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The authors introduce a new class of operators that are weakly decomposable relative to the identity, and some of their properties are derived; for example, these operators have the single valued extension property. The main result is that every generalized intertwining of an operator having property (δ) with such a weakly decomposable one is necessarily bounded whenever certain side conditions are satisfied. Examples also show that this class of weakly decomposable operators is not comparable by inclusion to the classical cases (e.g. decomposable operators).

1. Introduction.

In this note we shall generalize some results of Laursen and Neumann [**13**] on the automatic continuity of intertwinings of operators with certain spectral decomposition properties. Specifically, we prove that a linear map which is a generalized intertwining of an operator satisfying property (δ) with a second admissible operator that is weakly decomposable relative to the identity (WDI) is necessarily continuous (bounded) (provided the operator pair has no critical eigenvalue). The properties (δ) and WDI are both relaxations of the notion "decomposable" but in different directions; see below for details. We mention that some of our results and proofs have been improved and shortened by appeals to [**13**], which appeared after the original submission of the present paper.

Section 2 of the paper deals with definitions and other background needed for this study. Some results are proved which have their own independent interest. For example, we show that the notions of "admissible" and "subadmissible"operator (see [**13**, **15**]) are identical. Our Proposition 2.1 seems to be known, but the authors do not know where it may have been published.

In Section 3 we give the definition of our new "weakly" decomposable operators and then establish some of their elementary properties, among which is the single-valued extension property. We show through two examples that, unlike other subclasses with spectral decomposition, our new class is not comparable by inclusion to the class of decomposable operators.

In Section 4 we prove our main theorem (Theorem 4.1). The crucial result here is the admissibility of the WDI operator mentioned above. The last section gives some applications. We also want to thank the referee for helpful suggestions in revising this paper.

2. Preliminaries.

In this section we state some definitions and notations used in the paper. We also prove that the notions "subadmissible" and "admissible", introduced in [**13**, **15**], resp., are the same.

We write $T \in L(X)$ to mean that T is a bounded linear operator on the complex Banach space X, while $\sigma(T)$ and $\rho(T)$ denote its spectrum and resolvent set, resp. We recall that the analytic spectral manifolds for T are defined for the set F in the plane **C** by the formula

$$X(T,F) = \{x \in X : \sigma_T(x) \subset F\}$$

where $\sigma_T(x)$ denotes the local spectrum of T at x which itself is defined as the complement of the union of all the open sets in **C** on which are defined local analytic solutions of the equation $(\lambda - T)f(\lambda) = x$. We say that T has the single valued extension property (SVEP) if an analytic function $f : D \to X$ vanishes on D whenever it satisfies $(\lambda - T)f(\lambda) = 0$ on D.

Proposition 2.1. *If X(T, F) is closed for every closed set F, then T has* SVEP.

To prove Proposition 2.1 we need several lemmas.

Lemma 2.2 [**10**, p. 16]. *If F is closed and $\lambda_0 \in F$, $x_0 \in X$, and if $(\lambda_0 - T)x_0 \in X(T,F)$, then $x_0 \in X(T,F)$.*

Proof. Let f be an analytic X-valued function on some open set D with $D \cap F = \varnothing$ such that $(\lambda - T)f(\lambda) = (\lambda_0 - T)x_0$ for $\lambda \in D$. Then the function $h(\lambda) = (\lambda - \lambda_0)^{-1}[x_0 - f(\lambda)]$ satisfies

$$(\lambda - T)h(\lambda) = (\lambda - \lambda_0)^{-1}[(\lambda - \lambda_0)x_0 + (\lambda_0 - T)x_0 - (\lambda - T)f(\lambda)] = x_0$$

for $\lambda \in D$, which shows that $x_0 \in X(T,F)$, since D is an arbitrary open set disjoint from F. □

Lemma 2.3. *Let F be a fixed closed set. If X(T, F) is closed, then*

$$\partial\sigma(T|X(T,F)) \subset F,$$

where ∂S denotes the boundary of S.

Proof. Assume there exists $\lambda_0 \in \partial\sigma(T|X(T,F))$ but $\lambda_0 \notin F$. Then $(\lambda_0 - T)|X(T,F)$ is surjective, and since spectra are closed we infer that $(\lambda_0 - T)|X(T,F)$ is not injective. By [**6**, Corollary 2.4], $T|X(T,F)$ does not have SVEP. Thus [**6**, Cor. 2.3] ensures existence of a neighborhood δ of λ_0 and a nonzero analytic function f defined on δ with values in X such that

$$(2.1) \qquad (\lambda - T)f(\lambda) = 0 \quad (\lambda \in \delta).$$

We have $\rho(T|X(T,F)) \cap \delta \neq \varnothing$ because $\lambda_0 \in \partial\sigma(T|X(T,F))$. On other hand, (2.1) implies that $T|X(T,F)$ has eigenvalues in $\rho(T|X(T,F)) \cap \delta$, a contradiction which proves the lemma. □

