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ON H^P-SOLUTIONS OF THE BEZOUT EQUATION

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We obtain a sufficient condition on bounded holomorphic functions g_1, g_2 in the unit disk for the existence of f_1, f_2 in the Hardy space H^p such that $1 = f_1 g_1 + f_2 g_2$. The sharpness of this condition is also studied.

1. Let $\mathbb{D}$ be the unit disk in the complex plane, $\mathbb{T}$ its boundary. For $1 \leq p \leq \infty$, H^p denotes the Hardy-space of holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_p = \sup_r \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} < +\infty \qquad p < \infty$$

$$\|f\|_\infty = \sup_{|z|<1} |f(z)| .$$

It is well-known ([**7**, p. 57]) that if $f \in H^p$, the non-tangential maximal function

$$Mf(e^{i\theta}) = \sup\{|f(z)|;\ z \in \Gamma(\theta)\}$$

$\Gamma(\theta)$ being the Stolz angle with vertex $e^{i\theta}$, belongs to $L^p(\mathbb{T})$.

In this paper, given $g_1, g_2 \in H^\infty$, we study the Bezout equation $1 = f_1 g_1 + f_2 g_2$. Concretely, we are interested in knowing the precise condition on g_1, g_2 so that solutions $f_1, f_2 \in H^p$ exist.

If $|g|^2 = |g_1|^2 + |g_2|^2$, $|f|^2 = |f_1|^2 + |f_2|^2$, it follows from $1 = f_1 g_1 + f_2 g_2$ that $1 \leq |f|\,|g|$ and hence

$$\text{(C)} \qquad M(|g|^{-1}) \in L^p(\mathbb{T}) .$$

It can be easily seen that this condition is sufficient if g_1 or g_2 is an interpolating Blaschke product. Nevertheless, we show in Section 2 that it is not sufficient in general. In fact for each $\varepsilon > 0$ we exhibit $g_1, g_2 \in H^\infty$ such that $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ but no H^p solutions exist.

In Section 3 we obtain a general sufficient condition implying in particular the following:

Theorem 1. *Assume that for some $\varepsilon > 0$*

$$M\left(|g|^{-2}\,|\log|g||^{2+\varepsilon}\right) \in L^p(\mathbb{T}) .$$

Then there exist $f_1, f_2 \in H^p$ *such that* $1 = f_1 g_1 + f_2 g_2$.

In Section 4, it is shown that the same method gives the following improvement on the problem considered by Wolff and also by Cegrell in [**4**].

Theorem 2. *Let* $f, g_1, g_2 \in H^\infty$ *be such that*

$$|f| \leq \frac{|g|^2}{|\log|g||^{2+\varepsilon}} \qquad \text{for some} \quad \varepsilon > 0.$$

Then there are $f_1, f_2 \in H^\infty$ *such that* $f = f_1 g_1 + f_2 g_2$.

The proofs rely essentially on: (a) An L^p-version of Wolff's criteria for the existence of bounded solutions of the $\overline{\partial}$-equation, already used in [**1**]. (b) An improvement of Cegrell's result in [**4**] on gradients of bounded holomorphic functions.

Both theorems hold of course for more than two generators, using the Koszul complex technique as in [**7**, p. 364]. Theorem 1 holds as well in the setting of the unit ball, but some modifications are needed (see [**2**]).

Finally, we mention that similar results to those stated here have been independently obtained by K.C. Lin in [**8**] and [**9**]. The authors thank the referee for pointing this out to them.

2. Before proceeding, we recall that a positive measure μ on $\mathbb{D}$ is called a *Carleson measure* if there exists $K > 0$ such that

$$\mu(\{z : |z - e^{i\theta}| \leq r\}) \leq Kr \qquad e^{i\theta} \in \mathbb{T}, \quad r > 0.$$

The smallest of such K is called the Carleson norm of μ. Equivalently ([**7**, p. 32]) μ is a Carleson measure if and only if for all functions h in $\mathbb{D}$

$$\iint_{\mathbb{D}} |h| d\mu \leq c \int_0^{2\pi} M(h) d\theta.$$

In some particular cases it is quite easy to see that the condition (C) is sufficient. For instance, if g_1 is a Blaschke product with zeros z_n, the question is obviously equivalent to the interpolation problem

$$f_2(z_n) = \frac{1}{g_2(z_n)}, \quad \text{with} \quad f_2 \in H^p.$$

Indeed, $1 - f_2 g_2$ belongs then to H^p and vanishes on $\{z_n\}$, so $1 - f_2 g_2 = f_1 g_1$, $f_1 \in H^p$. In case g_1 is an interpolating Blaschke product, this interpolation problem has a solution if and only if

$$\sum_n \frac{1}{|g_2(z_n)|^p}(1 - |z_n|) < +\infty,$$

(see [**10**] and also [**5**]). Let δ_n denote the delta-mass at the point z_n. Since $\sum(1-|z_n|)\delta_n$ is a Carleson measure, (C) implies the above condition.

