Pacific Journal of Mathematics

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Volume 171 No. 2 December 1995

PACIFIC JOURNAL OF MATHEMATICS
Vol. 171, No. 2, 1995

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Recently, B.E.J. Dahlberg and C.E. Kenig considered the Neumann problem, $\Delta u = 0$ in D, $\partial u/\partial \nu = f$ on ∂D, for Laplace's equation in a Lipschitz domain D. One of their main results considers this problem when the data lies in the atomic Hardy space $H^1(\partial D)$ and they show that the solution has gradient in $L^1(\partial D)$. The aim of this paper is to establish an extension of their theorem for data in the Hardy space $H^p(\partial D)$, $1-\epsilon < p < 1$, where $0 < \epsilon < 1/n$ is a positive constant which depends only on m, the maximum of the Lipschitz constants of the functions which define the boundary of the domain. We also extend G. Verchota's and Dahlberg and Kenig's theorem on the potential representation of solutions of the Neumann problem to the range $1 - \epsilon < p < 1$. This has the interesting consequence that the double-layer potential is invertible on Hölder spaces $C^\alpha(\partial D)$ for α close to zero.

The techniques of this paper are a modification of those of Dahlberg and Kenig [**6**]. In Lemma 2.10 of [**6**], Varopoulos's extension lemma and $H^1(\partial D)$-$VMO(\partial D)$ duality are used to show that a harmonic function with nontangential maximal function in $L^1(\partial D)$ has normal derivative in $H^1(\partial D)$. This argument fails when $p < 1$, since we cannot realize $H^p(\partial D)$ as a dual space. To substitute for the use of their Lemma 2.10, we observe that solutions of the Dirichlet problem with $H^p_1(\partial D)$-data have normal derivative in $H^p(\partial D)$. This follows from Dahlberg and Kenig's construction. Then, we need to prove a uniqueness result in order to know that the functions produced by the single-layer potential are identical to the functions constructed in their existence theorem. We remark that we are also able to give a direct proof that $M(\nabla u) \in L^p(\partial D)$ implies $\partial u/\partial \nu \in H^p(\partial D)$ when $(n-1)/n < p \le 1$. This is done using atomic decomposition techniques of M. Wilson [**16**]. We remark that, after seeing a preliminary version of this paper, Wei Cao and E. Fabes [**1**] established similar results on the invertibility of the potential operators using an extension of the techniques in [**2**].

1. Existence.

We let $D \subset \mathbf{R}^n$ denote a connected Lipschitz domain. Thus for every $Q \in \partial D$, there is an $r > 0$, a coordinate system on $\mathbf{R}^n$ and a Lipschitz function

$\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ with $\phi(0) = 0$ and such that

$$Z(Q, 100\,r) \cap \partial D = \left\{(X' X_n) : X_n = \phi(X')\right\} \cap Z(Q, 100\,r)$$

and

$$Z(Q, 100\,r) \cap D = \{(X', X_n) : X_n > \phi(X)\} \cap Z(Q, 100\,r)$$

where

$$Z(Q, s) = \{(X', X_n) : |X' - Q'| < s,\ |X_n - Q_n| < (1 + 2m)s\}.$$

We call $Z(Q, s)$ a coordinate cylinder for ∂D. We note that since ∂D is compact, we may assume that ∂D is covered by a finite collection of coordinate cylinders whose radii r are bounded below by r_0.

Our results will only be proven for starshaped Lipschitz domains in $\mathbf{R}^n$, $n \geq 3$. This means that, after a translation, $0 \in \Omega$ and if $X \in \Omega$, then $rX \in \Omega$ for $0 \leq r < 1$. These assumptions are inherited from the work of Dahlberg and Kenig. It is easier to prove Theorems A and B quoted below for these special domains. It is not difficult to extend these results to more general domains, but we do not discuss this extension here.

We let $\Delta(Q_0, r) = \{P \in \partial D : |P - Q_0| < r\}$ and assume that r is less than $\operatorname{diam}(\partial D)$. We let $d = n - 1$ denote the dimension of ∂D. We say that a is an *atom* for $H^p(\partial D)$ if for some Q_0 and r we have

i) $\operatorname{supp} a \subset \Delta(Q_0, r)$

ii) $\displaystyle\int_{\Delta(Q_0,r)} a(Q)dQ = 0$

iii) $\|a\|_{L^2(\Delta(Q_0,r))} \leq cr^{-d(1/p-1/2)}$.

When $1 \geq p > \dfrac{d}{d+1}$, the space $H^p(\partial D)$ is defined as the collection

$$\left\{f : f = \sum \lambda_j a_j \text{ with } \sum \lambda_j^p < \infty\right\}$$

for some sequence of atoms a_j. The quasi-norm for $H^p(\partial D)$ given by

$$\|f\|^p_{H^p(\partial D)} = \inf \left\{\sum \lambda_j^p : f = \sum \lambda_j a_j\right\}.$$

We note that the infinite sums appearing here do not exist as functions. Rather one must view elements of $H^p(\partial D)$ as linear functionals on spaces of nice functions. In fact, the dual of $H^p(\partial D)$, $d/(d+1) < p < 1$, is the space of Hölder continuous function of exponent $\alpha(p) = d(1-p)/p$. Thus, the pairing between an element of $H^p(\partial D)$ and $C^{\alpha(p)}(\partial D)$ is defined. We will abuse notation by writing this pairing as an integral $\int_{\partial D} f u \, dQ$ for $f \in H^p(\partial D)$, and $u \in C^{\alpha(p)}(\partial D)$. We recall that $H^p(\partial D)$ is not a Banach space since the triangle inequality fails. However, we may define a metric on $H^p(\partial D)$ by $\|f - g\|^p_{H^p(\partial D)}$.