Corollary 2.4. *If F and X(T, F) are closed, then $\sigma(T|X(T,F)) \subset \widehat{F}$, where $\widehat{F}$ is the union of F and all bounded components of its complement.*

Lemma 2.5. *Suppose $D \subset \mathbf{C}$ is open and connected. If the equation $(\lambda - T)f(\lambda) = 0$ has a nonzero analytic solution on D and if X(T, F) is closed for each closed F with $F \cap D \neq \varnothing$, then $D \subset \sigma(T|X(T,F))$.*

Proof. Let $\lambda_0 \in F \cap D$. By Lemma 2.2, $f(\lambda_0) \in X(T,F)$. Moreover, since $Tf(\lambda) = \lambda f(\lambda)$ for all $\lambda \in D$, differentiation yields $Tf^{(n)}(\lambda) = \lambda f^{(n)}(\lambda) + nf^{(n+1)}(\lambda)$ for all $\lambda \in D$. In particular, $(\lambda_0 - T)f^{(n)}(\lambda_0) = -nf^{(n-1)}(\lambda_0)$. From Lemma 2.2 it follows again by an easy inductive argument that $f^{(n)}(\lambda_0) \in X(T,F)$ for $n = 0, 1, \ldots$. If $\{\lambda : |\lambda - \lambda_0| < \eta\} \subset D$, then

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\lambda_0)(\lambda - \lambda_0)^n$$

lies in X(T, F) for all λ with $|\lambda - \lambda_0| < \eta$. Hence $f(\lambda) \in X(T,F)$ for all $\lambda \in D$ by analytic continuation. Since f is nonzero and $(\lambda - T)f(\lambda) = 0$, it follows that $D \subset \sigma(T|X(T,F))$. □

Proof of Proposition 2.1. Suppose that $(\lambda - T)f(\lambda) = 0$ for some analytic X-valued function f on an open connected set D. If $f(\lambda_0) \neq 0$ for some $\lambda_0 \in D$, then choose $\eta > 0$ so that $f(\lambda) \neq 0$ on the disc $|\lambda - \lambda_0| < \eta$ lying in D. Put $D_0 = \{\lambda : |\lambda - \lambda_0| \leq \eta/2\}$. Then $\sigma(T|X(T,D_0)) \subset D_0$ by Corollary 2.4. But Lemma 2.5 implies $D \subset \sigma(T|X(T,F))$, or $D \subset D_0$, a contradiction. The result follows. □

Lemma 2.6. *Assume that Y is an invariant subspace of T. If X(T, F) is closed for every closed F, then $Y(T|Y,F)$ is also closed.*

Proof. By Proposition 2.1 T has SVEP, hence the conclusion follows from [**19**, Lemma 1.9]. □

We now recall that T is decomposable [**6**] (resp. weakly decomposable [**7**]) if for every open cover $\{G_i : 1 \leq i \leq n\}$ of the complex plane $\mathbf{C}$ there are T-invariant subspaces $M_1, M_2, \ldots, M_n$ such that $\sigma(T|M_i) \subset G_i$ for each i and $X = M_1 + \cdots + M_n$ (resp. X is the closed span of the M_i). If T is decomposable, then X(T, F) is closed for closed F. The operator T is called quasidecomposable [**7**] if it is weakly decomposable and X(T, F) is closed whenever F is.

Following [**13**] and its references [AE], [E], we say that an operator $T \in L(X)$ has property (δ) if for every open cover $\{U, V\}$ of $\mathbf{C}$ and for every $x \in X$ there exist a pair of elements $u, v \in X$ and a pair of analytic functions $f : \mathbf{C} \setminus U^- \to X$ and $g : \mathbf{C} \setminus V^- \to X$ such that

$$\begin{aligned} x &= u + v, \\ u &= (\lambda - T)f(\lambda) \quad \text{for all } \lambda \in \mathbf{C} \setminus U^-, \\ v &= (\lambda - T)g(\lambda) \quad \text{for all } \lambda \in \mathbf{C} \setminus U^-. \end{aligned}$$

The following notion of generalized intertwining can be found in [**15**] (see also [**5**, p. 48]). Let $T \in L(X)$ and $S \in L(Y)$, and let $\theta : X \to Y$ be a linear map. Let $C(S,T)\theta = S\theta - \theta T$ and define recursively

$$C^n(S,T)\theta = C^{n-1}(S,T)(S\theta - \theta T) = \sum_{k=0}^{n} \binom{n}{k} S^k \theta (-T)^{n-k}$$

for all n. If

$$\|C^n(S,T)\theta\|^{1/n} \to 0 \quad \text{as } n \to \infty, \tag{2.2}$$

we shall say that θ is a generalized intertwining of T with S. Clearly (2.2) contains the tacit assumption that $C^n(S,T)\theta$ is continuous for some, and hence almost all, n.

