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ON H^{P} -SOLUTIONS OF THE BEZOUT EQUATION

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We obtain a sufficient condition on bounded holomorphic functions g_1, g_2 in the unit disk for the existence of f_1, f_2 in the Hardy space H^p such that $1 = f_1g_1 + f_2g_2$. The sharpness of this condition is also studied.

1. Let \mathbb{D} be the unit disk in the complex plane, \mathbb{T} its boundary. For $1 \leq p \leq \infty$, H^p denotes the Hardy-space of holomorphic functions in \mathbb{D} such that

$$\|f\|_{p} = \sup_{r} \left(\frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^{p} d\theta\right)^{1/p} < +\infty \qquad p < \infty$$
$$\|f\|_{\infty} = \sup_{|z| < 1} |f(z)|.$$

It is well-known ([7, p. 57]) that if $f \in H^p$, the non-tangential maximal function

$$Mf(e^{i heta}) = \sup\{|f(z)|; \; z\in \Gamma(heta)\}$$

 $\Gamma(\theta)$ being the Stolz angle with vertex $e^{i\theta}$, belongs to $L^p(\mathbb{T})$.

In this paper, given $g_1, g_2 \in H^{\infty}$, we study the Bezout equation $1 = f_1g_1 + f_2g_2$. Concretely, we are interested in knowing the precise condition on g_1, g_2 so that solutions $f_1, f_2 \in H^p$ exist.

If $|g|^2 = |g_1|^2 + |g_2|^2$, $|f|^2 = |f_1|^2 + |f_2|^2$, it follows from $1 = f_1g_1 + f_2g_2$ that $1 \le |f| |g|$ and hence

(C)
$$M(|g|^{-1}) \in L^p(\mathbb{T}).$$

It can be easily seen that this condition is sufficient if g_1 or g_2 is an interpolating Blaschke product. Nevertheless, we show in Section 2 that it is not sufficient in general. In fact for each $\varepsilon > 0$ we exhibit $g_1, g_2 \in H^{\infty}$ such that $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ but no H^p solutions exist.

In Section 3 we obtain a general sufficient condition implying in particular the following:

Theorem 1. Assume that for some $\varepsilon > 0$

$$M\left(|g|^{-2} |\log |g||^{2+\varepsilon}\right) \in L^p(\mathbb{T}).$$

Then there exist $f_1, f_2 \in H^p$ such that $1 = f_1g_1 + f_2g_2$.

In Section 4, it is shown that the same method gives the following improvement on the problem considered by Wolff and also by Cegrell in [4].

Theorem 2. Let $f, g_1, g_2 \in H^{\infty}$ be such that

$$|f| \leq \frac{|g|^2}{|\log |g||^{2+\varepsilon}} \qquad for \ some \quad \varepsilon > 0 \,.$$

Then there are $f_1, f_2 \in H^{\infty}$ such that $f = f_1g_1 + f_2g_2$.

The proofs rely essentially on: (a) An L^p -version of Wolff's criteria for the existence of bounded solutions of the $\overline{\partial}$ -equation, already used in [1]. (b) An improvement of Cegrell's result in [4] on gradients of bounded holomorphic functions.

Both theorems hold of course for more than two generators, using the Koszul complex technique as in [7, p. 364]. Theorem 1 holds as well in the setting of the unit ball, but some modifications are needed (see [2]).

Finally, we mention that similar results to those stated here have been independently obtained by K.C. Lin in [8] and [9]. The authors thank the referee for pointing this out to them.

2. Before proceeding, we recall that a positive measure μ on \mathbb{D} is called a *Carleson measure* if there exists K > 0 such that

$$\mu(\{z: |z-e^{i heta}|\leq r\})\leq Kr \qquad e^{i heta}\in\mathbb{T},\quad r>0\,.$$

The smallest of such K is called the Carleson norm of μ . Equivalently ([7, p. 32]) μ is a Carleson measure if and only if for all functions h in \mathbb{D}

$$\iint_{\mathbb{D}} |h| d\mu \leq c \int_{0}^{2\pi} M(h) d heta$$
 .

