_n w_n)| \ge 1\} \quad \text{is an infinite set}$$

for each k. Let

$$(3) \qquad G_k(z, w) \ =\ g_k(zw) \qquad \text{for } (z, w) \in U^2.$$

By the proof of Corollary 2.1, we have $G_k \in \tilde{\mathcal{M}}$. Since

$$H^\infty(U) \ni q \;\to\; Q(z,w) \;=\; q(zw) \;\in\; \tilde{\mathcal{M}}$$

is a bounded linear map, by Lemma 3.6 G_k is a weakly null sequence in $\tilde{\mathcal{M}}$ and so is in $\tilde{B} + V$. Since $G \in (\tilde{B} + V)_b$, there exist sequences $\{\tilde{h}_k\}_k$ in $\tilde{B}$ and $\{c_k\}_k$ in V such that

$$(4) \qquad \left\| GG_k + \tilde{h}_k + c_k \right\|_{U^2} \to 0 \qquad \text{as } k \to \infty.$$

Since $G^* = c_k^* = 0$ a.e. on T^2, by considering radial limits, we have $\left\|\tilde{h}_k\right\|_{U^2} = \|h_k\|_{T^2} \to 0$ as $k \to \infty$. Hence by (4),

$$(5) \qquad \|GG_k + c_k\|_{U^2} \;\to\; 0 \qquad \text{as } k \to \infty.$$

Since $c_k \in W$, by our assumption $\lim_{n\to\infty} c_k(z_n, w_n) = 0$ for each k. Now we have

$$\begin{aligned}
\|GG_k + c_k\|_{U^2} &\geq \limsup_{n\to\infty} \; |G(z_n,w_n)G_k(z_n,w_n) + c_k(z_n,w_n)| \\
&\geq \limsup_{n\to\infty} \; |G(z_n,w_n)g_k(z_nw_n)| \quad \text{by (3)} \\
&\geq |\bar{G}(\zeta)|/2 \qquad \text{by (1) and (2).}
\end{aligned}$$

This contradicts (5). □

The following lemma follows Definition 5.2.

Lemma 5.4.

(i) $C_n(U^2) \cdot C_\infty(U^2) \;=\; C_n(U^2)$.

(ii) $\tilde{\mathcal{M}} + C_n(U^2)$ *is a closed subalgebra of* $C^\infty(U^2)$.

(iii) *Let* $\{\tilde{f}_k + c_k\}_k$ *be weakly null in* $\tilde{\mathcal{M}} + C_n(U^2)$. *Then* $\{\tilde{f}_k\}_k$ *is weakly null in* $\tilde{\mathcal{M}}$.

(iv) *Let* $\{f_k\}_k$ *be weakly null in* $\mathcal{M}$, $0 < r < 1$, *and* $\xi \in \tilde{T}$. *Then* $\left\|\tilde{f}_k\right\|_{D(\xi,r)} \to 0$ *as* $k \to \infty$.

Proof. (i) is trivial by the definition of $C_n(U^2)$.

(ii) By Lemma 4.1, $\tilde{\mathcal{M}} + C_n(U^2)$ is a closed subspace. By (i), it becomes an algebra.

(iii) It is easy to see that each function in $\tilde{\mathcal{M}} + C_n(U^2)$ is represented by the form $\tilde{f} + c$ uniquely, where $f \in \mathcal{M}$ and $c \in C_n(U^2)$. Then $\tilde{\mathcal{M}} + C_n(U^2) \ni \tilde{f} + c \to f \in \mathcal{M}$ is a bounded linear map. By Lemma 3.6, $\{f_k\}_k$ is weakly null in $\mathcal{M}$. Hence we get (iii).

(iv) Since $f_k \in \mathcal{M}$, by Corollary 2.2 we may consider that $\tilde{f}_k$ is continuous on $\bar{U}^2 \setminus T^2$. Since $\{f_k\}_k$ is weakly null in $\mathcal{M}$, $\{\tilde{f}_k\}_k$ is sup-norm bounded and converges to 0 pointwise in $\bar{U}^2 \setminus T^2$. Since $D(\xi, r) \subset \bar{U}^2 \setminus T^2$ and f_k is analytic in $D(\xi, r)$, $\left\|\tilde{f}_k\right\|_{D(\xi,r)} \to 0$ as $k \to \infty$. □

Lemma 5.5. *Let $f \in C_{n+1}(U^2)$ and let $\{\tilde{f}_k\}_k$ be a weakly null sequence in $\tilde{\mathcal{M}}$. Then $\left\|f\tilde{f}_k + C_n(U^2)\right\|_{U^2} \to 0$ as $k \to \infty$.*

Proof. Let $f \in C_{n+1}(U^2)$ and let $\{\tilde{f}_k\}_k$ be a weakly null sequence in $\tilde{\mathcal{M}}$. Then $\{\tilde{f}_k\}_k$ is sup-norm bounded. We may assume that

$$\left\|\tilde{f}_k\right\|_{U^2} \leq 1 \qquad \text{for every } k. \tag{6}$$

Since $\tilde{f}_k \in \tilde{\mathcal{M}}$, by Corollary 2.2 we may consider that $\tilde{f}_k$ is a continuous function on $\bar{U}^2 \setminus T^2$. Since $f \in C_{n+1}(U^2)$, $\bar{f} = 0$ on T^2. Then by Lemma 5.1 (i),

$$(f\tilde{f}_k)^{\bar{}}(\zeta) = |\tilde{f}_k(\zeta)|\, \bar{f}(\zeta) \qquad \text{for } \zeta \in \partial U^2. \tag{7}$$

Let $\delta > 0$ arbitrarily. Put

$$S = \{\zeta \in \partial U^2;\ \bar{f}(\zeta) \geq \delta\}. \tag{8}$$

Since $f \in C_{n+1}(U^2)$, there exist $E \in \Lambda_{n+1}$ and $r, 0 < r < 1$, such that

$$S \subset \cup \{D(\xi, r);\ \xi \in E\}. \tag{9}$$

Since $E \in \Lambda_{n+1}$, $E^{d(n)}$ is a finite set. Let

$$D = \cup \{D(\xi, r);\ \xi \in E^{d(n)}\}. \tag{10}$$

Then $D \subset \partial U^2, D \cap T^2 = \emptyset$, and by Lemma 5.4 (iv)

$$\left\|\tilde{f}_k\right\|_D \to 0 \qquad \text{as } k \to \infty. \tag{11}$$

By (7) and (11), there exists k_0 depending on δ such that

$$(f\tilde{f}_k)^{\bar{}} < \delta/2 \quad \text{on } D \qquad \text{for every } k > k_0. \tag{12}$$