In studying the exterior Neumann problem, it will be useful to introduce the space $\tilde{H}^p(\partial D)$. This is defined in the same manner as $H^p(\partial D)$, but we include the atom $\chi_{\partial D}$. We let $C^\alpha(\partial D)$, $0 < \alpha < 1$, denote the collection of equivalence classes of Hölder continuous functions which differ by a constant. The norm is given by

$$\|f\|_{C^\alpha(\partial D)} = \sup_{Q \neq P} \frac{|\bar{f}(P) - \bar{f}(Q)|}{|P - Q|^\alpha}$$

where the $\bar{f}$ is any representative of f. Finally, we define $\tilde{C}^\alpha(\partial D)$ as the space of functions for which the norm $\|f\|_{\tilde{C}^\alpha(\partial D)} = \|f\|_{L^\infty(\partial D)} + \|f\|_{C^\alpha(\partial D)}$ is finite. We let $C^\alpha_0(A)$ denote the set of functions in $C^\alpha(A)$ which have a compactly supported representative.

We study the following boundary value problems:

$$(NP) \quad \begin{cases} \Delta u = 0, & \text{in } D \\ \dfrac{\partial u}{\partial \nu} = f, & \text{on } \partial D \end{cases} \qquad \text{(DP)} \quad \begin{cases} \Delta u = 0, & \text{in } D \\ u = f, & \text{on } \partial D. \end{cases}$$

Since we will consider boundary values in (NP) which are not functions, we need to define the sense in which $\partial u/\partial \nu$ exists at the boundary. Let $f \in H^p(\partial D)$. We say that $\partial u/\partial \nu = f$ on ∂D if for each coordinate cylinder Z and compactly supported function $\psi \in C^\alpha(\partial D \cap Z)$, $\alpha = d(1/p - 1)$, we have

$$\lim_{\epsilon \to 0+} \int_{\partial D \cap Z} \psi(Q) \frac{\partial u_\epsilon}{\partial \nu}(Q) dQ = \int_{\partial D} \psi(Q) f(Q) dQ$$

where $u_\epsilon(X) = u(X + \epsilon e_n)$ is defined in a neighborhood of $Z \cap D$. We will also need to define tangential derivatives at the boundary. Let Z, ϕ be a coordinate cylinder. If f is smooth in a neighborhood of $Z \cap \partial D$, then we define tangential derivatives by

$$\frac{\partial f}{\partial T_i}(X', \phi(X')) = \frac{\partial}{\partial X_i} f(X', \phi(X')), \qquad i = 1, \ldots, n-1.$$

If u is smooth in D, we say that $\nabla_{tan} u$ exists in the $H^p(\partial D)$ sense if for each coordinate cylinder Z, there exists $f_1, \ldots, f_{n-1} \in H^p(\partial D)$ so that for all $\psi \in C^\alpha_0(\partial D \cap Z)$

$$\lim_{\epsilon \to 0+} \int_{\partial D \cap Z} \frac{\partial u}{\partial T_i}(Q + s e_n) \psi(Q) \, dQ = \int_{\partial D} f_i \psi, \qquad i = 1, \ldots, n-1.$$

An important part of the argument for inverting the layer potentials is the study of the Dirichlet problem when the data has one derivative in $H^p(\partial D)$. We call this space $H^p_1(\partial D)$ and give a precise definition by defining atoms. We say that A is an atom for $H^p_1(\partial D)$ if for some $Q_0 \in \partial D$ and $r > 0$, we have

i) $\operatorname{supp} A \subset \Delta(Q_0, r) \cap \partial D$

ii) $\|\nabla_{tan} A\|_{L^2(\partial D)} \le r^{-d(1/p-1/2)}$

and then define $H^p_1(\partial D)$ as the l^p-span of these atoms. We note that our definition of $H^1_1(\partial D)$ is slightly different than the one given in [**6**]. However, it is easy to see that the resulting spaces coincide.

We begin by stating Dahlberg and Kenig's existence results for solutions with atomic data. Their results for $H^1(\partial D)$ have a little wiggle room so they also apply to $H^p(\partial D)$. To state these results, we will use the nontangential maximum function. For a function v which is continuous on D, this is defined by

$$M(v)(P) = \sup_{X \in \Gamma(P)} |v(X)|$$

where $\Gamma(P)$ is the nontangential approach region

$$\Gamma(P) = \left\{ Y \in D : \ |Y - P| < 2\sqrt{1+m^2}\, \delta(Y) \right\}$$

and $\delta(Y)$ denotes the distance from Y to the boundary of D.

Theorem A. *Let a be an atom for $H^p(\partial D)$ and suppose that a is supported in $\Delta(Q_0, r)$. There exists $\eta > 0$ such that if $p > 2/(\eta + 2)$, then there is a unique solution of (NP) which satisfies*

i) $\displaystyle\int_{\partial D} M(\nabla u)(P)^2 \, dP \le C r^{-d(2/p-1)}$

ii) $\displaystyle\int_{\partial D} M(\nabla u)(P)^2 |P - Q_0|^{d(1+\eta)} dP \le c r^{d(\eta+2-2/p)}$

iii) *We have that $u|_{\partial D} \in H^p(\partial D)$ and if we normalize by setting $u(0) = 0$, then $\|u\|_{H^p_1(\partial D)} \le C$.*

Similarly, if $a \in \tilde{H}^p(\partial D)$, is an atom, then the solution of the exterior Neumann problem in $\mathbf{R}^n \setminus \bar{D}$ satisfies i) *and* ii). *If we normalize by requiring u to vanish at infinity, then we obtain the estimate of* iii) *also.*

We also quote the corresponding result for $H^p_1(\partial D)$-atoms.