If $X = Y$ and $S = T$, then we write $C^n(T)$ for $C^n(T,T)$. According to [**4**, p. 26], T is called well-decomposable (WD) if for each open cover $\{U, V\}$ of C there exist $P \in L(X)$ and T-invariant subspaces Y and Z such that $PX \subset Y$, $(I - P)X \subset Z$, $\sigma(T|Y) \subset U$, $\sigma(T|Z) \subset V$ and $C^n(T)P = 0$ for some n.

The question of automatic continuity is whether a generalized intertwining of T with S is continuous under certain conditions. This question is intimately related to the notion of "algebraic spectral subspace" [**11**]. For $T \in L(X)$, define $E_T(F)$ to be the maximal linear manifold $M \subset X$ such that $(\lambda - T)M = M$ for all $\lambda \notin F$; $E_T(F)$ is called an "algebraic spectral subspace" of T. In general, $E_T(F)$ need not be closed in X even if F is closed and T is decomposable [**13**], but we will be especially interested in the case where it is closed.

We say that $T \in L(X)$ is admissible [**13**, **15**] if $E_T(F)$ is closed whenever F is closed; T is said to be subadmissible [**13**] if there exist admissible $S \in L(Z)$ for some Banach space Z and a continuous linear injection $i : X \to Z$ with closed range such that $Si = iT$.

Proposition 2.7. *"Admissible" and "subadmissible" are equivalent.*

Proof. Obviously we need only show that every subadmissible operator is admissible. Let T be subadmissible with S, Z and i as above. Let $Y = i(X)$. Then Y is a closed subspace of Z and S-invariant because of $S|Y = iTi^{-1}$. As S is admissible, $E_S(F)$ is closed and hence $E_S(F) = Z(S, F)$ for every closed F by [**15**, p. 214]. By Lemma 2.6 $Y(S|Y, F)$ is closed. Moreover, S has the SVEP by Proposition 2.1 and hence $E_{S|Y}(F) = Y(S|Y, F)$ by [**13**, Lemma 1]. Then it is closed and so $S|Y$ is admissible. Hence T is also admissible because $T = i^{-1}(S|Y)i$. □

Remark. Proposition 2.7, which improves [**13**, Proposition 2], can be proved in another way. By [**11**, Corollary 3.6], if $E_S(F)$ is closed for every closed F, then S has the SVEP. Applying [**15**, p. 214], [**19**, Lemma 1.9] and [**13**, Lemma 1], we can prove that $S|Y$ is admissible, and hence also T.

We close this section with some other terminology used below. The operator T is algebraic if $p(T) = 0$ for some nonzero polynomial $p(\lambda)$. We say that the operator pair (T, S) has critical eigenvalue λ if λ is an eigenvalue of S and the dimension of the factor space $X/(\lambda - T)X$ is infinite.

3. Weak decomposability relative to the identity.

Definition 3.1. Let $T \in L(X)$. Then T is weakly decomposable relative to the identity if for each finite open cover $\{G_i : 1 \leq i \leq n\}$ of the complex plane

(i) there exist T-invariant subspaces $M_1, M_2, \ldots, M_n$ such that $\sigma(T|M_i) \subset G_i$ for each i and

(ii) for each pair $(j, i), i = 1, 2, \ldots, n,\ j = 1, 2, \ldots$, there is an operator P_{ji} in the commutant $\{T\}'$ such that

(a) $I = \text{WOT} - \lim_j(\sum_i P_{ji})$,

(b) $P_{ji}X \subset M_i$ $(1 \leq i \leq n,$ all $j)$ where in (a) the limit is that in the weak operator topology of L(X). For brevity we call T a WDI operator on X.

Theorem 3.2. *Every* WDI *operator has* SVEP.

Proof. Let T be WDI on X, and let $f : D \to X$ be analytic satisfying $(\lambda - T)f(\lambda) = 0$ for all $\lambda \in D$. We may clearly suppose that D is connected.