Hereafter we assume $k > k_0$. Let

$$B_k = \{\eta \in U^2;\ |(f\tilde{f}_k)(\eta)| \geq \delta\}. \tag{13}$$

Then B_k is a closed subset of U^2. We denote by $\bar{B}_k$ the closure of B_k in $\bar{U}^2$. By (6), we have

$$B_k \subset \{\eta \in U^2;\ |f(\eta)| \geq \delta\}. \tag{14}$$

By (8) and (14),

$$S \supset \bar{B}_k \cap \partial U^2. \tag{15}$$

By (12) and (13),

$$\bar{B}_k \cap D = \emptyset. \tag{16}$$

It is not difficult to find a function p in $C^\infty(U^2)$ such that

$$0 \leq p \leq 1 \qquad \text{on } U^2; \tag{17}$$

$$p = 1 \qquad \text{on } B_k; \tag{18}$$

$$\bar{p} = 0 \qquad \text{on } (\partial U^2) \setminus \bar{B}_k. \tag{19}$$

By (13), (17), and (18), we have $\left\|f\tilde{f}_k(1-p)\right\|_{U^2} \leq \delta$. Hence

$$\left\|(f\tilde{f}_k(1-p))^{-}\right\|_{\partial U^2} \leq \delta. \tag{20}$$

Here we prove that

$$f\tilde{f}_k p \in C_n(U^2). \tag{21}$$

Let $\sigma > 0$. By Lemma 5.1 (ii),

$$(f\tilde{f}_k p)^{-} \leq \bar{f}\bar{\tilde{f}}_k\bar{p} \qquad \text{on } \partial U^2. \tag{22}$$

Then by (15) and (19),

$$\{\zeta \in \partial U^2;\ (f\tilde{f}_k p)^{-}(\zeta) \geq \sigma\} \subset S. \tag{23}$$

We denote by E_0 the set of $\xi \in E$ such that

$$D(\xi, r) \cap \{\zeta \in \partial U^2;\ (f\tilde{f}_k p)^{-}(\zeta) \geq \sigma\} \neq \emptyset. \tag{24}$$

Since $E \in \Lambda_{n+1}$, by the definition of Λ_{n+1}, E is a closed subset. Since $f \in C_{n+1}(U^2)$, by Lemma 5.4 (i) $f\tilde{f}_k p \in C_{n+1}(U^2)$. Hence by Lemma 5.2, E_0 is a closed subset of E. By (9) and (23),

$$\{\zeta \in \partial U^2;\ (f\tilde{f}_k p)^{-}(\zeta) \geq \sigma\} \subset \cup\, \{D(\xi, r);\ \xi \in E_0\}. \tag{25}$$

By (16), (19), and (22), we have $(f\tilde{f}_k p)^{\tilde{}} = 0$ on D. Hence by (10) and (24), $E_0 \cap E^{d(n)} = \emptyset$. Therefore

$$(E_0)^{d(n)} \subset E_0 \cap E^{d(n)} = \emptyset.$$

This means that $(E_0)^{d(n-1)}$ is a finite set and $E_0 \in \Lambda_n$. By (25), we get (21).

Now we have

$$\begin{aligned}\left\|f\tilde{f}_k + C_n(U^2)\right\|_{U^2} &= \left\|f\tilde{f}_k(1-p) + f\tilde{f}_k p + C_n(U^2)\right\|_{U^2} \\ &= \left\|f\tilde{f}_k(1-p) + C_n(U^2)\right\|_{U^2} \qquad \text{by (21)} \\ &\le \left\|f\tilde{f}_k(1-p) + C_0(U^2)\right\|_{U^2} \\ &= \left\|(f\tilde{f}_k(1-p))^{\tilde{}}\right\|_{\partial U^2} \qquad \text{by Lemma 5.1 (iii)} \\ &\le \delta \qquad \text{by (20).}\end{aligned}$$

Since the above inequality holds for every $\delta > 0$ and $k > k_0$ (k_0 depends on δ), we have $\left\|f\tilde{f}_k + C_n(U^2)\right\|_{U^2} \to 0$ as $k \to \infty$. This completes the proof. □

Proof of Theorem 5.1. It is sufficient to prove that

$$(\tilde{\mathcal{M}} + C_n(U^2))_b = \tilde{\mathcal{M}} + C_{n+1}(U^2) \qquad \text{for } n \ge 0.$$

First we shall prove that $\tilde{\mathcal{M}} + C_{n+1}(U^2) \subset (\tilde{\mathcal{M}} + C_n(U^2))_b$. By Lemma 5.4 (ii), $\tilde{\mathcal{M}} + C_n(U^2)$ is a closed algebra. Hence $\tilde{\mathcal{M}} + C_n(U^2) \subset (\tilde{\mathcal{M}} + C_n(U^2))_b$. To prove $C_{n+1}(U^2) \subset (\tilde{\mathcal{M}} + C_n(U^2))_b$, let $f \in C_{n+1}(U^2)$ and $\{\tilde{f}_k + c_k\}_k$ is a weakly null sequence in $\tilde{\mathcal{M}} + C_n(U^2)$. By Lemma 5.4 (iii), $\{\tilde{f}_k\}_k$ is weakly null in $\tilde{\mathcal{M}}$. Then

$$\begin{aligned}&\left\|f(\tilde{f}_k + c_k) + \tilde{\mathcal{M}} + C_n(U^2)\right\|_{U^2} \\ &\qquad = \left\|f\tilde{f}_k + \tilde{\mathcal{M}} + C_n(U^2)\right\|_{U^2} \quad \text{by Lemma 5.4 (i)} \\ &\qquad \le \left\|f\tilde{f}_k + C_n(U^2)\right\|_{U^2} \\ &\qquad \to 0 \quad \text{as } k \to \infty \quad \text{by Lemma 5.5.}\end{aligned}$$

Hence $f \in (\tilde{\mathcal{M}} + C_n(U^2))_b$. Thus we have $\tilde{\mathcal{M}} + C_{n+1}(U^2) \subset (\tilde{\mathcal{M}} + C_n(U^2))_b$.