Theorem B. *Let A be an atom for $H^p_1(\partial D)$ which is supported in $\Delta(Q_0, r)$. There exists $\eta > 0$ such that if $p > 2/(\eta + 2)$, then the L^2_1-solution of the Dirichlet problem with data A satisfies*

i) $\displaystyle\int_{\partial D} M(\nabla u)(P)^2 \, dP \le C r^{-d(2/p-1)}$

ii) $\int_{\partial D} M(\nabla u)(P)^2 |P - Q_0|^{d(1+\eta)} dP \leq C r^{d(\eta+2-2/p)}$

iii) $\partial u/\partial \nu \in H^p(\partial D)$ *and* $\|\partial u/\partial \nu\|_{H^p(\partial D)} \leq C$

iv) u *has tangential derivatives in the* H^p*-sense.*

Theorem A and B are established in [**6**]. The estimates iii) are not explicitly stated in their paper. However, they follow easily from i) and ii) via the idea of a molecule (see [**4**]). The statements for the exterior problems may be obtained from the interior problems (in a different domain) using the Kelvin transform.

As immediate corollaries of Theorems A and B, we obtain the solvability of the boundary value problems with data in $H_1^p(\partial D)$ and $H^p(\partial D)$. In these theorems and in much of the rest of this article, we will restrict p to the range $1 - \delta_m < p \leq 1$ where $1/n > \delta_m > 0$ is determined by the following three conditions: 1) $\delta_m < 1 - 2/(\eta + 2)$ where η is as in Theorems A and B. 2) If $p > 1 - \delta_m$, then we must be able to solve the Dirichlet problem with data in the dual space, $C^{\alpha(p)}(\partial D)$, $\alpha(p) = d(1/p - 1)$, and obtain a solution in $C^{\alpha(p)}(\bar{D})$ (see Lemma 2.3). 3) The Neumann Green's function for domains lying above the graph of a function with Lipschitz constant m must lie in $C^{\alpha(p)}$ away from the singularity. See [**6**] or Theorem 2.8 below for the construction of this Green's function.

Theorem C. *Let* $1 > p > 1 - \delta_m$ *and suppose that* $f \in H^p(\partial D)$. *Then the interior Neumann problem with data* f *has a solution* u *which satisfies*

$$\|u\|_{H_1^p(\partial D)} + \|M(\nabla u)\|_{L^p(\partial D)} \leq C\|f\|_{H^p(\partial D)}$$

when u *is normalized by* $u(0) = 0$. *For the exterior problem, we allow* f *in* $\tilde{H}^p(\partial D)$ *and we normalize by setting* $u(\infty) = 0$. *This solution satisfies the estimate*

$$\|u\|_{H_1^p(\partial D)} + \|M(\nabla u)\|_{L^p(\partial D)} \leq C\|f\|_{\tilde{H}^p(\partial D)}.$$

Furthermore, the normal and tangential derivatives of u *exist in the sense described above.*

Theorem D. *Let* $1 - \delta_m < p < 1$ *and suppose* $f \in H_1^p(\partial D)$. *Then the interior Dirichlet problem with data* f *has a solution in* D *which satisfies*

$$\|M(\nabla u)\|_{L^p(\partial D)} + \left\|\frac{\partial u}{\partial \nu}\right\|_{H^p(\partial D)} \leq C\|f\|_{H^p(\partial D)}.$$

For the exterior problem, we have

$$\|M(\nabla u)\|_{L^p(\partial D)} + \left\|\frac{\partial u}{\partial \nu}\right\|_{\tilde{H}^p(\partial D)} \leq C\|f\|_{H^p(\partial D)}.$$

In each case, the normal and tangential derivatives exist in the sense described above.

We close this section with a theorem whose proof is due to M. Wilson. We observe that if u is harmonic in a Lipschitz domain D, $M(\nabla u)$ lies in $L^p(\partial D)$, $p > (n-1)/n$, then we may define $\partial u/\partial \nu$ as a linear functional on Lipschitz functions on the boundary. In fact, if ψ is supported in a coordinate cylinder Z, then

(1.1)
$$\lim_{\epsilon\to 0^+} \int_{\partial D\cap Z} \psi(Q)\frac{\partial u}{\partial \nu}(Q+\epsilon e_n)\,dQ = \lim_{\epsilon\to 0^+}\int_D \nabla\psi(Y)\nabla u(Y+\epsilon e_n)\,dY.$$

Using Lemma 2.1 below, one can see that

$$\int_{\partial D\cap Z} |\nabla u(Q+re_n)|\,dQ \le r^{-d(1-p)/p}\|M(\nabla u)\|_{L^p(\partial D)}.$$

Hence, for $p > (n-1)/n$ the integral on the right of (1.1) converges as $\epsilon \to 0^+$ (see also the proof of Theorem 2.9).