Next let G_1, G_2 be disjoint open discs in D, and let H_1 be open such that $\{G_1, H_1\}$ covers $\mathbf{C}$ and $G_1 \setminus H_1^- \neq \varnothing$. By Definition 3.1 there are T-invariant subspaces M_1, N_1 such that

$$\sigma(T|M_1) \subset G_1 \text{ and } \sigma(T|N_1) \subset H_1$$

and sequences $P_j, Q_j \in \{T\}'$ such that

$$\langle x - P_j x - Q_j x, u\rangle < j^{-1} \text{ and } P_j X \subset M_1 \text{ and } Q_j X \subset N_1$$

for all $x \in X$, $u \in X^*$. Since $(\lambda - T)Q_j f(\lambda) = 0$ for λ in $G_1 \setminus H_1^-$, we clearly have $Q_j f(\lambda) = 0$ for all such λ. Hence $\langle f(\lambda) - P_j f(\lambda),\ u\rangle \to 0$ as $j \to \infty$, and thus $f(\lambda)$ lies in the weak closure of M_1. Because M_1 is convex, $f(\lambda) \in M_1$, so $f(\lambda) \in M_1$ for all $\lambda \in D$ by analytic continuation. We can similarly find M_2 with $\sigma(T|M_2) \subset G_2$ and $f(\lambda) \in M_2$ for all $\lambda \in D$. Since the set G_1 and G_2 are convex, $\sigma(T|M_1 \cap M_2) \subset G_1 \cap G_2 = \varnothing$, hence $M_1 \cap M_2$ is the zero subspace and $f = 0$ on D. This completes the proof. □

Theorem 3.3. *If T is* WDI *on X, then the manifold $X(T, F)$ is closed whenever F is closed.*

Proof. For λ in the complement of F define

$$G_\lambda = \{\mu : |\mu - \lambda| < (1/2)\,\mathrm{dist}(\lambda, F)\},$$
$$H_\lambda = \{\mu : |\mu - \lambda| > (1/3)\,\mathrm{dist}(\lambda, F)\}.$$

Since $\{G_\lambda, H_\lambda\}$ covers $\mathbf{C}$, let X_λ and Y_λ be T-invariant subspaces such that $\sigma(T|X_\lambda) \subset H_\lambda$ and $\sigma(T|Y_\lambda) \subset G_\lambda$, and let P_j, $Q_j \in \{T\}'$ satisfy Definition 3.1(ii). We prove $X(T,F) \subset X_\lambda$. For if $x \in X(T,F)$, then $\langle x - P_j x - Q_j x,\ u\rangle \to 0$ as $j \to \infty$ for all $u \in X^*$. But $Q_j x \in Y_\lambda$ for all j. Now $\sigma_T(x)$ exists because of Theorem 3.2, so for each j

$$\sigma_T(Q_j x) \subset \sigma_T(x) \cap G_\lambda \subset F \cap G_\lambda = \varnothing.$$

Hence $Q_j x = 0$ [**5**, p. 2] and thus x lies in the weak (and hence norm) closure of X_λ (as in Theorem 3.2). Moreover, we have $X(T,F) \subset \cap\{X_\lambda : \lambda \notin F\}$. On the other hand, if x_1 lies in the last intersection, then $\sigma_T(x_1) \subset H_\lambda$ (all $\lambda \notin F$). So $\sigma_T(x_1) \subset \cap\{H_\lambda : \lambda \notin F\} = F$, i.e. $x_1 \in X(T,F)$. Hence $X(T,F)$, as the intersection of closed subspaces, is closed itself. □

Remark. The reader may wonder why the results of §2 were not applied to prove Theorem 3.3. The reason is that the defining properties of WDI operators do not allow us to conclude directly that the manifolds X(T, F) are closed for closed F.

Corollary 3.4. *Every* WDI *operator is quasidecomposable.*

Proof. Let T be WDI on X. By [**7**] it suffices to show that X is closed span of $X(T, G_i^-)$ where $\{G_i: 1 \leq i \leq n\}$ is an open cover of $\mathbf{C}$. But Definition 3.1 shows that the manifold $X(T, G_1^-) + \cdots + X(T, G_n^-)$ is weakly dense in X, hence it is also norm dense. □

Example 3.5. In [**2**] E. Albrecht constructed a certain l^1 sum of function spaces, and he proved that multiplication by the independent variable is quasidecomposable on this space but not decomposable. Here we sketch a proof that Albrecht's example is in fact a WDI operator, i.e. there exist sufficiently many operators in its commutant to guarantee part (ii) of Definition 3.1.

Let D be the closed unit disc, and let $B^0(D)$ be the set of all continuous complex-valued functions on D. Now let $B^1(D)$ consist of those $f \in B^0(D)$ which have a $\overline{\partial}$-distributional derivative $g \in B^0(D)$ (here $\overline{\partial}$ is the differential operator $\overline{\partial} = (1/2)(\partial/\partial x + i\partial/\partial y)$). Next inductively define $B^j(D)$, $j = 2, 3, \ldots$, in a similar way. With the usual Montel norm each $B^j(D)$ is a Banach space. Let X be the l^1-direct sum of all $B^j(D)$, $j = 0, 1, \ldots$, and define T on X by formula

$$T(f_j(\lambda)) = (\lambda f_j(\lambda)), \quad \lambda \in D.$$

By a rather long argument Albrecht proved that T is quasidecomposable (see [**2**]).