In some particular cases it is quite easy to see that the condition (C) is sufficient. For instance, if g_1 is a Blaschke product with zeros z_n , the question is obviously equivalent to the interpolation problem

$$f_2(z_n)=rac{1}{g_2(z_n)}\,,\quad ext{with}\quad f_2\in H^p\,.$$

Indeed, $1 - f_2 g_2$ belongs then to H^p and vanishes on $\{z_n\}$, so $1 - f_2 g_2 = f_1 g_1$, $f_1 \in H^p$. In case g_1 is an interpolating Blaschke product, this interpolation problem has a solution if and only if

$$\sum_{n} \frac{1}{|g_2(z_n)|^p} (1 - |z_n|) < +\infty,$$

(see [10] and also [5]). Let δ_n denote the delta-mass at the point z_n . Since $\sum (1 - |z_n|)\delta_n$ is a Carleson measure, (C) implies the above condition.

Next, we give examples showing that condition (C) is far from being sufficient.

Theorem 3. Given $1 \leq p < \infty$ and $\varepsilon > 0$, there exist bounded analytic functions g_1, g_2 with $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$, such that there exist no $f_1, f_2 \in H^p$ satisfying $f_1g_1 + f_2g_2 \equiv 1$.

Proof. We will denote by $\rho(z, w)$ the pseudo-hyperbolic distance in the unit disc, $\rho(z, w) = |z - w| |1 - \overline{w}z|^{-1}$, $z, w \in \mathbb{D}$ and $f^{(j)}$ the *j*-th derivative of a function f. Let N be a natural number such that $(N + 1)\varepsilon > 1$.

Let $z_n = 1 - 2^{-n}$, $n \ge 1$, and take an H^{∞} -interpolating sequence $\{\alpha_n\}$, $0 < \rho(\alpha_n, z_n)$ decreasing to 0, satisfying

(1)
$$\sum_{n} (1-|z_n|)\rho(\alpha_n,z_n)^{-(N+1)p(2-\varepsilon)} < \infty,$$

(2)
$$\sum_{n} (1 - |z_n|) \rho(\alpha_n, z_n)^{-(2N+1)p} = \infty.$$

Let B_1 and B_2 be the Blaschke products with zeros $\{z_n\}$ and $\{\alpha_n\}$. From now on, the letter *c* will denote different constants independent on *n*. Since B_2 is an interpolating Blaschke product, one has

$$\inf_n \rho(z,\alpha_n) \ge |B_2(z)| \ge c \, \inf_n \rho(z,\alpha_n) \,, \quad |z| < 1 \,,$$

(see [7, p. 404]).

Now as in [3] take $g_i = B_i^{N+1}$, i = 1, 2. Let I_n be the arc on the unit circle centered at 1 of length $2(1 - |z_n|) = 2^{-n+1}$. In estimating $|g(z)|^{-1}$, for $z \in \Gamma(\theta)$, the worst case occurs when z is one of the $\{z_n\}$ or $\{\alpha_n\}$. Since $\rho(\alpha_n, z_n)$ is decreasing, for $e^{i\theta} \in I_n \setminus I_{n+1}$ one has

$$M(|g|^{-1}) \le rac{c}{
ho(lpha_{n+1}, z_{n+1})^{N+1}}$$
 .

Hence, condition (1) implies $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$. Now, assume there exist $f_1, f_2 \in H^p$ satisfying $f_1g_1 + f_2g_2 \equiv 1$. Then,

$$f_2^{(N)}(z_n) = (B_2^{-N-1})^{(N)}(z_n), \quad n \ge 1.$$

Write

$$B_{2,n}(z) = \prod_{k \neq n} \frac{-\alpha_k}{|\alpha_k|} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \qquad B_2^{-N-1}(z) = h(z)k(z)$$

where $h(z) = (1 - \overline{\alpha}_n z)^{N+1} B_{2,n}(z)^{-N-1}, \ k(z) = (z - \alpha_n)^{-N-1}$. Then $(B_2^{-N-1})^{(N)}(z) = \sum_{j=0}^N \binom{N}{j} h^{(j)}(z) k^{(N-j)}(z).$

Using Cauchy's formula on the disk of center z_n and radius $4^{-1}(1 - |z_n|)$, one gets

$$|h^{(j)}(z_n)| \le c \, \frac{(1-|\alpha_n|)^{N+1}}{(1-|z_n|)^j} \le c(1-|z_n|)^{N+1-j}$$

and hence

$$\begin{aligned} &|h^{(j)}(z_n)| \, |k^{(N-j)}(z_n)| \leq c \, |z_n - \alpha_n|^{-2N+j-1} (1 - |z_n|)^{N+1-j} \\ &\leq c \rho(z_n, \alpha_n)^{-2N+j-1} (1 - |z_n|)^{-N} \, . \end{aligned}$$