Next we shall prove that $(\tilde{\mathcal{M}} + C_n(U^2))_b \subset \tilde{\mathcal{M}} + C_{n+1}(U^2)$. Let $g \in (\tilde{\mathcal{M}} + C_n(U^2))_b$. By Lemma 4.2, there exists the radial limit g^* a.e. on T^2 and $g^* \in \mathcal{M}_b$. By Theorem 3.1, we have $g^* \in \mathcal{M}$. To prove $g - (g^*)^{\tilde{}} \in C_{n+1}(U^2)$, suppose that $g - (g^*)^{\tilde{}} \notin C_{n+1}(U^2)$. We shall lead a contradiction. We put

$$G = g - (g^*)^{\tilde{}} \in C^\infty(U^2).$$

Then by the first paragraph, $G \in (\tilde{\mathcal{M}} + C_n(U^2))_b$. Here we can use Lemma 5.3. Then

$$\bar{G} = 0 \quad \text{on } T^2. \tag{26}$$

Since $G \notin C_{n+1}(U^2)$, there is $\delta > 0$ such that

$$\{\zeta \in \partial U^2;\ \bar{G}(\zeta) \geq \delta\} \not\subset \cup\{D(\xi, r);\ \xi \in E\} \tag{27}$$

for every $E \in \Lambda_{n+1}$ and r with $0 < r < 1$. Since $\bar{G}$ is upper semicontinuous on ∂U^2, by (26) there exists r_0 with $0 < r_0 < 1$ such that

$$\{\zeta \in \partial U^2;\ \bar{G}(\zeta) \geq \delta\} \subset \cup\{D(\xi, r_0);\ \xi \in \tilde{T}\}.$$

We denote by E_0 the set of ξ in $\tilde{T}$ such that

$$D(\xi, r_0) \cap \{\zeta \in \partial U^2;\ \bar{G}(\zeta) \geq \delta\} \neq \emptyset. \tag{28}$$

By Lemma 5.2, E_0 is a closed subset of $\tilde{T}$, and

$$\{\zeta \in \partial U^2;\ \bar{G}(\zeta) \geq \delta\} \subset \cup\{D(\xi, r_0);\ \xi \in E_0\}.$$

Hence by (27), $E_0 \notin \Lambda_{n+1}$. Then $(E_0)^{d(n)}$ is an infinite set. Take a distinct convergent sequence $\{\lambda_k\}_k$ in $(E_0)^{d(n)}$. Here for the sake of simplicity, we assume that $E_0 \subset T \times 0$. Put $\lambda_k = (e^{i\theta_k}, 0)$. Then $\{e^{i\theta_k}\}_k$ is a convergent sequence in T. By Lemma 3.4, we may assume moreover that there is a weakly null sequence $\{g_k(e^{i\theta})\}_k$ in $A(T)$ such that $g_k(e^{i\theta_k}) = 1$ for every k. We consider that $g_k(e^{i\theta})$ is a function in $A(T^2)$. Then g_k is weakly null in $\tilde{\mathcal{M}} + C_n(U^2)$, and

$$\tilde{g}_k = g_k(e^{i\theta}) \quad \text{on } e^{i\theta} \times \bar{U}. \tag{29}$$

By Lemma 5.1 (i),

$$(G\tilde{g}_k)^{-}(\zeta) = |\tilde{g}_k(\zeta)|\, \bar{G}(\zeta) \quad \text{for } \zeta \in \partial U^2. \tag{30}$$

Let

$$E_k = \{\xi \in E_0;\ |\tilde{g}_k(\xi)| \geq 1/2\}. \tag{31}$$

Then E_k is a closed subset of E_0, and by (29) $(e^{i\theta_k}, 0) \in E_k$ for each k. Since $(e^{i\theta_k}, 0) = \lambda_k \in (E_0)^{d(n)}$, by (31) $\lambda_k \in (E_k)^{d(n)}$. Hence $(E_k)^{d(n-1)}$ is an infinite set. Therefore $E_k \notin \Lambda_n$ for every k. This means that $\cup\{D(\xi, r_0); \xi \in E_k\}$ is not contained in $\cup\{D(\xi, r_0); \xi \in F\}$ for every $F \in \Lambda_n$. Hence by Definition 5.2,

$$\inf_{\xi \in E_k} \|\bar{c}\|_{D(\xi, r_0)} = 0 \quad \text{for every } c \in C_n(U^2). \tag{32}$$

Since $E_k \subset E_0$, by (28)

$$\|\bar{G}\|_{D(\xi,r_0)} \geq \delta \quad \text{for every } \xi \in E_k.$$

Hence by (29), (30), and (31), we have

$$\|(G\tilde{g}_k)^{-}\|_{D(\xi,r_0)} \geq \delta/2 \quad \text{for every } \xi \in E_k. \tag{33}$$

Since $G \in (\tilde{\mathcal{M}} + C_n(U^2))_b$ and $\{\tilde{g}_k\}_k$ is weakly null in $\tilde{\mathcal{M}} + C_n(U^2)$, there exists a sequence $\{\tilde{h}_k + c_k\}_k$ in $\tilde{\mathcal{M}} + C_n(U^2)$ such that

$$\left\|G\tilde{g}_k + \tilde{h}_k + c_k\right\|_{U^2} \to 0.$$

Since $G^* = c_k^* = 0$ a.e. on T^2, $\left\|\tilde{h}_k\right\|_{U^2} = \|h_k\|_{T^2} \to 0$. Hence

$$\|G\tilde{g}_k + c_k\|_{U^2} \to 0 \quad \text{as } k \to \infty. \tag{34}$$

But we have

$$\begin{aligned}
\|G\tilde{g}_k + c_k\|_{U^2} &\geq \|(G\tilde{g}_k + c_k)^{-}\|_{\partial U^2} \\
&\geq \sup_{\xi \in E_k} \|(G\tilde{g}_k + c_k)^{-}\|_{D(\xi,r_0)} \\
&\geq \sup_{\xi \in E_k} \left\{\|(G\tilde{g}_k)^{-}\|_{D(\xi,r_0)} - \|\bar{c}_k\|_{D(\xi,r_0)}\right\} \quad \text{by Lemma 5.1 (iv)} \\
&\geq \delta/2 - \inf_{\xi \in E_k} \|\bar{c}_k\|_{D(\xi,r_0)} \quad \text{by (33)} \\
&= \delta/2 \quad \text{by (32)}.
\end{aligned}$$