To see that T is WDI, let $\{G_1, \ldots, G_m\}$ be an open cover of D (or $\mathbf{C}$) and choose a system of C^∞-functions $\{\phi_k\}$ with $\operatorname{supp}\phi_k \subset G_k^-$ and $\phi_1 + \phi_2 + \cdots + \phi_m = 1$ on D. For $(f_j) \in X$ and $n = 0, 1, \ldots$, define

$$f_j^{(n)} = \begin{cases} f_j & \text{if } j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

Then clearly $(f_j^{(n)}) \to (f_j)$ in the norm of X as $n \to \infty$. Also for $1 \leq k \leq m$ and $n = 0, 1, \ldots$, define the multiplier-truncation operator P_{kn} by $P_{kn}(f_j) = (\phi_k f_j^{(n)})$. By Albrecht's results $P_{kn} \in \{T\}'$ and

$$\left(\sum_{k=1}^{m} P_{kn}\right)(f_j) = (f_j^{(n)}) \to (f_j),$$

that is , the sums $\sum_k P_{kn}$ tend to the identity in the strong operator topology, so also (WOT). Finally, the condition $\operatorname{supp}\phi_k \subset G_k^-$ implies that $P_{kn}X \subset X(T, G_k^-)$ for each $k = 1, 2, \ldots$. Hence 3.1(ii) is satisfied, so T is weakly decomposable relative to the identity.

Example 3.6. In [**1**] Albrecht gave another example of an operator $T = S + Q$ where S is generalized scalar [**5**] and $Q \in \{S\}'$ and $Q^2 = 0$. In fact, he proved that each $V \in \{T\}'$ has zero square and all their products vanish. It follows that T is decomposable, but T is not WDI. For if Definition 3.1 were satisfied, then for $j = 1, 2, \dots$, there are $P_{j1}, P_{j2} \in \{T\}'$ such that $V_j = P_{j1} + P_{j2} \to I$ (WOT). For $j = k$ fixed, choose $x \in X$, $u \in X^*$ with $\langle V_k x,\ u\rangle = 1$. But then $0 = \langle V_j V_k x,\ u\rangle \to \langle V_k x,\ u\rangle = 1$ $(j \to \infty)$, so T is not WDI.

Examples 3.5 and 3.6 show that the classes of WDI and decomposable operators are not comparable by inclusion; this fact distinguishes WDI operators from all other previously studied classes of operators with spectral decomposition, which include well-decomposable operators.

It is easy to see that a WDI operator having property (δ) is decomposable. On the other hand, if T is the operator given by Albrecht in [**3**], then by the argument of [**3**, p. 12(B)] T has no nontrivial spectral maximal space contained in $X(T, [0,1])$, i.e. every spectral maximal space contained in $X(T, [0,1])$ is either $\{0\}$ or $X(T, [0,1])$ itself. Now if T is WDI, then the restriction $T|X(T, [0,1])$ is also WDI, so we would have the contradiction that T contains a proper spectral maximal subspace of $X(T, [0,1])$. This shows that the intersection of the classes WDI and (δ) is a proper subclass of decomposable operators.

We close this section with a more detailed discussion of the relation between WDI and WD operators; we first generalize slightly the notion of WDI.

Definition 3.7. We call $T \in L(X)$ weakly well-decomposable (WWD) if for each finite open cover $\{G_i :\ 1 \le i \le n\}$ of $\mathbf{C}$

(i) Definition 3.1(i) holds and

(ii) there exists a positive integer m such that for each pair $(j, i)(i = 1, 2, \dots, n;\ j = 1, 2, \dots)$ there is an operator P_{ji} satisfying $C^m(T)P_{ji} = 0$ and Definition 3.1(ii, a and b).

We claim that Theorem 3.2 and 3.3 remain true for WWD operators. For let T be a WWD operator on X. In the proof of Theorem 3.2 given above, we need only show that the equality $(\lambda - T)f(\lambda) = 0$ implies that $Q_j f(\lambda) = 0$ for $\lambda \in G_1 \setminus H_1^-$. From the equality $(\lambda - T)f(\lambda) = 0$ we have

$$\begin{aligned}(\lambda - T)^m Q_j f(\lambda) &= \sum_{k=0}^{m} (-1)^{m-k} T^{m-k} Q_j \lambda^k f(\lambda)\\ &= \sum_{k=0}^{m} (-1)^{m-k} T^{m-k} Q_j T^k f(\lambda)\\ &= C^m(T) Q_j f(\lambda) = 0\end{aligned}$$

for $\lambda \in G_1 \setminus H_1^-$. Then $Q_j f(\lambda) = 0$ for all such λ, since $(\lambda - T)^m$ is injective. Continuing the argument as in Theorem 3.2, we conclude that T has SVEP.