Next, we give examples showing that condition (C) is far from being sufficient.

Theorem 3. *Given* $1 \le p < \infty$ *and* $\varepsilon > 0$, *there exist bounded analytic functions* g_1, g_2 *with* $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$, *such that there exist no* $f_1, f_2 \in H^p$ *satisfying* $f_1 g_1 + f_2 g_2 \equiv 1$.

Proof. We will denote by $\rho(z,w)$ the pseudo-hyperbolic distance in the unit disc, $\rho(z,w) = |z-w|\,|1-\overline{w}z|^{-1}$, $z, w \in \mathbb{D}$ and $f^{(j)}$ the j-th derivative of a function f. Let N be a natural number such that $(N+1)\varepsilon > 1$.

Let $z_n = 1 - 2^{-n}$, $n \ge 1$, and take an H^∞-interpolating sequence $\{\alpha_n\}$, $0 < \rho(\alpha_n, z_n)$ decreasing to 0, satisfying

$$\sum_n (1-|z_n|)\rho(\alpha_n, z_n)^{-(N+1)p(2-\varepsilon)} < \infty\,, \tag{1}$$

$$\sum_n (1-|z_n|)\rho(\alpha_n, z_n)^{-(2N+1)p} = \infty\,. \tag{2}$$

Let B_1 and B_2 be the Blaschke products with zeros $\{z_n\}$ and $\{\alpha_n\}$. From now on, the letter c will denote different constants independent on n. Since B_2 is an interpolating Blaschke product, one has

$$\inf_n \rho(z, \alpha_n) \ge |B_2(z)| \ge c \inf_n \rho(z, \alpha_n)\,, \quad |z| < 1\,,$$

(see [**7**, p. 404]).

Now as in [**3**] take $g_i = B_i^{N+1}$, $i = 1, 2$. Let I_n be the arc on the unit circle centered at 1 of length $2(1-|z_n|) = 2^{-n+1}$. In estimating $|g(z)|^{-1}$, for $z \in \Gamma(\theta)$, the worst case occurs when z is one of the $\{z_n\}$ or $\{\alpha_n\}$. Since $\rho(\alpha_n, z_n)$ is decreasing, for $e^{i\theta} \in I_n \setminus I_{n+1}$ one has

$$M(|g|^{-1}) \le \frac{c}{\rho(\alpha_{n+1}, z_{n+1})^{N+1}}\,.$$

Hence, condition (1) implies $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$. Now, assume there exist $f_1, f_2 \in H^p$ satisfying $f_1 g_1 + f_2 g_2 \equiv 1$. Then,

$$f_2^{(N)}(z_n) = (B_2^{-N-1})^{(N)}(z_n)\,, \quad n \ge 1\,.$$

Write

$$B_{2,n}(z) = \prod_{k \ne n} \frac{-\overline{\alpha}_k}{|\alpha_k|} \frac{z-\alpha_k}{1-\overline{\alpha}_k z}\,, \qquad B_2^{-N-1}(z) = h(z)k(z)$$

where $h(z) = (1-\overline{\alpha}_n z)^{N+1} B_{2,n}(z)^{-N-1}$, $k(z) = (z-\alpha_n)^{-N-1}$. Then

$$(B_2^{-N-1})^{(N)}(z) = \sum_{j=0}^{N} \binom{N}{j} h^{(j)}(z) k^{(N-j)}(z) .$$

Using Cauchy's formula on the disk of center z_n and radius $4^{-1}(1-|z_n|)$, one gets

$$|h^{(j)}(z_n)| \le c \frac{(1-|\alpha_n|)^{N+1}}{(1-|z_n|)^j} \le c(1-|z_n|)^{N+1-j}$$

and hence

$$|h^{(j)}(z_n)|\,|k^{(N-j)}(z_n)| \le c\,|z_n-\alpha_n|^{-2N+j-1}(1-|z_n|)^{N+1-j}$$

$$\le c\rho(z_n,\alpha_n)^{-2N+j-1}(1-|z_n|)^{-N} .$$

For $j=0$, one gets

$$|h(z_n)|\,|k^{(N)}(z_n)| \ge c\,(1-|z_n|)^{N+1}|z_n-\alpha_n|^{-2N-1}$$

$$\ge c\rho(\alpha_n,z_n)^{-2N-1}(1-|z_n|)^{-N} .$$

Therefore, for large n,

$$(3) \qquad |f_2^{(N)}(z_n)| = |(B_2^{-N-1})^{(N)}(z_n)| \ge c\,(1-|z_n|)^{-N}\rho(\alpha_n,z_n)^{-2N-1} .$$