This contradicts (34). This completes the proof. □

In Theorem 5.1, we proved that the higher Bourgain algebras of $\tilde{\mathcal{M}} + C_0(U^2)$ are all different. Next we shall study $(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)}$. We prove that $(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)} = (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$ for every n. To describe its Bourgain algebra, we need to introduce a new space $W(U^2)$. Recall the Definition 4.1, for a function f in $C^\infty(U^2)$ and $\zeta \in \partial U^2$,

$$\omega(f,\zeta) = \sup\left\{\limsup_{n\to\infty} |f(\zeta_n) - f(\xi_n)|;\ \zeta_n, \xi_n \in U^2, \zeta_n, \xi_n \to \zeta\right\}.$$

Definition 5.3. We denote by $W(U^2)$ the space of functions f in $C^\infty(U^2)$ such that
(i) $\bar{f} = 0$ on T^2;
(ii) for $\delta > 0$, $\{\zeta \in \partial U^2;\ \omega(f,\zeta) > \delta\}$ is a finite set.

Roughly speaking, $W(U^2)$ is the set of functions in $C^\infty(U^2)$ whose boundary functions are continuous except finite sets. We note that $C_{T^2}(\bar{U}^2)$ is contained in $W(U^2)$ properly. It is not difficult to see the following.

Lemma 5.6. *Let $f \in W(U^2)$. Then* $\|f + C_{T^2}(\bar{U}^2)\| \leq \|\omega(f,\zeta)\|_{\partial U^2}$.

Since $\omega(f,\zeta)$ is an upper semicontinuous function in $\zeta \in \partial U^2$, we have

Lemma 5.7. *Let $f \in W(U^2)$ and let $\{\zeta_n\}_n$ be a distinct sequence in ∂U^2. Then* $\liminf_{n\to\infty} \omega(f,\zeta_n) = 0$.

Now we have the following theorem.

Theorem 5.2. $(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_{b(n)} = (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b = \tilde{\mathcal{M}} + W(U^2)$.

Proof. Step 1. We shall prove $\tilde{\mathcal{M}} + W(U^2) \subset (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$. By Corollary 2.2, for each function h in $\mathcal{M}$ we may consider that $\tilde{h}$ is a continuous function on $\bar{U}^2 \setminus T^2$. Then $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$ becomes a closed subalgebra of $C^\infty(U^2)$. Hence $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2) \subset (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$.

To prove $W(U^2) \subset (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$, let $f \in W(U^2)$ and let $\{\tilde{f}_n + c_n\}_n$ be a weakly null sequence in $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$. Then by Lemma 3.3 there is a constant $K > 0$ such that

$$\left\|\tilde{f}_n + c_n\right\|_{U^2} < K \qquad \text{for every } n. \tag{35}$$

By the definition of $W(U^2)$, $\bar{f} = 0$ on T^2 and there is a sequence $\{\zeta_k\}_k$ in $\partial U^2 \setminus T^2$ such that $\{\zeta \in \partial U^2; \omega(f,\zeta) \neq 0\} = \{\zeta_k\}_k$. By Lemma 5.7,

$$\omega(f,\zeta_k) \to 0 \qquad \text{as } k \to \infty. \tag{36}$$

Since $\{\tilde{f}_n + c_n\}_n$ is weakly null,

$$(\tilde{f}_n + c_n)(\zeta_k) \to 0 \quad (n \to \infty) \qquad \text{for each } k. \tag{37}$$

Since $\bar{f} = 0$ on T^2, $(f(\tilde{f}_n + c_n))^{-} = 0$ on T^2. Therefore by Lemma 4.5 (iii),

$$\begin{aligned}\left\|\omega(f(\tilde{f}_n + c_n),\zeta)\right\|_{\partial U^2} &= \sup_k \omega(f,\zeta_k)\ |(\tilde{f}_n + c_n)(\zeta_k)| \\ &\to 0 \quad (n \to \infty) \qquad \text{by (35), (36), and (37).}\end{aligned}$$

Then by Lemma 5.6,

$$\left\|f(\tilde{f}_n + c_n) + C_{T^2}(\bar{U}^2)\right\|_{U^2} \to 0 \qquad \text{as } n \to \infty.$$

Hence $f \in (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$. Thus $\tilde{\mathcal{M}} + W(U^2) \subset (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$.

Step 2. Next we prove $(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b \subset \tilde{\mathcal{M}} + W(U^2)$. Let $g \in (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$. By Lemma 4.2, there exists the radial limit g^* a.e. on T^2 and $g^* \in \mathcal{M}_b = \mathcal{M}$. By the first paragraph, $g - (g^*)^{\tilde{}} \in (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$. We put

$$G \;=\; g - (g^*)^{\tilde{}} \;\in\; C_\infty(U^2).$$

We prove $G \in W(U^2)$. To prove this, suppose that $G \notin W(U^2)$. Since $G \in (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$, by Lemma 5.3 we have $\bar{G} = 0$ on T^2. Then for some $\delta > 0$, there is a distinct sequence $\{\zeta\}_n$ in $(\partial U^2)\setminus T^2$ such that $\omega(G, \zeta_n) > 2\delta$ for every n. We may assume that $\{\zeta_n\}_n$ is a convergent sequence. Since $C_{T^2}(\bar{U}^2)$ is a C^*- algebra, it is not difficult to find a weakly null sequence $\{h_n\}_n$ in $C_{T^2}(\bar{U}^2)$ such that $h_n(\zeta_n) = 1$ for every n. Since every function in $\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2)$ is continuous on $(\partial U^2) \setminus T^2$, by Lemma 4.5 we have

$$\left\| Gh_n + \tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2) \right\|_{U^2} \;\geq\; \omega(Gh_n, \zeta_n)/2 \;=\; \omega(G, \zeta_n)/2 \;>\; \delta.$$

This means that $G \notin (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$. But $G \in (\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b$. This is a contradiction.