As for Theorem 3.3, we need only verify the inclusion $\sigma_T(Q_j x) \subset \sigma_T(x)$. Let x(.) be the local resolvent of T at x. Then

$$(3.1) \qquad (\lambda - T)x(\lambda) = x \quad (\lambda \notin \sigma_T(x)).$$

Differentiating (3.1) k times, we obtain

$$(3.2) \qquad (\lambda - T)x^{(k)}(\lambda) = -kx^{(k-1)}(\lambda).$$

Now define the following analytic function on $C \setminus \sigma_T(x)$:

$$y(\lambda) = \sum_{k=0}^{m-1} (-1)^k C^k(T) Q_j \frac{x^{(k)}(\lambda)}{k!}.$$

Then

$$\begin{aligned} Ty(\lambda) &= \sum_{k=0}^{m-1} (-1)^k T C^k(T) Q_j \frac{x^{(k)}(\lambda)}{k!} \\ &= \sum_{k=0}^{m-1} (-1)^k C^{k+1}(T) Q_j \frac{x^{(k)}(\lambda)}{k!} + \sum_{k=0}^{m-1} (-1)^k C^k(T) Q_j T \frac{x^{(k)}(\lambda)}{k!}, \end{aligned}$$

hence for $\lambda \notin \sigma_T(x)$, from (3.2) and $C^m(T)Q_j = 0$,

$$\begin{aligned} (\lambda - T)y(\lambda) &= \sum_{k=0}^{m-1} (-1)^k C^k(T) Q_j (\lambda - T) \frac{x^{(k)}(\lambda)}{k!} \\ &\quad - \sum_{k=0}^{m-1} (-1)^k C^{k+1}(T) Q_j \frac{x^{(k)}(\lambda)}{k!} \\ &= Q_j(\lambda - T)x(\lambda) + \sum_{k=1}^{m-1} (-1)^{k+1} C^k(T) Q_j \frac{x^{(k-1)}(\lambda)}{(k-1)!} \\ &\quad + \sum_{k=0}^{m-1} (-1)^{k+1} C^{k+1}(T) Q_j \frac{x^{(k)}(\lambda)}{k!} = Q_j x, \end{aligned}$$

which proves the result. Thus X(T, F) is closed for all closed F.

The following example shows that there are WWD operators which are not WDI.

Example 3.8. Let T_1 be the operator defined in [**4**, p. 26(2)] and denote the domain of T_1 by Y. Then it was proved in [**4**, pp. 26-27] that T_1 is WD but not decomposable relative to the identity. Now let T_2 be the operator in Ex. 3.5 acting on Z. Then $T = T_1 \oplus T_2$ on $X = Y \oplus Z$ is WWD, but we now show that T is not WDI. By [**4**, p. 27] $Y = Y_0 \oplus Y_1$ (in [**4**, p. 27], Y, Y_0, Y_1 are denoted X, X_0, X_1 resp.). Suppose that T is WDI. By Definition 3.1, there exists $\{P_j\} \subset \{T\}'$ such that each $x \in X$ with $\sigma_T(x) \subset \{\lambda : |\lambda| \leq 1/3\}$ can be realized as

$$\text{WOT} - \lim_{j \to \infty} P_j x = x, \tag{3.3}$$

and all $x \in X$ with $\sigma_T(x) \subset \{\lambda : |\lambda| \geq 2/3\}$ satisfies

$$P_j x = 0. \tag{3.4}$$

Let $Q \in L(X)$ be the projection of X onto Y along Z, and put $R_j = QP_jQ|Y$. Then R_j commutes with T_1, and hence by [**1**, Prop. 2.6]

$$R_j(f,g) = (a_j f + b_j \partial g + c_j g, a_j g) \text{ for all } (f,g) \in Y, \tag{3.5}$$

where $a_j, b_j, c_j \in Y_0$ and a_j is analytic on the interior of the unit disc D. If $(0,g) \in Y$ with $\sigma_{T_1}((0,g)) \subset \{\lambda : |\lambda| \geq 2/3\}$, then $R_j(0,g) = 0$ by (3.4); hence $a_j = 0$ by analyticity. It follows from (3.5) that

$$R_j(f,g) = (b_j \partial g + c_j g, 0) \text{ for all } (f,g) \in Y. \tag{3.6}$$

Choose now $(f,0) \in Y$ with $\sigma_{T_1}((f,0)) \subset \{\lambda : |\lambda| \leq 1/3\}$ and $f \neq 0$. Then (3.3) and (3.6) imply that

$$(f,0) = \text{WOT} - \lim_{j \to \infty} R_j(f,0) = 0.$$

This contratiction proves that T is not WDI. But T cannot be WD either because T_2 is not decomposable by Ex. 3.5.

The previous example shows that the class of WWD operators properly contains the class of WDI operators as well as the WD operators.