Since $f_2 \in H^p$, the function

$$F(e^{i\theta}) = \left(\int_{\Gamma(\theta)} |f_2^{(N)}(z)|^2(1-|z|)^{2N-2}dm(z)\right)^{1/2}$$

belongs to $L^p(\mathbb{T})$ ([**11**, p. 216]). For $e^{i\theta} \in I_n \setminus I_{n+1}$, since $D_n = \{z \in \mathbb{D} : |z-z_n| < 4^{-1}(1-|z_n|)\} \subset \Gamma(\theta)$, one has

$$|F(e^{i\theta})|^2 \ge \int_{D_n} |f_2^{(N)}(z)|^2(1-|z|)^{2N-2}dm(z)$$

$$\ge c\,(1-|z_n|)^{2N}|f_2^{(N)}(z_n)|^2, \quad e^{i\theta} \in I_n \setminus I_{n+1} .$$

Using (3) and $F \in L^p(\mathbb{T})$, one gets

$$\infty > \sum (1-|z_n|)\rho(\alpha_n,z_n)^{-(2N+1)p}$$

and this contradicts (2). □

3. In this section we will prove a generalization of Theorem 1 stated in the introduction.

Lemma 1. *If g is holomorphic on $\overline{\mathbb{D}}$ and $0 < p \le 2$,*

$$\int_0^{2\pi} \{|g(e^{i\theta})|^p - |g(0)|^p\} \frac{d\theta}{2\pi} \le 4 \int_0^{2\pi} |g(e^{i\theta}) - g(0)|^p \frac{d\theta}{2\pi}.$$

Proof. This is a general statement for a probability measure $d\mu$ on X and measurable φ

$$\int_X |\varphi|^p \, d\mu - \left| \int_X \varphi \, d\mu \right|^p \le 4 \int_X \left| \varphi - \int_X \varphi \, d\mu \right|^p d\mu.$$

First notice that this is trivial for $p \le 1$ (with constant 1) and that for $p = 2$ there is equality with constant 1, too. In general, and assuming without loss of generality that $\int_X \varphi \, d\mu = 1$, it follows for real φ integrating the inequality

$$|\varphi|^p - 1 \le 3|\varphi - 1|^p + p(\varphi - 1).$$

For complex-valued $\varphi = \varphi_1 + i\varphi_2$, it follows from $|\varphi|^p \le |\varphi_1|^p + |\varphi_2|^p$ (for positive φ the inequality holds with constant 1). □

We start with a generalization of a result in [**4**]. Although we need it only for H^∞ functions we state it in full generality, for $BMOA$ functions (see [**7**] for definitions). We denote by $\|g\|_*$ the BMO norm of $g(e^{i\theta})$.

Let $d\mu$ be a positive measure on $[0,1)$ such that $\int_0^1 \frac{d\mu(\alpha)}{\alpha^2} < +\infty$, and write

$$\widetilde{\mu}(x) = \int_0^1 x^\alpha d\mu(\alpha), \quad x > 0.$$

Lemma 2. *If $g \in BMOA$, $\frac{|g'|^2}{|g|^2}\widetilde{\mu}(|g|^2)(1 - |z|^2)$ is a Carleson measure with Carleson norm bounded by $K\|g\|_*$, K depending on μ.*

Proof. We consider the function

$$G(z) = \int_0^1 \frac{|g(z)|^{2\alpha}}{\alpha^2} d\mu(\alpha), \qquad |z| < 1.$$

For $\alpha > 0$, a computation shows that $\Delta |g|^{2\alpha} = 4\alpha^2 |g'|^2 \, |g|^{2\alpha - 2}$ when $g \ne 0$, hence

$$\Delta G = 4|g'|^2 \, |g|^{-2} \widetilde{\mu}(|g|^2).$$

Without loss of generality we can assume that g is holomorphic on $\overline{\mathbb{D}}$. We argue like in [7, p. 327]. Let $z_1, \dots, z_N$ be the non-zero zeros in $\mathbb{D}$, and let Ω_ε be the domain $\mathbb{D} \setminus \bigcup_{j=0}^{N} \Delta_j$ where $\Delta_0 = \{|z| \le \varepsilon\}$, $\Delta_j = \{|z - z_j| \le \varepsilon\}$, $j = 1, \dots, N$.