For j = 0, one gets

$$\begin{aligned} &|h(z_n)| \, |k^{(N)}(z_n)| \geq c \, (1-|z_n|)^{N+1} |z_n - \alpha_n|^{-2N-1} \\ &\geq c \rho(\alpha_n, z_n)^{-2N-1} (1-|z_n|)^{-N} \, . \end{aligned}$$

Therefore, for large n,

(3)
$$|f_2^{(N)}(z_n)| = |(B_2^{-N-1})^{(N)}(z_n)| \ge c (1-|z_n|)^{-N} \rho(\alpha_n, z_n)^{-2N-1}$$

Since $f_2 \in H^p$, the function

$$F(e^{i heta}) = \left(\int_{\Gamma(heta)} |f_2^{(N)}(z)|^2 (1-|z|)^{2N-2} dm(z)
ight)^{1/2}$$

belongs to $L^p(\mathbb{T})$ ([11, p. 216]). For $e^{i\theta} \in I_n \setminus I_{n+1}$, since $D_n = \{z \in \mathbb{D} : |z - z_n| < 4^{-1}(1 - |z_n|)\} \subset \Gamma(\theta)$, one has

$$|F(e^{i\theta})|^2 \ge \int_{D_n} |f_2^{(N)}(z)|^2 (1-|z|)^{2N-2} dm(z)$$

$$\ge c (1-|z_n|)^{2N} |f_2^{(N)}(z_n)|^2, \quad e^{i\theta} \in I_n \setminus I_{n+1}.$$

Using (3) and $F \in L^p(\mathbb{T})$, one gets

$$\infty > \sum (1 - |z_n|)\rho(\alpha_n, z_n)^{-(2N+1)p}$$

 \Box

and this contradicts (2).

3. In this section we will prove a generalization of Theorem 1 stated in the introduction.

Lemma 1. If g is holomorphic on $\overline{\mathbb{D}}$ and 0 ,

$$\int_{0}^{2\pi} \{|g(e^{i\theta})|^{p} - |g(0)|^{p}\}\frac{d\theta}{2\pi} \le 4\int_{0}^{2\pi} |g(e^{i\theta}) - g(0)|^{p}\frac{d\theta}{2\pi}$$

Proof. This is a general statement for a probability measure $d\mu$ on X and measurable φ

$$\int_X |\varphi|^p \, d\mu - \left| \int_X \varphi \, d\mu \right|^p \le 4 \int_X \left| \varphi - \int_X \varphi \, d\mu \right|^p \, d\mu \, .$$

First notice that this is trivial for $p \leq 1$ (with constant 1) and that for p = 2 there is equality with constant 1, too. In general, and assuming without loss of generality that $\int_X \varphi \, d\mu = 1$, it follows for real φ integrating the inequality

$$|arphi|^p-1\leq 3|arphi-1|^p+p(arphi-1)$$
 .

For complex-valued $\varphi = \varphi_1 + i\varphi_2$, it follows from $|\varphi|^p \leq |\varphi_1|^p + |\varphi_2|^p$ (for positive φ the inequality holds with constant 1).

We start with a generalization of a result in [4]. Although we need it only for H^{∞} functions we state it in full generality, for *BMOA* functions (see [7] for definitions). We denote by $||g||_*$ the *BMO* norm of $g(e^{i\theta})$.

Let $d\mu$ be a positive measure on [0,1) such that $\int_0^1 \frac{d\mu(\alpha)}{\alpha^2} < +\infty$, and write

$$\widetilde{\mu}(x)=\int_{0}^{1}x^{lpha}d\mu(lpha)\,,\quad x>0\,.$$

Lemma 2. If $g \in BMOA$, $\frac{|g'|^2}{|g|^2} \tilde{\mu}(|g|^2)(1-|z|^2)$ is a Carleson measure with Carleson norm bounded by $K||g||_*$, K depending on μ .