4. Automatic continuity.

Theorem 4.1. *Let $T \in L(X)$ have property (δ) and let $S \in L(Y)$ be* WDI. *Then* (i) *and* (ii) *are equivalent.*

(i) *Every linear $\theta : X \to Y$ that is a generalized intertwining of T with S is bounded.*

(ii) *(T,S) has no critical eigenvalue and either T is algebraic or $E_S(\varnothing) = \{0\}$.*

The implication (i) $\Rightarrow$ (ii) always holds by [**17**, Lemma 3.2 and Theorem 3.6]. The proof of the converse relies on the following lemmas.

Lemma 4.2. *If T is algebraic, then $\sigma(T)$ is finite, hence T is decomposable.*

Proof. Let $p(\lambda)$ be a nonzero polynomial such that $p(T) = 0$. The spectral mapping theorem implies that $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\} = \sigma(p(T)) = \{0\}$, hence $\sigma(T)$ is contained in the finite set of zeros of $p(\lambda)$. □

The next lemma is an analog of [**14**, Proposition 4].

Lemma 4.3. *Suppose S is* WDI *on X and F is closed in* **C**. *Then*

$$(4.1)\ \cap_{V\supset F}[X(S,V^-) + E_S(\varnothing)] \subset E_S(F) \subset \cap_{V\supset F}[X(S,V^-) + E_S(\varnothing)].$$

Proof. Since $X(S,V^-)+E_S(\varnothing) \subset E_S(V^-)$ and since $\cap_{V\supset F} E_S(V^-) = E_S(F)$ by [**11**, Theorem 2.6], the first inclusion in (4.1) is immediate. To prove the second inclusion, let V be open with $F \subset V$ and let U be open such that $F \subset U \subset U^- \subset V$. Let $P_j, Q_j \in \{S\}'$ be operators satisfying Definition 3.1(ii) for the cover $\{V, \mathbf{C}\backslash U^-\}$ of **C**, i.e. $P_j + Q_j \to I(\text{WOT})$ and $P_j X \subset X(S,V^-)$ and $Q_j X \subset X(S, \mathbf{C} \setminus U)$ for all j. For $\lambda \notin F$,

$$(\lambda - S)Q_j E_S(F) = Q_j E_S(F),$$

so we obtain $Q_j E_S(F) \subset E_S(F)$ by the maximality of $E_S(F)$. On the other hand,

$$Q_j E_S(F) \subset Q_j X \subset X(S, \mathbf{C} \setminus U) \subset E_S(\mathbf{C} \setminus U).$$

From [**11**, Theorem 2.6] again,

$$Q_j E_S(F) \subset E_S(F) \cap E_S(\mathbf{C} \setminus U) = E_S(\varnothing).$$

Thus for each $x \in E_S(F)$ and $u \in X^*$, the equality

$$\langle x, u\rangle = \lim \langle (P_j + Q_j)x, u\rangle$$

implies that $x \in [X(S,V^-) + E_S(\varnothing)]^-$. This proves the second inclusion.

□

Corollary 4.4. *Let S be* WDI. *If $E_S(\varnothing) = \{0\}$, then $E_S(F) = X(S,F)$ for each closed F, and thus S is admissible.*

Proof. The condition $E_S(\varnothing) = \{0\}$ together with (4.1) shows $E_S(F) = X(S,F)$. So Theorem 3.3 shows that S is admissible. □

Proof of (ii) $\Rightarrow$ (i) *of Theorem* 4.1. If (T, S) has no critical eigenvalue and $E_S(\varnothing) = \{0\}$, we use Corollary 4.4 and [**13**, Corollary 9] to reach (i). If (T, S) has no critical eigenvalue and T is algebraic, then we may assume that S has a nontrivial divisible subspace (otherwise we are reduced to the case $E_S(\varnothing) = \{0\}$), hence we may use Lemma 4.2 and [**15**, Corollary 3.3] to reach (i). □

Corollary 4.5. *Assume $T \in L(X)$ has property (δ) and S is the* WDI *operator in Example 3.5. Then any linear generalized intertwining of T with S is continuous.*

Proof. By Theorem 4.1 it suffices to prove that (T, S) has no critical eigenvalue and $E_S(\varnothing) = \{0\}$. The former is evident since S has no eigenvalues. To prove the latter, let L be any linear manifold such that $(\mu - S)L = L$ for all μ. We prove $L = \{0\}$. Let $|\mu| < 1$. If $y \in L, y \neq 0$, then $y = (f_j)$ with $f_j \in B^j(D)$, and there is $u \in L$ with $u = (g_j)$ and $y = (\mu - S)u$. Hence $(\mu - S)g_j(z) = f_j(z)$ for all j and $|z| \leq 1$. We may suppose that $f_k(\lambda) = 1$ for some k and $|\lambda| \leq 1$. But then g_k is discontinuous at $z = \lambda$, contradicting the construction of S. □