By Green's formula applied to the function G in Ω_ε

$$4 \iint_{\Omega_\varepsilon} |g'|^2 \, |g|^{-2} \widetilde{\mu}(|g|^2) \log \frac{1}{|z|} dA(z) = \int_0^{2\pi} G(e^{i\theta}) d\theta$$

$$- \sum_{j=0}^{N} \int_{\partial \Delta_j} \left(\frac{\partial}{\partial n} G \right) \log \frac{1}{|z|} - |G| \frac{\partial}{\partial n} \left(\log \frac{1}{|z|} \right) ds\,.$$

Let $r = |z - z_j|$; then for z close to z_j,

$$|g'(z)|^2 |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \le C r^{-2} \widetilde{\mu}(r^2)\,.$$

Now, the hypothesis on μ translates to

$$\int_0^1 \frac{\widetilde{\mu}(x)}{x} |\log x| dx < +\infty\,.$$

Hence $|g'|^2 |g|^{-2} \widetilde{\mu}(|g|^2) \log \dfrac{1}{|z|}$ is integrable on $\mathbb{D}$. Also, for $|z - z_j| = \varepsilon$

$$|G(z)| \le c \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha^2} \, d\mu(\alpha)$$

which tends to 0 when $\varepsilon \to 0$, and

$$|\nabla G(z)| \le \frac{c}{\varepsilon} \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha} \, d\mu(\alpha)$$

which also tends to zero when multiplied by $\varepsilon |\log \varepsilon|$. At zero we obtain $-2\pi |G(0)|$ as limit when $\varepsilon \to 0$. Therefore

$$\frac{2}{\pi} \iint_{\mathbb{D}} |g'(z)|^2 \, |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \log \frac{1}{|z|} \, dA(z)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{|g(e^{i\theta})|^{2\alpha}}{\alpha^2} \, d\mu(\alpha) \, d\theta - \int_0^1 \frac{|g(0)|^{2\alpha}}{\alpha^2} \, d\mu(\alpha)$$

$$= \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} \{ |g(e^{i\theta})|^{2\alpha} - |g(0)|^{2\alpha} \} \frac{d\theta}{2\pi} \le \quad \text{(by Lemma 1)}$$

$$\le 4 \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} |g(e^{i\theta}) - g(0)|^{2\alpha} \frac{d\theta}{2\pi} \le \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \, (c\|g\|_*)^{2\alpha}\,.$$

If ψ_w is an automorphism of the disc, applying this inequality to $g \circ \psi_w$, changing variables in the area integral and using the invariance of the BMO norm we get

$$\sup_{w\in\mathbb{D}} \iint_{\mathbb{D}} |g'(z)|^2 |g(z)|^{-2} \tilde{\mu}(|g(z)|^2) \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}\, dA(z)$$

$$\leq c \int_0^1 \frac{\|g\|_*^{2\alpha}}{\alpha^2}\, d\mu(\alpha) < +\infty\,,$$

and the result follows, by [**7**, p. 239]. □

Taking for μ a delta-mass at ε we get Cegrell's result [**4**] that $|g'|^2|g|^{\varepsilon-2}(1-|z|^2)$ is a Carleson measure. Taking $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$ one gets that

$$\frac{|g'|^2}{|g|^2 |\log|g||^{2+\varepsilon}} (1-|z|^2)$$

is a Carleson measure for every $\varepsilon > 0$.

Next lemma is the L^p-version of Wolff's criteria for bounded solutions of the $\overline{\partial}$-equation ([**7**, p. 322]).

Lemma 3. *Let $1 \leq p \leq \infty$, let G be a C^1 function in $\overline{\mathbb{D}}$ such that:*

(a) *$G = \varphi_1 \psi_1$, where $M(\varphi_1) \in L^p$ and $|\psi_1|^2 \log \frac{1}{|z|}$ is a Carleson measure.*

(b) *$\partial G = \varphi_2 \psi_2$, where $M(\varphi_2) \in L^p$ and $|\psi_2| \log \frac{1}{|z|}$ is a Carleson measure.*

Then there exists a C^1 function u in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ such that

$$\frac{\partial u}{\partial \overline{z}} = G$$

and

$$\int_0^{2\pi} |u(e^{i\theta})|^p \, d\theta \leq C$$

where C depends only of the L^p-norms of $M(\varphi_1), M(\varphi_2)$ and the Carleson norms of the measures in (a), (b).