Proof. We consider the function

$$G(z) = \int_0^1 \frac{|g(z)|^{2\alpha}}{\alpha^2} d\mu(\alpha), \qquad |z| < 1.$$

For $\alpha > 0$, a computation shows that $\Delta |g|^{2\alpha} = 4\alpha^2 |g'|^2 |g|^{2\alpha-2}$ when $g \neq 0$, hence

$$\Delta G = 4|g'|^2 |g|^{-2} \widetilde{\mu}(|g|^2) \,.$$

Without loss of generality we can assume that g is holomorphic on \mathbb{D} . We argue like in [7, p. 327]. Let z_1, \ldots, z_N be the non-zero zeros in \mathbb{D} , and let Ω_{ε} be the domain $\mathbb{D} \setminus \bigcup_{j=0}^{N} \Delta_j$ where $\Delta_0 = \{|z| \leq \varepsilon\}, \ \Delta_j = \{|z - z_j| \leq \varepsilon\}, \ j = 1, \ldots, N.$

By Green's formula applied to the function G in Ω_{ε}

$$4 \iint_{\Omega_{\varepsilon}} |g'|^2 |g|^{-2} \widetilde{\mu}(|g|^2) \log \frac{1}{|z|} dA(z) = \int_0^{2\pi} G(e^{i\theta}) d\theta$$
$$-\sum_{j=0}^N \int_{\partial \Delta_j} \left(\frac{\partial}{\partial n} G\right) \log \frac{1}{|z|} - |G| \frac{\partial}{\partial n} \left(\log \frac{1}{|z|}\right) ds \,.$$

Let $r = |z - z_j|$; then for z close to z_j ,

$$|g'(z)|^2 |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \le Cr^{-2} \widetilde{\mu}(r^2)$$

Now, the hypothesis on μ translates to

$$\int_0^1 \frac{\widetilde{\mu}(x)}{x} |\log x| dx < +\infty \,.$$

Hence $|g'|^2 |g|^{-2} \widetilde{\mu}(|g|^2) \log \frac{1}{|z|}$ is integrable on \mathbb{D} . Also, for $|z - z_j| = \varepsilon$

$$|G(z)| \le c \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha^2} d\mu(\alpha)$$

which tends to 0 when $\varepsilon \to 0$, and

$$|\nabla G(z)| \leq \frac{c}{\varepsilon} \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha} \, d\mu(\alpha)$$

which also tends to zero when multiplied by $\varepsilon |\log \varepsilon|$. At zero we obtain $-2\pi |G(0)|$ as limit when $\varepsilon \to 0$. Therefore

$$\begin{split} &\frac{2}{\pi} \iint_{\mathbb{D}} |g'(z)|^2 |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \log \frac{1}{|z|} dA(z) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{|g(e^{i\theta})|^{2\alpha}}{\alpha^2} d\mu(\alpha) d\theta - \int_0^1 \frac{|g(0)|^{2\alpha}}{\alpha^2} d\mu(\alpha) \\ &= \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} \{|g(e^{i\theta})|^{2\alpha} - |g(0)|^{2\alpha}\} \frac{d\theta}{2\pi} \leq \quad \text{(by Lemma 1)} \\ &\leq 4 \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} |g(e^{i\theta}) - g(0)|^{2\alpha} \frac{d\theta}{2\pi} \leq \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} (c||g||_*)^{2\alpha} \,. \end{split}$$

If ψ_w is an automorphism of the disc, applying this inequality to $g \circ \psi_w$, changing variables in the area integral and using the invariance of the *BMO* norm we get

$$\begin{split} \sup_{w \in \mathbb{D}} \iint_{\mathbb{D}} |g'(z)|^2 |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2} \, dA(z) \\ &\leq c \int_0^1 \frac{\|g\|_*^{2\alpha}}{\alpha^2} \, d\mu(\alpha) < +\infty \,, \end{split}$$

and the result follows, by [7, p. 239].

Taking for μ a delta-mass at ε we get Cegrell's result [4] that $|g'|^2 |g|^{\varepsilon-2}(1-|z|^2)$ is a Carleson measure. Taking $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$ one gets that

$$\frac{|g'|^2}{|g|^2|\log|g||^{2+\varepsilon}}(1-|z|^2)$$

is a Carleson measure for every $\varepsilon > 0$.

Next lemma is the L^p -version of Wolff's criteria for bounded solutions of the $\overline{\partial}$ -equation ([7, p. 322]).

Lemma 3. Let $1 \le p \le \infty$, let G be a C^1 function in $\overline{\mathbb{D}}$ such that:

(a) $G = \varphi_1 \psi_1$, where $M(\varphi_1) \in L^p$ and $|\psi_1|^2 \log \frac{1}{|z|}$ is a Carleson measure. (b) $\partial G = \varphi_2 \psi_2$, where $M(\varphi_2) \in L^p$ and $|\psi_2| \log \frac{1}{|z|}$ is a Carleson measure.