Step 3. Next we prove $(\tilde{\mathcal{M}} + W(U^2))_b \;=\; \tilde{\mathcal{M}} + W(U^2)$. By Steps 1 and 2, we have

$$(\tilde{\mathcal{M}} + C_{T^2}(\bar{U}^2))_b \;=\; \tilde{\mathcal{M}} + W(U^2).$$

Then $\tilde{\mathcal{M}} + W(U^2)$ is a closed subalgebra of $C^\infty(U^2)$. Hence $\tilde{\mathcal{M}} + W(U^2) \subset (\tilde{\mathcal{M}} + W(U^2))_b$.

To prove the converse inclusion, let $h \in (\tilde{\mathcal{M}} + W(U^2))_b$. By Lemma 4.2, $h^* \in \mathcal{M}_b = \mathcal{M}$. By the above fact, $h - (h^*)^{\tilde{}} \in (\tilde{\mathcal{M}} + W(U^2))_b$. We put

$$H \;=\; h - (h^*)^{\tilde{}} \;\in\; C^\infty(U^2).$$

We prove $H \in W(U^2)$. To prove this, suppose that $H \notin W(U^2)$. Since $H \in (\tilde{\mathcal{M}} + W(U^2))_b$, by Lemma 5.3 we have $\bar{H} = 0$ on T^2. Then for some $\sigma > 0$, there is a distinct convergent sequence $\{\zeta\}_n$ in $(\partial U^2) \setminus T^2$ such that

$$\omega(H, \zeta_n) \;>\; \sigma \qquad \text{for every } n. \tag{38}$$

Let $\zeta_n \to \zeta_0 \in \partial U^2$. Since $\omega(H, \zeta)$ is upper semicontinuous in $\zeta \in \partial U^2$, $\omega(H, \zeta_0) \geq \sigma$ and $\zeta_0 \in \partial U^2 \setminus T^2$. Take a sequence of open subsets $\{V_n\}_n$ of $\bar{U}^2$ such that

$$\zeta_n \in V_n,\; V_n \cap T^2 \;=\; \emptyset, \;\text{ and } \bigcap_{n=1}^{\infty} V_n \;=\; \{\zeta_0\}. \tag{39}$$

Take a sequence of open subsets $\{W_n\}_n$ of $\bar{U}^2$ such that

$$\zeta_n \in W_n \subset V_n \quad \text{and } W_n \cap W_k \;=\; \emptyset \;\text{ if } n \neq k. \tag{40}$$

Take a sequence of functions $\{h_n\}_n$ in $C_{T^2}(\bar{U}^2)$ such that

$$\|h_n\|_{U^2} \;=\; 1,\; h_n(\zeta_n) \;=\; 1, \quad \text{and } h_n \;=\; 0 \;\text{ on } \bar{U}^2 \setminus W_n. \tag{41}$$

We divide the set of integers into disjoint infinite subsets,

$$\bigcup_{k=1}^{\infty} \{n_{k,j}\}_j \;=\; \{1,2,\dots\} \quad \text{and } \{n_{k,j}\}_j \cap \{n_{i,j}\}_j = \emptyset \;\text{ if } k \neq i.$$

Let

$$H_k \;=\; \sum_{j=1}^{\infty} h_{n_{k,j}} \qquad \text{on } U^2.$$

By (39), (40), and (41), $H_k \in C^\infty(U^2)$, H_k can be extended continuously on $\partial U^2 \setminus \{\zeta_0\}$, and $\bar{H}_k = 0$ on T^2. Hence $H_k \in W(U^2)$. Moreover we have

$$H_k(\zeta_{n_{k,j}}) \;=\; 1 \qquad \text{for every } j. \tag{42}$$

Since

$$\sum_{k=1}^{\infty} |H_k| \;\leq\; \sum_{n=1}^{\infty} |h_n| \;\leq\; 1 \qquad \text{on } U^2,$$

by Lemma 4.3 $\{H_k\}_k$ is a weakly null sequence in $C^\infty(U^2)$, and so is in $\tilde{\mathcal{M}} + W(U^2)$.

Since $H \in (\tilde{\mathcal{M}} + W(U^2))_b$, there is a sequence $\{\tilde{g}_k + c_k\}_k$ in $\tilde{\mathcal{M}} + W(U^2)$ such that

$$\|HH_k + \tilde{g}_k + c_k\|_{U^2} \;\to\; 0.$$

Since $H^* \;=\; c_k^* \;=\; 0$ on T^2, by considering the radial limits, $\|\tilde{g}_k\|_{U^2} = \|g_k\|_{T^2} \to 0$. Hence

$$\|HH_k + c_k\|_{U^2} \;\to\; 0. \tag{43}$$

But we have

$$\begin{aligned}
&\|HH_k + c_k\|_{U^2} \geq \limsup_{j\to\infty} \; \omega(HH_k + c_k, \zeta_{n_{k,j}})/2 && \text{by Lemma 4.5 (i)}\\
&\geq \limsup_{j\to\infty} \; (\omega(HH_k, \zeta_{n_{k,j}}) \;-\; \omega(c_k, \zeta_{n_{k,j}}))/2 && \text{by Lemma 4.5 (ii)}\\
&\geq (\sigma \;-\; \liminf_{j\to\infty} \; \omega(c_k, \zeta_{n_{k,j}}))/2 && \text{by (38), (42) and Lemma 4.5 (iii)}\\
&= \sigma/2 && \text{by Lemma 5.7.}
\end{aligned}$$

This contradicts (43). Hence $(\tilde{\mathcal{M}}+W(U^2))_b \subset \tilde{\mathcal{M}}+W(U^2)$. This completes the proof. □

Remark 5.1. In Theorem 5.1, we proved that $\tilde{\mathcal{M}} + C_1(\bar{U}^2) \neq (\tilde{\mathcal{M}} + C_1(\bar{U}^2))_b$. In Theorem 5.2, we proved that $\tilde{\mathcal{M}} + W(U^2) = (\tilde{\mathcal{M}} + W(U^2))_b$. The spaces $C_1(\bar{U}^2)$ and $W(U^2)$ are similar in these definitions. But these spaces consist of much different kind of functions. A boundary function of a function in $C_1(\bar{U}^)$ vanishes on ∂U^2 except a countable set. A boundary function of a function in $W(U^2)$ is continuous on ∂U^2 except a countable set.