Proof. We adapt Wolff's proof for the case $p = \infty$. Let q be the conjugate exponent of p, $1 < q \leq \infty$. By duality,

$$\inf\left\{ \|b\|_p : \frac{\partial b}{\partial \overline{z}} = G \right\} = \sup\left\{ \left| \frac{1}{2\pi} \int_0^{2\pi} F k d\theta \right| : k \in H_0^q,\ \|k\|_q \leq 1 \right\}$$

where F is a priori solution, say the one given by the Cauchy kernel, which is continuous on $\overline{\mathbb{D}}$. By Green's formula

$$\frac{1}{2\pi}\int_0^{2\pi} Fk d\theta = \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta(Fk)\log\frac{1}{|z|}dA(z)$$

$$= \frac{2}{\pi}\iint_{\mathbb{D}} k'(z)G(z)\log\frac{1}{|z|}\,dA(z) + \frac{2}{\pi}\iint_{\mathbb{D}} k(z)\frac{\partial G}{\partial z}\log\frac{1}{|z|}dA(z) = I_1 + I_2\,.$$

We will prove now that if $|\psi|^2\log\dfrac{1}{|z|}$ is a Carleson measure with constant K, then

$$\text{(4)}\qquad \iint_{\mathbb{D}} |k'(z)|\,|\varphi(z)|\,|\psi(z)|\,\log\frac{1}{|z|}\,dA(z) \le C\|k\|_q\|M\varphi\|_p K$$

where C is an absolute constant. This will imply the required bound for I_1. For $p=\infty$, $q=1$ this holds true as shown by Wolff reducing the situation to $k=g^2$ with $g\in H^2$. Alternatively a real-analysis proof can be obtained using the inequality, following from [**6**, Th. 1],

$$\iint_{\mathbb{D}} |k'(z)|\,|\psi(z)|\log\frac{1}{|z|}\,dA(z) \le \int_0^{2\pi} A(k)(e^{i\theta})^{1/2}C(\psi)(e^{i\theta})\,d\theta$$

where $A(k)$ is the area function of k

$$A(k)(e^{i\theta}) = \left(\iint_{\Gamma(e^{i\theta})} |k'(z)|^2\,dA(z)\right)^{1/2}$$

and $C(\psi)$ is given by

$$C(\psi)(e^{i\theta}) = \sup_{e^{i\theta}\in I}\left(\frac{1}{|I|}\iint_{\hat{I}} |\psi|^2\log\frac{1}{|z|}\,dA\right)^{1/2}$$

$\hat{I}$ being the tent over I. This method applies to situations where there is no factorization.

For $p=1$, $q=\infty$, we use Schwarz inequality to bound the left member of (4) by

$$\left(\iint_{\mathbb{D}} |\varphi|\,|k'|^2\log\frac{1}{|z|}\,dA(z)\right)^{1/2}\left(\iint_{\mathbb{D}} |\varphi|\,|\psi|^2\log\frac{1}{|z|}dA(z)\right)^{1/2}$$

If $k\in BMOA$, $|k'|^2\log\dfrac{1}{|z|}dA$ is a Carleson measure; since Carleson measures operate on functions with integrable non-tangential maximal function, (4) follows for $p=1$, $q=\infty$. Next, consider the operator, for fixed ψ

$$\varphi \mapsto L_\varphi$$

where $L_\varphi(k) = \iint_{\mathbb{D}} k'\varphi\psi \log \frac{1}{|z|} dA(z)$; let T^p_∞ be the tent space ([6])

$$T^p_\infty = \{\varphi : M(\varphi) \in L^p\}.$$

We have shown that L is bounded from T^1_∞ to $(BMOA)^*$ and from T^∞_∞ to $(H^1)^*$. By interpolation, we conclude that L is bounded from T^p_∞ to $(H^q)^*$ i.e.

$$|L_\varphi(k)| \le C\|k\|_q \, \|M\varphi\|_p K$$

(alternatively, φ can be replaced by the harmonic extension of $M\varphi$ and argue with the L^p-spaces rather than the tent spaces).