Theorem 1.2. *Let $(n-1)/n < p \le 1$ and suppose that u is harmonic with $M(\nabla u) \in L^p(\partial D)$, then we may find an atomic decomposition of $\partial u/\partial \nu$ into $H^p(\partial D)$ atoms. In particular, $\partial u/\partial\nu$ is in $H^p(\partial D)$. For the exterior domain, we obtain the normal derivative is in $\tilde{H}^p(\partial D)$.*

This may be proven using the techniques of M. Wilson from [**16**]. His argument works without alteration in domains lying above the graph of a Lipschitz function. We leave the details of general domains to the reader. We note that this theorem provides a different proof of the estimates for the normal derivative in Theorem D. It would be interesting to see if the estimates of the boundary values of u in Theorem C can be obtained this way.

2. Uniqueness.

In this section, we show that the solutions described in section one are unique. This also depends on the ideas developed in [**6**]. However, there are some technical difficulties in dealing with the case $p < 1$. Our main new tool is Lemma 2.2 which allows us to estimate the L^1-norm of $M(u)$ in terms of the L^p-norm of $M(\nabla u)$, $p = (n-1)/n$. This is a version of the Hardy-Littlewood theorem on fractional integration. In Lemma 2.2 below, we prove a sharp version of this result. We will show that for harmonic u, we can always control $\|M(u)\|_{pd/(d-p)}$ by $\|M(\nabla u)\|_p$, when $p < n-1 = d$.

In our first lemma, we let $\fint_E f$ denote the average $|E|^{-1} \int_E f$. Similar results are known for classical Hardy spaces. In [**9**], Krantz proves a fractional integration result for atomic Hardy spaces. Earlier, Stein and Weiss observe that the theorem on fractional integration holds for the Hardy spaces which they define in [**13**]. In one dimension, the result dates back to Hardy and Littlewood.

Lemma 2.1. *Let w be harmonic in D and let Z be a coordinate cylinder and let α be a multi-index with nonnegative entries, then*

$$\left|\frac{\partial^\alpha w}{\partial X^\alpha}(X)\right| \leq C\delta(X)^{-d/p-|\alpha|}\|M(w)\|_{L^p(\partial D)}.$$

Proof. Using interior estimates for harmonic functions, we have

$$\begin{aligned}\frac{\partial^\alpha w}{\partial X^\alpha}(X) &\leq \frac{C_p}{\delta(X)^{|\alpha|}}\left(\fint_{B(X,\delta(X)/2)} |\nabla w(Y)|^p dY\right)^{1/p} \\ &\leq C\delta(X)^{-|\alpha|}\left(\fint_{\Delta(\hat{X},C\delta(X))} M(\nabla w)(Q)^p\, dQ\right)^{1/p} \\ &\leq C_{p,m,\alpha}\delta(X)^{-d/p-|\alpha|}\|M(\nabla w)\|_p\end{aligned}$$

where $\hat{X}$ denotes a point on ∂D satisfying $\delta(X) = |X - \hat{X}|$. □

Lemma 2.2. *Let D be a connected Lipschitz domain and suppose that u is harmonic in D. Let X^* be a fixed point in D and suppose that $u(X^*) = 0$. For $p < d$ and $p^* = dp/(d-p)$ we have*

$$\|M(u)\|_{L^{p^*}(\partial D)} \leq C\|M(\nabla u)\|_{L^p(\partial D)}$$

where the constant C depends on the distance of X^* to the boundary, p and the Lipschitz character of ∂D.

Proof. We prove the corresponding result for a domain

$$D = \{(X', X_n) : X_n > \phi(X')\}$$

which lies above the graph of a Lipschitz function ϕ. From Lemma 2.1, we have $|\nabla u(X)| \leq C\delta(X)^{-d/p}$. It follows that $\lim_{X_n\to\infty} u(X', X_n)$ exists and is independent of X'. Thus we may add a constant to u and obtain that u vanishes at infinity. Also, after replacing u by $u_\epsilon(X) = u(X + \epsilon e_n)$, we may assume that $M(u) \in L^{p^*}(\partial D)$.

We will need the area integral which is defined by

$$A(u)(Q)^2 = \int_{\Gamma'(Q)} |\nabla u(Y)|^2 |Y-Q|^{2-n}\,dY$$

where $\Gamma'(Q) \supset \Gamma(Q)$ is a strictly larger cone defined by

$$\left\{X \in D : \delta(X) < 4\sqrt{1+m^2}|X-Q|\right\}.$$

To estimate $M(u)$, we use the fundamental theorem of calculus and Hölder's inequality to obtain that for each $\eta > 0$,

$$\begin{aligned}
|u(X', X_n)| &\le \int_{X_n}^{\infty} \left|\frac{\partial u}{\partial X_n}(X', s)\right| ds \\
&\le \left(\int_{X_n}^{\infty} (s-\phi(X'))|\nabla u(X', s)|^2\,ds\right)^{\frac{1}{2}2\eta} \\
&\quad \times \left(\int_{X_n}^{\infty} (s-\phi(X'))^{-\eta/(1-\eta)}|\nabla u(X', s)|^{\frac{1-2\eta}{1-\eta}}\,ds\right)^{\frac{1-\eta}{1-2\eta}(1-2\eta)} \\
&\equiv B_1(X)^{2\eta} B_2(X)^{1-2\eta}.
\end{aligned}$$

Since $B_1(X)$ is essentially the g-function, we have $B_1(X) \le CA(u)(Q)$ for $X \in \Gamma(Q)$.