Then there exists a C^1 function u in \mathbb{D} , continuous on $\overline{\mathbb{D}}$ such that

$$\frac{\partial u}{\partial \overline{z}} = G$$

and

$$\int_0^{2\pi} \lvert u(e^{i\theta}) \rvert^p \, d\theta \leq C$$

where C depends only of the L^p -norms of $M(\varphi_1), M(\varphi_2)$ and the Carleson norms of the measures in (a), (b).

Proof. We adapt Wolff's proof for the case $p = \infty$. Let q be the conjugate exponent of $p, 1 < q \le \infty$. By duality,

 \Box

where F is a priori solution, say the one given by the Cauchy kernel, which is continuous on $\overline{\mathbb{D}}$. By Green's formula

$$\frac{1}{2\pi} \int_0^{2\pi} Fk d\theta = \frac{1}{2\pi} \iint_{\mathbb{D}} \Delta(Fk) \log \frac{1}{|z|} dA(z)$$
$$= \frac{2}{\pi} \iint_{\mathbb{D}} k'(z) G(z) \log \frac{1}{|z|} dA(z) + \frac{2}{\pi} \iint_{\mathbb{D}} k(z) \frac{\partial G}{\partial z} \log \frac{1}{|z|} dA(z) = I_1 + I_2.$$

We will prove now that if $|\psi|^2 \log \frac{1}{|z|}$ is a Carleson measure with constant K, then

(4)
$$\iint_{\mathbb{D}} |k'(z)| |\varphi(z)| |\psi(z)| \log \frac{1}{|z|} dA(z) \leq C ||k||_q ||M\varphi||_p K$$

where C is an absolute constant. This will imply the required bound for I_1 . For $p = \infty$, q = 1 this holds true as shown by Wolff reducing the situation to $k = g^2$ with $g \in H^2$. Alternatively a real-analysis proof can be obtained using the inequality, following from [6, Th. 1],

$$\iint_{\mathbb{D}} |k'(z)| \, |\psi(z)| \log \frac{1}{|z|} \, dA(z) \leq \int_{0}^{2\pi} A(k) (e^{i\theta})^{1/2} C(\psi) (e^{i\theta}) \, d\theta$$

where A(k) is the area function of k

$$A(k)(e^{i heta}) = \left(\iint_{\Gamma(e^{i heta})} |k'(z)|^2 \, dA(z)
ight)^{1/2}$$

and $C(\psi)$ is given by

$$C(\psi)(e^{i\theta}) = \sup_{e^{i\theta} \in I} \left(\frac{1}{|I|} \iint_{\hat{I}} |\psi|^2 \log \frac{1}{|z|} dA\right)^{1/2}$$

 \hat{I} being the tent over I. This method applies to situations where there is no factorization.

For p = 1, $q = \infty$, we use Schwarz inequality to bound the left member of (4) by

$$\left(\iint_{\mathbb{D}} |\varphi| \, |k'|^2 \log \frac{1}{|z|} \, dA(z)\right)^{1/2} \left(\iint_{\mathbb{D}} |\varphi| \, |\psi|^2 \log \frac{1}{|z|} dA(z)\right)^{1/2}$$

If $k \in BMOA$, $|k'|^2 \log \frac{1}{|z|} dA$ is a Carleson measure; since Carleson measures operate on functions with integrable non-tangential maximal function, (4) follows for p = 1, $q = \infty$. Next, consider the operator, for fixed ψ

$$\varphi \mapsto L_{\varphi}$$

where
$$L_{\varphi}(k) = \iint_{\mathbb{D}} k' \varphi \psi \log \frac{1}{|z|} dA(z)$$
; let T_{∞}^{p} be the tent space ([6])
 $T_{\infty}^{p} = \{\varphi : M(\varphi) \in L^{p}\}.$

We have shown that L is bounded from T_{∞}^1 to $(BMOA)^*$ and from T_{∞}^{∞} to $(H^1)^*$. By interpolation, we conclude that L is bounded from T_{∞}^p to $(H^q)^*$ i.e.

$$|L_{\varphi}(k)| \le C ||k||_q ||M\varphi||_p K$$

(alternatively, φ can be replaced by the harmonic extension of $M\varphi$ and argue with the L^{p} -spaces rather than the tent spaces).

It remains to bound I_2 . But

$$|I_2| \leq \iint_{\mathbb{D}} |k(z)| \left| arphi_2(z)
ight| \left| \psi_2(z)
ight| \log rac{1}{|z|} dA(z)$$

and this is easier: just note that $M(k\varphi_2) \leq M(k)M(\varphi_2)$ is in L^1 and use again that Carleson measures operate on such functions.