6. Bourgain algebras of the polydisk algebra.

In [9], the first author dertermined the Bourgain algebra of the disk algebra $A(T)$ relative to $L^\infty(T)$. Succeedingly in [3], Cima, Stroethoff and Yale determined the Bourgain algebra of the disk algebra $A(\bar{U})$ relative to $C^\infty(U)$. In this section, we briefly study the Bourgain algebras of the polydisk algebra.

Let X be the maximal ideal space of $L^\infty(T^2)$. We may consider that $L^\infty(T^2) = C(X)$ by Gelfand transform. For $\lambda = (\lambda_1, \lambda_2) \in T^2$, we put

$$X_\lambda = \{x \in X;\ z(x) = \lambda_1, w(x) = \lambda_2\}.$$

For $f \in L^\infty(T^2)$, let

$$\omega_0(f, \lambda) = \sup\{|f(\lambda_1) - f(\lambda_2)|;\ \lambda_1, \lambda_2 \in X_\lambda\}.$$

Definition 6.1. We denote by $V(T^2)$ the space of $f \in L^\infty(T^2)$ such that $\{\lambda \in T^2; \omega_0(f, \lambda) > \epsilon\}$ is a finite set for every $\epsilon > 0$.

In [9, Corollary 1], the first author proved that

$$(1) \qquad A(T^2)_{b(n)} = A(T^2)_b = (H^\infty \cap V)(T^2) \qquad \text{for every } n \geq 1$$

relative to $L^\infty(T^2)$. First we have the following proposition.

Proposition 6.1. *$(H^\infty \cap V)(T^2) \subset \mathcal{M}$, and $(H^\infty \cap V + C)(T^2)$ is a closed subalgebra of $L^\infty(T^2)$.*

Proof. Let $f \in (H^\infty \cap V)(T^2)$. By the definition of $V(T^2)$, there is a sequence $\{\lambda_n\}_n$ in T^2 such that $\{\lambda_n\}_n = \{\lambda \in T^2; \omega_0(f, \lambda) \neq 0\}$. Hence we may consider that f is continuous on $T^2 \setminus \{\lambda_n\}_n$. Therefore for a convergent sequence $\{e^{i\theta_n}\}_n$ in T to $e^{i\theta_0}$, $f(e^{i\theta_n}, e^{i\psi})$ converges to $f(e^{i\theta_0}, e^{i\psi})$ pointwise for every $e^{i\psi} \in T$ except a countable set. Thus we have $I_k(f)(e^{i\theta_n}) \to I_k(f)(e^{i\theta_0})$ for every $k \geq 0$. Since $I_k(f) \in H^\infty(T)$, this means that $I_k(f) \in A(T)$ for $k \geq 0$. By the same way, $J_k(f) \in A(T)$ for $k \geq 0$. By Theorem 2.1, $f \in \mathcal{M}$.

Let $f = F_n + e^{in(\theta+\psi)} G_n$ be in the form (A_3). Since $f \in \mathcal{M}$, by Corollary 2.3 we have $F_n \in A(T^2)$. Hence by the definition of $V(T^2)$ we have $G_n \in (H^\infty \cap V)(T^2)$. This means that $(H^\infty \cap V)(T^2)$ is $*$-invariant. By Theorem 2.2, $(H^\infty \cap V + C)(T^2)$ is a closed subalgebra of $L^\infty(T^2)$. $\square$

By (1) and Theorem 4.1, we have

Proposition 6.2. $A(\bar{U}^2)_b = ((H^\infty \cap V)(T^2)\tilde{\ })_b = (H^\infty \cap V)(T^2)\tilde{\ } + C_0(U^2)$.

To prove the results in Section 5, we only used the following properties of $\mathcal{M}$,

(a) $\mathcal{M}$ is a closed subalgebra of $L^\infty(T^2)$;

(b) $\mathcal{M}_b = \mathcal{M}$;

(c) for every f in $\mathcal{M}$, $\tilde{f}$ can be extended continuously on $\bar{U}^2 \setminus T^2$;

(d) for a sequence $\{\zeta_n\}_n$ in U^2 such that ζ_n converges to some point in T^2, there exists a weakly null sequence $\{g_k\}_k$ in $\mathcal{M}$ such that $\{\zeta_n; |g_k(\zeta_n)| \geq 1\}$ is an infinite set for each k.

By (1) and Proposition 6.1, (a), (b), and (c) are true for $(H^\infty \cap V)(T^2)$ instead of $\mathcal{M}$. We shall show in Proposition 6.3 that (d) is true for $(H^\infty \cap V)(T^2)$ instead of $\mathcal{M}$. Therefore all results in Section 5 are true for $(H^\infty \cap V)(T^2)$ instead of $\mathcal{M}$ and we have the following two theorems.

Theorem 6.1.

(i) $((H^\infty \cap V)(T^2)\tilde{\ } + C_0(U^2))_{b(n)} = (H^\infty \cap V)(T^2)\tilde{\ } + C_n(U^2)$ *for every* $n \geq 0$.

(ii) $A(\bar{U}^2)_{b(n)} = (H^\infty \cap V)(T^2)\tilde{\ } + C_{n-1}(U^2)$ *for every* $n \geq 1$.

(iii) *The n-th Bourgain algebras of $A(\bar{U}^2)$ are all distinct.*

Theorem 6.2. $((H^\infty \cap V)(T^2)\tilde{\ } + C_{T^2}(\bar{U}^2))_{b(n)} = ((H^\infty \cap V)(T^2)\tilde{\ } + C_{T^2}(\bar{U}^2))_b = (H^\infty \cap V)(T^2)\tilde{\ } + W(U^2)$ *for* $n \geq 1$.