To study the function $B_2(X)$, we let $B_{X',s}$ denote the ball $B((X', s), c_m[s-\phi(X')])$ where c_m is chosen so that $B_{X',s}$ lies in D and we let $\Delta_{X',s} = \Delta(Q, C[s-\phi(X')])$. If we choose C sufficiently large, then we have

$$\begin{aligned}
B_2(X)^{\frac{1-2\eta}{1-\eta}} &\le C_\eta \int_{X_n}^{\infty} (s-\phi(X'))^{-\eta/(1-\eta)} \fint_{B_{X',s}} |\nabla u(Y)|^{\frac{1-2\eta}{1-\eta}}\,dY\,ds \\
&\le C_{\eta,m} \int_{X_n}^{\infty} (s-\phi(X'))^{-\eta/(1-\eta)} \fint_{\Delta_{X',s}} M(\nabla u)(P)^{\frac{1-2\eta}{1-\eta}}\,dP\,ds.
\end{aligned}$$

Changing the order of integration in this last integral, we obtain

$$\begin{aligned}
B_2(X)^{\frac{1-2\eta}{1-\eta}} &\le C \int_{\partial D} M(\nabla u)(P)^{\frac{1-2\eta}{1-\eta}} |P-Q|^{\frac{1-2\eta}{1-\eta}-d}\,dP \\
&\equiv F(Q)^{\frac{1-2\eta}{1-\eta}}, \qquad X \in \Gamma(Q).
\end{aligned}$$

By the Hardy-Littlewood theorem on fractional integration [**12**, p. 119], we have

$$\|F\|_{L^{p^*}(\partial D)} \le C\|M(\nabla u)\|_{L^p(\partial D)}$$

when $p < d$, $\frac{1}{2} > \eta > 0$ and, if $p \le 1$, $\eta > (1-p)/(2-p)$. Combining our estimates for B_1 and B_2, we have

$$\|M(u)\|_{L^{p^*}(\partial D)} \le C\|A(u)\|_{L^{p^*}(\partial D)}^{2\eta} \|M(\nabla u)\|_{L^p(\partial D)}^{1-2\eta}.$$

Now, we may use Dahlberg's estimate [5] for the area integral, $\|A(u)\|_{L^p(\partial D)} \leq C\|M(u)\|_{L^p(\partial D)}$, and our *a priori* assumption that $M(u)$ is in $L^{p^*}(\partial D)$ to obtain the estimate of the theorem. □

Our final preliminary result studies the action of the generalized Riesz transforms on $C^\alpha(\partial D)$ when D is a Lipschitz domain. We restrict our attention to domains lying above the graph of a Lipschitz function and introduce the notation $D_\phi = \{(X', X_n) : X_n > \phi(X')\}$ with $\|\nabla\phi\|_\infty = m$.

Lemma 2.3. *Let $\phi : \mathbf{R}^{n-1} \to \mathbf{R}$ be Lipschitz. There exists α_0 depending only on m such that for $f \in C^\alpha(\partial D_\phi)$, $0 < \alpha < \alpha_0$, we may find a harmonic gradient $(w^1, ..., w^n)$ satisfying $w^n(Q) = f(Q)$,*

$$\frac{\partial w^i}{\partial X_j} = \frac{\partial w^j}{\partial X_i} \quad i, j = 1, \dots, n$$

$$\sum_{i=1}^{n} \frac{\partial w^i}{\partial X_i} = 0$$

$$\Delta w^i = 0, \qquad i = 1, \dots, n.$$

Furthermore, each of these functions is Hölder continuous in $\bar{D}_\phi$ and satisfies

$$\|w^i\|_{C^\alpha(\bar{D}_\phi)} \leq C\|f\|_{C^\alpha(\partial D_\phi)}, \qquad i = 1, \dots, n.$$

This is fairly standard, thus our proof will be brief.

Proof. We let w^n be the solution of the Dirichlet problem in D_ϕ with data f. We have $\|w^n\|_{C^\alpha(\bar{D}_\phi)} \leq C\|f\|_{C^\alpha(\partial D_\phi)}$. Next, we apply interior estimates to the harmonic function $w^n(\cdot) - w^n(X)$ on the ball $B(X, \delta(X)/2)$ to obtain that

$$\left|\frac{\partial^\beta w^n}{\partial X^\beta}(X)\right| \leq C\delta(X)^{\alpha - |\beta|}\|w^n\|_{C^\alpha(\bar{D}_\phi)}, \qquad |\beta| \geq 1. \tag{2.4}$$

The converse also holds for any function which is in $C^1_{\text{loc}}(D_\phi)$:

$$\|u\|_{C^\alpha(\bar{D}_\phi)} \leq C_{\alpha,m} \sup_{X \in D_\phi} \delta(X)^{1-\alpha}|\nabla u(X)|. \tag{2.5}$$

We can define the conjugate functions by the formula

$$\nabla w^i(X) = -\int_{X_n}^{\infty} \nabla \frac{\partial w^n}{\partial X_i}(X', s)ds.$$

The estimate (2.4) guarantees that this integral is converges and (2.5) implies the functions w^i are Hölder continuous. □

Lemma 2.6. *If $(w^1, \dots, w^n)$ is a harmonic gradient satisfying $w^n \geq 0$, w^n vanishes continuously on $\partial D \setminus \Delta(Q_0, r)$, and $\lim_{X_n \to \infty} w^i(X', X_n) = 0$ for each i then*

$$\sup\{|w^i(X)| : \operatorname{dist}(X, \Delta(Q_0, r)) \geq r\} \leq C_{r,q}\|M(w^n)\|_q.$$

Proof. We first observe that Chebyshev's inequality implies that

$$|w^n(Q_0 + re_n)| \leq C_r\|M(w^n)\|_{L^q(\partial D)}.$$

Using the boundary Harnack principle (see [**8**, Lemma 5.4], for example), we obtain that for some $\alpha > 0$

$$w^n(X) \leq C_{r,\alpha}\left(\frac{\delta(X)}{r}\right)^\alpha w^n(Q_0 + re_n), \qquad X \in D \setminus B(Q_0, 2r). \tag{2.7}$$

Applying the maximum principle in $D_\phi \setminus B(Q_0, 2r)$ yields that w^n is bounded there.