It remains to bound I_2. But

$$|I_2| \le \iint_{\mathbb{D}} |k(z)|\,|\varphi_2(z)|\,|\psi_2(z)| \log \frac{1}{|z|} dA(z)$$

and this is easier: just note that $M(k\varphi_2) \le M(k)M(\varphi_2)$ is in L^1 and use again that Carleson measures operate on such functions. □

Note that the lemma holds if $G, \partial G$ are linear combinations $\sum \varphi_i \psi_i$ with φ_i, ψ_i as above.

Theorem 4. *Let $g_1, g_2 \in H^\infty$ such that $|g|^2 = |g_1|^2 + |g_2|^2 > 0$. Let μ and $\widetilde{\mu}$ be as above. Assume that*

$$M\left(\frac{1}{|g|^2}\frac{1}{\widetilde{\mu}(|g|^2)}\right) \in L^p(\mathbb{T}).$$

Then there are $f_1, f_2 \in H^p$ such that $f_1 g_1 + f_2 g_2 = 1$.

Proof. By a standard regularization argument we may assume that g_1, g_2 are holomorphic on $\overline{\mathbb{D}}$. The smooth solutions

$$\varphi_i = \frac{\overline{g_i}}{|g|^2}, \quad i = 1, 2$$

satisfy $M(\varphi_i) \in L^p$ and the general holomorphic solutions are given by

$$f_1 = \varphi_1 + u g_2 \qquad f_2 = \varphi_2 - u g_1$$

where u satisfies

$$\frac{\partial u}{\partial \overline{z}} = \frac{\overline{g_1'}\,\overline{g_2} - \overline{g_1}\,\overline{g_2'}}{|g|^4} \stackrel{def}{=} G.$$

We need only check that G satisfies the hypothesis of Lemma 3. For (a) we can take, by Lemma 2, ψ_1 to be

$$\psi_1 = \frac{\overline{g_i'}}{|g|}\,\widetilde{\mu}(|g|^2)^{1/2}, \quad |\psi_1| \le \frac{|g_i'|}{|g_i|}\widetilde{\mu}(|g_i|^2)^{1/2}$$

and

$$\varphi_1 = \frac{\overline{g_j}}{|g|^3 \widetilde{\mu}(|g|^2)^{1/2}}, \quad |\varphi_1| \le \frac{1}{|g|^2 \widetilde{\mu}(|g|^2)^{1/2}}.$$

Similarly, ∂G is a linear combination of terms of type

$$\frac{\overline{g_i'}\,\overline{g_j'}}{|g|^6} g_k g_l$$

and we may take

$$\psi_2 = \frac{\overline{g_i'}\,\overline{g_j'}}{|g|^2} \widetilde{\mu}(|g|^2), \quad |\psi_2| \le \frac{|g_i'|^2}{|g_i|^2} \widetilde{\mu}(|g_i|^2) + \frac{|g_j'|^2}{|g_j|^2} \widetilde{\mu}(|g_j|^2)$$

$$\varphi_2 = \frac{g_k g_l}{|g|^4 \widetilde{\mu}(|g|^2)}, \quad |\varphi_2| \le \frac{1}{|g|^2 \widetilde{\mu}(|g|^2)}$$

using again Lemma 2. □

We note as a particular case of the theorem, corresponding to $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$, the sufficient condition

$$M\left(\frac{|\log|g||^{2+\varepsilon}}{|g|^2}\right) \in L^p$$

stated in the introduction.

4. Lemma 2 can be used as well to improve Cegrell's result on the equation $f = f_1 g_1 + f_2 g_2$:

Theorem 5. *If $f, g_1, g_2 \in H^\infty$ satisfy*

$$|f| \le |g|^2 \widetilde{\mu}(|g|^2)$$

there exist $f_1, f_2 \in H^\infty$ such that $f = f_1 g_1 + f_2 g_2$.

Proof. In this case it must be shown that the equation $\overline{\partial} u = G$ where

$$G = f \frac{\overline{g_1'}\,\overline{g_2} - \overline{g_1}\,\overline{g_2'}}{(|g_1|^2 + |g_2|^2)^2}$$

has a bounded solution. In this case

$$|G| \le \frac{|g_1'| + |g_2'|}{|g|} \widetilde{\mu}(|g|^2)^{1/2} \le \frac{|g_1'|}{|g_1|} \widetilde{\mu}(|g_1|^2)^{1/2} + \frac{|g_2'|}{|g_2|} \widetilde{\mu}(|g_2|^2),$$

and $|G|^2(1 - |z|)$ is indeed a Carleson measure; similarly for ∂G.

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