Proposition 6.3. *For a sequence $\{\zeta_n\}_n$ in U^2 which converges to some point in T^2, there exists a weakly null sequence $\{g_k\}_k$ in $(H^\infty \cap V)(T^2)$ such that $\{\zeta_n; |g_k(\zeta_n)| \geq 1\}$ is an infinite set for each k. Moreover for every $\delta > 0$, $\{\zeta_n; |g_k(\zeta_n)| < \delta\}$ is an infinite set for each k.*

Proof. First, by induction we constract a sequence of functions $\{F_j\}_j$ in $A(\bar{U}^2)$ satisfying some additional conditions. Let $\Gamma = \{\zeta_n\}_n$ and $\zeta_n \to \lambda$ for some $\lambda \in T^2$. Since $\{\lambda\}$ is a peak point for $A(\bar{U}^2)$ (see [**12**, p. 132]), there exists f in $A(\bar{U}^2)$ such that

$$\|f\|_{\bar{U}^2} = 1, \; f(\lambda) = 1, \quad \text{and } |f| < 1 \quad \text{on } \bar{U}^2 \setminus \{\lambda\}. \tag{2}$$

Let $\{\sigma_n\}_n$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \sigma_n < 1/4. \tag{3}$$

Choose $\xi_1 \in \Gamma$ such that $|f(\xi_1)| > 3/4$. Take a positive integer t_1 such that $|f(1-f^{t_1})(\xi_1)| > 3/4$. Let $F_1 = f(1-f^{t_1})$. By (2), we have $\|F_1\|_{\bar{U}^2} \leq 2$. Since $F_1(\lambda) = 0$, there is an open subset V_1 of $\bar{U}^2$ such that

$$\lambda \in V_1, \; \zeta_1 \notin V_1, \quad \text{and} \quad \|F_1\|_{V_1} < \sigma_1.$$

Next take a positive integer s_2 such that $|f^{s_2}| < \sigma_2$ on $\bar{U}^2 \setminus V_1$. Choose $\xi_2 \in \Gamma$ such that $\xi_2 \in V_1$ and $|f^{s_2}(\xi_2)| > 3/4$. Take a positive integer t_2 such that $|f^{s_2}(1-f^{t_2})(\xi_2)| > 3/4$. Let $F_2 = f^{s_2}(1-f^{t_2})$. Then $\|F_2\|_{\bar{U}^2} \leq 2$ and $|F_2| < 2\sigma_2$ on $\bar{U}^2 \setminus V_1$. Since $F_2(\lambda) = 0$, there is an open subset V_2 of $\bar{U}^2$ such that

$$\lambda \in V_2 \subset V_1, \; \xi_2 \notin V_2, \; \|F_1\|_{V_2} < \sigma_2/2, \quad \text{and} \quad \|F_2\|_{V_2} < \sigma_2/2.$$

Continue these processes succeedingly. As a result, we can get sequences $\{F_n\}_n$ in $A(\bar{U}^2)$, $\{\xi_n\}_n$ in Γ, and open subsets $\{V_n\}_n$ of $\bar{U}^2$, we put $V_0 = \bar{U}^2$, such that

$$\lambda \in V_{n+1} \subset V_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} V_n = \{\lambda\}; \tag{4}$$

$$\xi_n \in V_{n-1} \setminus V_n; \tag{5}$$

$$\|F_n\|_{\bar{U}^2} \leq 2; \tag{6}$$

$$|F_n(\xi_n)| > 3/4; \tag{7}$$

$$\|F_n\|_{V_k} < \sigma_k/k \qquad \text{for } k \geq n; \tag{8}$$

$$\|F_n\|_{(\bar{U}^2 \setminus V_k)} < 2\sigma_n \qquad \text{for } k < n. \tag{9}$$

Now we devide the set of integers into infinite disjoint subsets;

$$\bigcup_{k=1}^{\infty} \{n_{k,j}\}_j = \{1,2,\dots\} \quad \text{and} \quad \{n_{k,j}\}_j \cap \{n_{i,j}\}_j = \emptyset \quad \text{if } k \neq i.$$

Let

$$g_k = 4\sum_{j=1}^{\infty} F_{n_{k,j}} \quad \text{for each } k.$$

For each m, we have

$$\limsup_{n\to\infty} \left(\sum_{j=n}^{\infty} \|F_j\|_{(\bar{U}^2 \setminus V_m)} \right) \leq \limsup_{n\to\infty} \sum_{j=n}^{\infty} 2\sigma_j \quad \text{by (9)}$$
$$= 0 \quad \text{by (3)}.$$

Hence by (4), $\sum_{j=1}^{\infty} F_{n_{k,j}}$ converges uniformly on each compact subset of $\bar{U}^2 \setminus \{\lambda\}$, so that

(10) $$g_k \text{ is continuous on } \bar{U}^2 \setminus \{\lambda\}.$$

For each k, we have

(11) $$\sum_{j\neq k} \|F_j\|_{(V_{k-1}\setminus V_k)} \leq 2\sum_{j=k-1}^{\infty} \sigma_j,$$

for

$$\sum_{j\neq k} \|F_j\|_{(V_{k-1}\setminus V_k)} \leq \sum_{j=1}^{k-1} \|F_j\|_{V_{k-1}} + \sum_{j=k+1}^{\infty} \|F_j\|_{(\bar{U}^2\setminus V_k)}$$
$$\leq \sigma_{k-1} + \sum_{j=k+1}^{\infty} 2\sigma_j \quad \text{by (8) and (9)}.$$

Therefore by (3)

(12) $$\sum_{j\neq k} \|F_j\|_{(V_{k-1}\setminus V_k)} < \frac{1}{2}.$$

Hence by (6) we have $\sum_{j=1}^{\infty} \|F_j\|_{(\bar{U}^2\setminus\{\lambda\})} \leq 5/2$, so that $g_k \in H^\infty(U^2)$. By (10), we have

$$g_k^* \in (H^\infty \cap V)(T^2).$$

Since $\sum_{k=1}^{\infty} \|g_k\|_{U^2} \leq \sum_{j=1}^{\infty} \|F_j\|_{U^2} \leq 5/2$, by Lemma 4.3 $\{g_k\}_k$ is a weakly null sequence in $H^\infty(U^2)$. Hence $\{g_k^*\}_k$ is weakly null in $(H^\infty \cap V)(T^2)$. Also for each i, we have

$$
\begin{aligned}
|g_k(\xi_{n_{k,i}})| &\geq 4\{|F_{n_{k,i}}(\xi_{n_{k,i}})| - \sum_{j \neq i} |F_{n_{k,j}}(\xi_{n_{k,i}})|\} \\
&\geq 4\{3/4 - \sum_{t \neq n_{k,i}} \|F_t\|_{(V_{n_{k,i}-1} \setminus V_{n_{k,i}})} \quad \text{by (5) and (7)} \\
&\geq 4(3/4 - 1/2) = 1 \quad \text{by (12)}.
\end{aligned}
$$

Hence $\{\zeta \in \Gamma; |g_k(\zeta)| \geq 1\}$ is an infinite set.