To obtain the boundedness of w^i we observe that (2.7) implies that

$$|\nabla w^n(X)| \leq C_{r,\alpha}\delta(X)^{\alpha-1}\|M(w^n)\|_q, \ X \in D_\phi \setminus B(Q_0, 2r).$$

While Lemma 2.1 gives

$$|\nabla w^n(X)| \leq C_{q,m}\delta(X)^{-1-d/q}\|M(w^n)\|_{L^q(\partial D)}.$$

Using these estimates and writing w^n as an integral of its derivative,

$$w^i(X) = -\int_{X_n}^{\infty} \frac{\partial w^n}{\partial X_i}(X', s)\,ds,$$

gives the boundedness of the functions w^i. □

We are now ready to give our uniqueness result for the Neumann problem.

Theorem 2.8 (Uniqueness in NP). *If $1 - \delta_m < p < 1$, u satisfies*

$$\begin{cases} \Delta u = 0 \\ M(\nabla u) \in L^p(\partial D) \end{cases}$$

and $\partial u/\partial \nu$ vanishes in the H^p-sense, then u is a constant.

Proof. We fix a coordinate cylinder (Z, ϕ) and let $X^* = (X', 2\phi(X') - X_n)$ be reflection in the graph of ϕ. We let $G(X, Y)$ be the Green's function $G(X, Y) + G(X, Y^*) = N(X, Y)$ be the Neumann kernel for D_ϕ constructed in [**6**, p. 447]. We have that $N(X,Q) \in C^\alpha(\partial D)$ for $0 < \alpha < \alpha(m)$. This follows since $G(X,Y)$ is the fundamental solution in $\mathbf{R}^n$ of an operator whose coefficients are bounded and measurable [**10**, **11**].

We let ψ be a cutoff function satisfying $\chi_{1/2Z} \le \psi \le \chi_Z$. We have

$$\begin{aligned}\psi(X)u_\epsilon(X) = &\int_D N(X,Y)u_\epsilon(Y)\Delta\psi(y)dY \\ &+2\int_D N(X,Y)\nabla u_\epsilon(Y)\nabla\psi(Y)dY \\ &+\int_{\partial D} N(X,Q)u_\epsilon(Q)\frac{\partial\psi}{\partial\nu}(Q)dQ \\ &+\int_{\partial D} N(X,Q)\frac{\partial u_\epsilon(Q)}{\partial\nu}\psi(Q)\,dQ \\ =& A(X)+B(X)+C(X)+D(X).\end{aligned} \tag{2.9}$$

To establish uniqueness, we first show that u is bounded in $\frac{1}{4}Z$. We observe that $N(X,Y) \le C[|X-Y|^{2-n} + |X^*-Y|^{2-n}]$ for $X \in D$, [**10**]. Thus, we have

$$|C(X)| \le \int_{Z\cap\partial D} |u_\epsilon(Q)|\,dQ \text{ dist}\left(X, \left(\frac{1}{2}Z\right)^c\right)^{2-n}. \qquad X \in \frac{1}{4}Z$$

Next, we observe that Lemma 2.1 implies that

$$\begin{aligned}\int_{Z\cap D} |\nabla u_\epsilon|\,dX &\le C\|M(\nabla u_\epsilon)\|_p^{1-p} \int_0^{2r_0}\int_{\partial D} M(\nabla u_\epsilon)(P)\,dP\, r^{-d(1-p)/p}\,dr \\ &\le C\|M(\nabla u)\|_p, \qquad \text{when } p > (d-1)/d.\end{aligned}$$

This gives

$$|B| \le C\|M(\nabla u_\epsilon)\|_p \cdot \text{dist}\,(X, 1/2Z^c)^{2-n}.$$

The term D vanishes as $\epsilon \to 0^+$ because the normal derivative vanishes in the H^p-sense. This follows because $N(X,\cdot)$ is in $C^\alpha(\partial D)$. The term A is easy to estimate and we omit the details. The estimates on A through C and the vanishing of D imply that u is bounded.

To see that u is constant, we choose f in the Hardy space $H^1(\partial D)$. By Theorem C, there exists a solution ψ to (NP), with data f and $M(\nabla\psi)$ in $L^1(\partial D)$. We let $u_\epsilon(X) = u((1-\epsilon)X)$ and consider

$$\int_{\partial D} u_\epsilon(Q)\frac{\partial\psi_\eta}{\partial\nu}(Q) - \psi_\eta(Q)\frac{\partial u_\epsilon}{\partial\nu}(Q)\,dQ = 0.$$

We note that

$$\int \partial u_\epsilon/\partial\nu\, \psi_\eta \to 0$$

as $\epsilon \to 0^+$ since ψ_η is smooth. Also,

$$\int u_\epsilon \partial\psi_\eta/\partial\nu \to \int u\partial\psi_\eta/\partial\nu$$

since our claim that u is bounded implies that u has nontangential limits a.e, (see [**8**], for example). Finally,

$$0 = \int_{\partial D} u\frac{\partial\psi_\eta}{\partial\nu} \to \int_{\partial D} u\frac{\partial\psi}{\partial\nu}\,dQ$$

as $\eta \to 0^+$ because u is bounded and $\|\partial_\nu\psi_\eta - \partial_\nu\psi\|_{L^1(\partial D)}$. Thus we have shown that

$$\int_{\partial D} uf = 0$$

for every $f \in H^1(\partial D)$. Finally, we know that bounded harmonic functions in Lipschitz domains are the Poisson integral of their boundary values. Hence, u is constant. □