Remark 1. Lemma 4.3 remains valid for WWD operators. Let S be such an operator, and let F, U, V be the sets in Lemma 4.3. Assume P_j, Q_j are the operators satisfying Definition 3.7(ii). Then $P_j + Q_j \to I(\mathrm{WOT}), P_jX \subset X(S, V^-)$ and $Q_jX \subset X(S, \mathbf{C} \setminus U)$ for all j. By [**15**, Cor. 1.2] $Q_jE_S(F) \subset E_S(F)$, hence $Q_jE_S(F) \subset X(S, \mathbf{C} \setminus U) \cap E_S(F) \subset E_S(\mathbf{C} \setminus U) \cap E_S(F) = E_S(\varnothing)$. Hence the second inclusion in (4.1) follows, and since the first inclusion is evident, we conclude that Lemma 4.3, Theorem 4.1 and Cor. 4.4 all hold for WWD operators.

Remark 2. In [**9**, Theorem 1] and [**10**, Theorem 1.2.1] the authors proved the following theorem.

Theorem 4.6. *Let $T \in L(X)$. Then T is decomposable if and only if for every open cover $\{G, H\}$ of $\mathbf{C}$ there is a linear map $P : X \to X$ such that*

(i) *if $x \in X$, then $\sigma_T(Px) \subset G^-$ and $\sigma_T(x - Px) \subset H^-$;*

(ii) *if F is closed in $G \setminus H^-$ and $x \in X(T, F)^-$, then $Px = x$;*

(iii) *if K is closed in $H \setminus G^-$ and $x \in X(T, K)^-$, then $Px = 0$.*

According to [**13**, p. 329] it is still an open question whether a decomposable operator with no nontrivial divisible subspaces is admissible. But if the linear map P in Theorem 4.6 commutes with T, then Lemma 4.3 and hence Corollary 4.4 remain valid: T is admissible. Here the problem reduces to that of constructing a P satisfying (i)-(iii) of Theorem 4.6 and commuting with T.

5. Further considerations.

In this section we give several applications of the foregoing results. We say that an operator is "weakly subdecomposable relative to the identity" if it is the restriction of a WDI operator to an invariant subspace; "subscalar" operator is a similar restriction of a generalized scalar operator.

Theorem 5.1. *Assume $T \in L(X)$ has property (δ) and $S \in L(Y)$ is weakly subdecomposable relative to the identity. If (T, S) has no critical eigenvalue, $E_V(\varnothing) = \{0\}$ and $S = V|Y$ for some* WDI *operator $V \in L(W)$, then every generalized intertwining of T with S is continuous.*

Proof. Corollary 4.4 and Proposition 2.7 imply that S is admissible. The theorem now follows from [**13**, Corollary 9]. □

Corollary 5.2. *Assume T has property (δ) and S is subscalar. If S has no eigenvalue, then every generalized intertwining of T with S is continuous. In particular, if S is hyponormal with no eigenvalue, then every intertwining of T with S is continuous.*

Proof. Let $V \in L(W)$ be a generalized scalar extension of S. Then V is WDI [**5**, p. 94] and $E_V(\varnothing) = \{0\}$. The last equality follows from a result of Vrbova [**18**] that

$$W(V,F) = \cap\{(\lambda - V)^p W : \lambda \notin F\} \tag{5.1}$$

for some fixed $p > 0$. Let Z be a linear manifold with $(\lambda - V)Z = Z$ for all λ. Then

$$Z = (\lambda - V)^p Z \subset (\lambda - V)^p W \quad (\lambda \in \mathbf{C}).$$

Hence $Z \subset W(V, \varnothing)$ by (5.1). This shows $E_S(\varnothing) = \{0\}$, so the conclusion follows from Theorem 5.1. The last assertion follows from Putinar's theorem [**16**] that hyponormal operators are subscalar.

Our final result requires knowing that the generalized scalar operator T is "completely regular" [**5**, p. 110] if for every $A \in L(X)$ the condition $AX(T, F) \subset X(T, F)$ for each closed F implies that A commutes with one of the spectral distributions of T (hence A and T commute). □

Proposition 5.3. *Let $T \in L(X)$ be a completely regular generalized scalar operator, and suppose that T has no eigenvalues. If θ is a generalized intertwining of T, then θ commutes with T.*

Proof. By Corollary 5.2 $\theta \in L(X)$, hence by [**5**, Theorem 4.4.5] $\theta X(T, F) \subset X(T, F)$ for every closed F in **C**. Thus $\theta E_T(F) \subset E_T(F)$ for every such

F, because by Corollary 4.4 and previous proof the algebraic and analytic spectral manifolds of a generalized scalar operator agree on the closed sets in **C** (hence $E_T(\varnothing) = \{0\}$). The complete regularity of T yields the desired conclusion. □

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