At last, let k, m be positive integers with $k \neq m$. Then $n_{k,j} \neq n_{m,i}$ for every i and j. For each i, there exists a positive integer p_i such that $\xi_{n_{m,i}} \in V_{p_i - 1} \setminus V_{p_i}$. If $i \to \infty$, then $p_i \to \infty$. Hence by (11), we have

$$
\begin{aligned}
|g_k(\xi_{n_{m,i}})| &\leq 4(\sum_{j=1}^{\infty} |F_{n_{k,j}}(\xi_{n_{m,i}})|) \\
&\leq 4(2 \sum_{j=p_i-1}^{\infty} \sigma_j) \\
&\to 0 \quad \text{as } i \to \infty \quad \text{by (3)}.
\end{aligned}
$$

Therefore for every $\epsilon > 0$, $\{\zeta \in \Gamma; |g_k(\zeta)| < \epsilon\}$ is an infinite set. This completes the proof. □

In the rest of this section, we study $(A(\bar{U}^2) + C_0(U^2))_{b(n)}$ and $(A(\bar{U}^2) + C_{T^2}(\bar{U}^2))_{b(n)}$ relative to $C^\infty(U^2)$. By the same way as the construction of the families $\{\Lambda_n\}_n$ of closed subsets of $\tilde{T}$, we can define the families $\{\Gamma_n\}_n$ of closed subsets of T^2. Let Γ_1 be the set of finite subsets of T^2. Consider that the empty set is contained in Γ_1. Let Γ_n be the set of closed subsets E of T^2 such that $E^d \in \Gamma_{n-1}$. We have the following definition which is similar to Definiton 5.2'.

Definition 6.1. We denote by $C'_n(U^2), n \geq 1$, the space of functions f in $C^\infty(U^2)$ such that for each $\delta > 0$, there exists $E \in \Lambda_n$ such that $\{\zeta \in \partial U^2; \bar{f}(\zeta) \geq \delta\} \subset \cup \{D(\xi, 1); \xi \in E\}$ and $\{\zeta \in T^2; \bar{f}(\zeta) \geq \delta\} \in \Gamma_n$.

Then $C'_n(U^2) \subset C'_{n+1}(U^2)$ and $C'_n(U^2) \neq C'_{n+1}(U^2)$. Of course, $C_n(U^2)$ is strictly contained in $C'_n(U^2)$. We can prove the following theorem.

Theorem 6.3.

(i) $(A(\bar{U}^2) + C_0(U^2))_{b(n)} = (H^\infty \cap V)(T^2)\tilde{} + C'_n(U^2)$.

(ii) $(A(\bar{U}^2) + C_0(U^2))_{b(n)} \neq (A(\bar{U}^2) + C_0(U^2))_{b(n+1)}$ *for every* $n \geq 0$.

To describe $(A(\bar{U}^2) + C_{T^2}(\bar{U}^2))_{b(n)}$, we need other new spaces.

Definition 6.2. We denote by $W_n(U^2)$ the space of functions f in $C^\infty(U^2)$ such that for each $\delta > 0$,

(i) $\{\zeta \in T^2;\ \bar{f}(\zeta) > \delta\} \ \in\ \Gamma_n$;

(ii) $\{\zeta \in \partial U^2 \setminus T^2;\ \omega(f,\zeta) > \delta\}$ is a finite set.

Then $W_n(U^2) \subset W_{n+1}(U^2)$ and $W_n(U^2) \neq W_{n+1}(U^2)$. We can prove the following theorem.

Theorem 6.4.

(i) $(A(\bar{U}^2) + C_{T^2}(\bar{U}^2))_{b(n)} \ =\ (H^\infty \cap V)(T^2)\tilde{} + W_n(U^2)$.

(ii) $(A(\bar{U}^2) + C_{T^2}(\bar{U}^2))_{b(n)} \neq (A(\bar{U}^2) + C_{T^2}(\bar{U}^2))_{b(n+1)}$ *for every* $n \geq 0$.

We leave both proofs of Theorems 6.3 and 6.4 for the reader. The ideas to prove these are the same as the ones used in the proofs of Theorems 5.1 and 5.2. Theorem 6.4 (ii) is an interesting fact which contrasts with Theorem 5.2.

References

[1] J. Cima, S. Janson and K. Yale, *Completely continuous Hankel operators on H^∞ and Bourgain algebras*, Proc. Amer. Math. Soc., **105** (1989), 121-125.

[2] J. Cima, K. Stroethoff and K. Yale, *Bourgain algebras on the unit disk*, Pacific J. Math., **160** (1993), 27-41.

[3] J. Cima, K. Stroethoff and K. Yale, *Bourgain algebras of the disk algebra*, Proc. Roy. Irish Acad. Sect. A, **94** (1994), 19-23.

[4] J. Cima and R. Timoney, *The Dunford-Pettis property for certain planner uniform algebras*, Michigan Math. J., **34** (1987), 99-104.

[5] J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[6] P. Gorkin and K. Izuchi, *Bourgain algebras on the maximal ideal space of H^∞*, to appear in Rocky Mount. J. Math.

[7] P. Gorkin, K. Izuchi and R. Mortini, *Bourgain algebras of Douglas algebras*, Canad. J. Math., **44** (1992), 797-804.

[8] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[9] K. Izuchi, *Bourgain algebras of the disk, polydisk, and ball algebras*, Duke Math. J., **66** (1992), 503-519.

[10] K. Izuchi and Y Izuchi, *Algebras of bounded analytic functions containing the disk algebra*, Canad. J. Math., **38** (1986), 87-108.

[11] K. Izuchi, K. Stroethoff and K. Yale, *Bourgain algebras of spaces of harmonic functions*, Michigan Math. J., **41** (1994), 309-321.

[12] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.

[13] W. Rudin, *Spaces of type $H^\infty + C$*, Ann. Inst. Fourier Grenoble, **25** (1975), 99-125.

[14] D. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc., **79** (1973), 286-299.

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