Theorem 2.10. *Suppose that $\Delta u = 0$, $M(\nabla u) \in L^p(\partial D)$ and p lies between $1 - \delta_m$ and 1. If the tangential derivatives of u vanish in the H^p-sense and the nontangential limits of u vanish on ∂D, then $u = 0$ in D.*

Proof. We will begin by showing that for each nonnegative $f \in C_0^\alpha(Z \cap \partial D)$, we have

$$\lim_{\epsilon\to 0^+} \int_{\partial D} \frac{\partial u_\epsilon}{\partial\nu}(Q)f(Q)dQ \tag{2.11}$$

exists and the limit satisfies

$$\left|\int_{\partial D} \frac{\partial u}{\partial\nu} f\,dQ\right| \le \|M(w)\|_{L^2(\partial D)} \cdot \|M(\nabla u)\|_{L^p(\partial D)} \tag{2.12}$$

where w is the solution of $\Delta w = 0$, $w = f$ on ∂D. Since $\|M(w)\|_2 \le C\|f\|_2$, the second inequality implies that $\partial u/\partial\nu$ is in $L^2(\partial D)$.

Towards establishing (2.11) and (2.12), we fix a coordinate cylinder Z and let $f \in C_0^\alpha(\partial D \cap Z)$ be nonnegative. We choose a smooth cutoff function ψ which satisfies $\chi_{2Z} \le \psi \le \chi_{4Z}$. For η, $\epsilon > 0$ we have

$$\begin{aligned}
&\int_{\partial D} \psi w_\epsilon \frac{\partial u_\eta}{\partial\nu}\,dQ \\
&= \int_{\partial D} u_\eta \frac{\partial\psi}{\partial\nu} w_\epsilon + u_\eta\psi\frac{\partial w_\epsilon}{\partial\nu}\,dQ \\
&\quad -\int_D u_\eta w_\epsilon \Delta\psi + 2u_\eta \nabla\psi\cdot\nabla w_\epsilon\,dX \\
&\equiv \int_{\partial D} A(Q) + B(Q)\,dQ - \int_D C_1(X) + C_2(X)\,dX.
\end{aligned}$$

We observe that $\dfrac{\partial\psi}{\partial\nu}$ is supported in $(4Z \setminus 2Z) \cap \partial D$ hence

$$A(Q) \leq M(u)(Q)\|M(w)\|_2$$

by Lemma 2.6. To bound the integrand, $C_2(X)$, we note that $\nabla\psi$ is supported in $4Z \setminus 2Z$, hence we may use the observation of Lemma 2.6 that $|\nabla w(X)| \leq C\delta(X)^{\alpha-1}\|M(w)\|_2$ in $4Z \setminus 2Z$ to estimate

$$C_2(Q + re_n) \leq M(u)(Q)r^{\alpha-1}\|M(w^n)\|_{L^2(\partial D)}.$$

The estimate for $C_1(X)$ is also easy.

This leaves the main term $\int_{\partial D} B$ to be understood. We let $w^n, w^{n-1}, \dots, w^1$ be the harmonic gradient determined by f (see Lemma 2.3) and write

$$\begin{aligned}
&\int_{\partial D} \psi(Q)u_\eta(Q)\frac{\partial w^n_\epsilon}{\partial\nu}(Q)\,dQ \\
&\quad= \int_{\mathbf{R}^{n-1}} \psi(X', \phi(X'))u_\eta(X', \phi(X'))\left(-\frac{\partial w^n_\epsilon}{\partial X_n}(X', \phi(X'))\right. \\
&\qquad\qquad \left. + \sum_{i=1}^{n-1} \phi_{X_i}(X')\frac{\partial w^n_\epsilon}{\partial X_i}(X', \phi(X'))\right)\,dX' \\
&\quad= \int_{\mathbf{R}^{n-1}} \psi(X', \phi(X'))u_\eta(X', \phi(X'))\left(\sum_{i=1}^{n-1}\frac{\partial}{\partial X_i}[w^i_\epsilon(X', \phi(X'))]\right)\,dX' \\
&\quad= -\int_{\partial D} w^i_\epsilon(Q)u_\eta(Q)\frac{\partial\psi}{\partial T_i}(Q) + \psi(Q)w^i_\epsilon(Q)\frac{\partial u_\eta}{\partial T_i}(Q)\,dQ \\
&\quad\equiv -\int_{\partial D} B_1(Q) + B_2(Q)\,dQ.
\end{aligned}$$

By Lemma 2.6, we have

$$B_1(Q) \leq M(u)(Q)\|M(w^n)\|_{L^2(\partial D)}.$$

Our hypothesis that $\partial u/\partial T_i$ vanishes implies that $\int B_2$ vanishes as $\epsilon \to 0^+$. Thus, we may let $\epsilon \to 0^+$ in each of these expressions and then let η go to zero to obtain (2.11) and (2.12). This uses Lemma 2.2 to bound the L^1-norm of $M(u)$.

Our next step is to show that the nontangential maximal function of u is in $L^2(\partial D)$. This and our assumption that u has nontangential limits of 0 a.e. on ∂D are sufficient to imply that u vanishes identically on ∂D.