Note that the lemma holds if $G, \partial G$ are linear combinations $\sum \varphi_i \psi_i$ with φ_i, ψ_i as above.

Theorem 4. Let $g_1, g_2 \in H^{\infty}$ such that $|g|^2 = |g_1|^2 + |g_2|^2 > 0$. Let μ and $\tilde{\mu}$ be as above. Assume that

$$M\left(rac{1}{|g|^2}rac{1}{\widetilde{\mu}(|g|^2)}
ight)\in L^p(\mathbb{T})\,.$$

Then there are $f_1, f_2 \in H^p$ such that $f_1g_1 + f_2g_2 = 1$.

Proof. By a standard regularization argument we may assume that g_1, g_2 are holomorphic on $\overline{\mathbb{D}}$. The smooth solutions

$$arphi_i = rac{\overline{g}_i}{|g|^2}\,, \quad i=1,2$$

satisfy $M(\varphi_i) \in L^p$ and the general holomorphic solutions are given by

$$f_1 = \varphi_1 + ug_2$$
 $f_2 = \varphi_2 - ug_1$

where u satisfies

$$rac{\partial u}{\partial \overline{z}} = rac{\overline{g_1'} \ \overline{g_2} - \overline{g_1} \ \overline{g_2'}}{|g|^4} \stackrel{def}{=} G$$

We need only check that G satisfies the hypothesis of Lemma 3. For (a) we can take, by Lemma 2, ψ_1 to be

$$\psi_1 = rac{\overline{g_i'}}{|g|} \, \widetilde{\mu}(|g|^2)^{1/2} \,, \quad |\psi_1| \leq rac{|g_i'|}{|g_i|} \widetilde{\mu}(|g_i|^2)^{1/2}$$

and

$$arphi_1 = rac{\overline{g_j}}{|g|^3 \widetilde{\mu}(|g|^2)^{1/2}}\,, \quad |arphi_1| \leq rac{1}{|g|^2 \widetilde{\mu}(|g|^2)^{1/2}}\,.$$

Similarly, ∂G is a linear combination of terms of type

$$\frac{\overline{g_i'} \ \overline{g_j'}}{|g|^6} g_k g_l$$

and we may take

$$egin{aligned} \psi_2 &= rac{\overline{g'_i} \ \overline{g'_j}}{|g|^2} \widetilde{\mu}(|g|^2) \,, \quad |\psi_2| \leq rac{|g'_i|^2}{|g_i|^2} \widetilde{\mu}(|g_i|^2) + rac{|g'_j|^2}{|g_j|^2} \widetilde{\mu}(|g_j|^2) \ arphi_2 &= rac{g_k g_l}{|g|^4 \widetilde{\mu}(|g|^2)} \,, \quad |arphi_2| \leq rac{1}{|g|^2 \widetilde{\mu}(|g|^2)} \end{aligned}$$

using again Lemma 2.

We note as a particular case of the theorem, corresponding to $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$, the sufficient condition

$$M\left(rac{|\log|g||^{2+arepsilon}}{|g|^2}
ight)\in L^p$$

stated in the introduction.

4. Lemma 2 can be used as well to improve Cegrell's result on the equation $f = f_1g_1 + f_2g_2$:

Theorem 5. If $f, g_1, g_2 \in H^{\infty}$ satisfy

 $|f| \leq |g|^2 \widetilde{\mu}(|g|^2)$

there exist $f_1, f_2 \in H^{\infty}$ such that $f = f_1g_1 + f_2g_2$.

Proof. In this case it must be shown that the equation $\overline{\partial} u = G$ where

$$G = f \frac{\overline{g_1'} \ \overline{g_2} - \overline{g_1} \ \overline{g_2'}}{(|g_1|^2 + |g_2|^2)^2}$$

has a bounded solution. In this case

$$|G| \leq \frac{|g_1'| + |g_2'|}{|g|} \widetilde{\mu}(|g|^2)^{1/2} \leq \frac{|g_1'|}{|g_1|} \widetilde{\mu}(|g_1|^2)^{1/2} + \frac{|g_2'|}{|g_2|} \widetilde{\mu}(|g_2|^2) \,,$$

and $|G|^2(1-|z|)$ is indeed a Carleson measure; similarly for ∂G .

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