As in Theorem 2.8, we fix a coordinate cylinder Z, a cutoff function ψ and let $N(X, Y)$ be the Neumann Green's function for a graph domain D_ϕ

for which $D \cap 2Z = D_\phi \cap 2Z$. We let $u_\epsilon(X) = u(X + \epsilon e_n)$ and apply Green's formula to obtain

$$\psi u_\epsilon(X) = \int_{\partial D} N(X,Q)\left(\psi(Q)\frac{\partial u_\epsilon}{\partial\nu} + u_\epsilon(Q)\frac{\partial\psi}{\partial\nu}(Q)\right) dQ$$
$$+ \int_D N(X,Y)(u_\epsilon(Y)\Delta\psi(Y) + \nabla u_\epsilon(Y)\cdot\nabla\psi(Y))\, dY.$$

We have just shown that we may let $\epsilon \to 0^+$ in the term involving $\partial u_\epsilon/\partial\nu$. Since Lemma 2.2 implies $M(u_\epsilon) \in L^1(\partial D)$ and as in the proof of Theorem 2.8 we have ∇u_ϵ and u_ϵ in $L^1(D)$, we may let $\epsilon \to 0^+$ in the representation formula for ψu_ϵ and obtain

$$\begin{aligned}\psi u(X) &= \int_{\partial D} N(X,Q)\psi(Q)\frac{\partial u}{\partial\nu}\, dQ \\ &\quad + \int_D N(X,Y)(u(Y)\Delta\psi(Y) + \nabla u(Y)\cdot\nabla\psi(Y))\, dY \\ &= A(X) + B(X).\end{aligned}$$

As in Theorem 2.8, the term $B(X)$ is clearly bounded in $1/4Z$, say. To estimate $B(X)$, we use our observation above that $\partial u/\partial\nu$ is in L^2 of the boundary. In fact, since $N(X,Y) \le C|X-Y|^{2-n}$ in D, it is easy to see that the nontangential maximal function of $A(X)$ is in $L^2(D)$. This establishes our claim about $M(u)$ and hence the Theorem follows. □

Remark. The calculation used to estimate the term B in the study of $\partial u/\partial\nu$ was used by G. Verchota in [**15**].

3. Layer potentials.

In this section, we show that the solutions of the Neumann problem constructed in Section 2 may also be represented as single-layer potentials. This representation follows from the estimates of Theorem C and D via an argument of G. Verchota [**14, 15**]. Using the potential representation of the solutions of the Neumann problem, we immediately obtain a potential representation for solutions of the Dirichlet problem with data in $C^\alpha(\partial D)$ (or $\tilde{C}^\alpha(\partial D)$ for the exterior Dirichlet problem).

We begin by defining these potentials and recalling their mapping properties. We let

$$\Gamma(X) = \frac{1}{(n-2)\omega_n|X|^{n-2}}$$

denote the fundamental solution of Laplace's equation in $\mathbf{R}^n$, $n \ge 3$. Here, ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. We define the single-layer potential of $f \in H^p(\partial D)$, by

$$\mathcal{S}(f)(X) = \int_{\partial D} \Gamma(X - Q) f(Q) dQ, \quad X \in \mathbf{R}^n \setminus \partial D.$$

Note that we may also define $\mathcal{S}(f)$ on ∂D as an element of $L^{pd/(d-p)}(\partial D)$. This is the familiar Hardy-Littlewood theorem on fractional integration which extends easily to the setting of atomic Hardy spaces. Next, we define the double-layer potential by

$$\mathcal{D}(f)(X) = \int_{\partial D} \frac{\partial \Gamma}{\partial \nu(Q)} (X - Q) f(Q) dQ.$$

Notice that if $X \notin \bar{D}$, then

$$\int_{\partial D} \frac{\partial \Gamma}{\partial \nu(Q)} (X - Q) dQ = \int_D \Delta_Y \Gamma(X - Y) dY = 0.$$

Thus the double-layer potential of an equivalence class $\{f(Q) + r : \ r \in \mathbf{R}\}$ in $C^\alpha(\partial D)$ is well-defined. To discuss the boundary values of the potentials, we introduce the boundary potential operators:

$$\mathcal{K}^*(f)(P) = \text{p.v.} \int_{\partial D} \frac{\partial \Gamma}{\partial \nu(P)} (P - Q) f(Q) \, dQ$$

and

$$\mathcal{K}(f)(P) = \text{p.v.} \int_{\partial D} \frac{\partial \Gamma}{\partial \nu(Q)} (P - Q) f(Q) dQ.$$

The boundedness of $\mathcal{K} : \ H^p(\partial D) \to H^p(\partial D)$ (and on $\tilde{H}^p(\partial D)$) is a consequence of the results of Coifman, Meyer and McIntosh [**3**] on the Cauchy integral on Lipschitz curves (see also [**7**, **14**]).

Hence, we let $\nu = \nu^+$ denote the outer normal to D and $\nu^- = -\nu^+$ denote the outer normal to $D^* = R^n \setminus \partial D$. We let $\mathcal{S}^+(f)$ and $\mathcal{S}^-(f)$ denote the restrictions of $\mathcal{S}(f)$ to D and D^* respectively. Similarly, we let $\mathcal{D}^+(f)$ and $\mathcal{D}^-(f)$ denote the restrictions of $\mathcal{D}(f)$ to D and D^*. We summarize the boundary behavior of these operators in our next two results.

Theorem 3.1. *Let* $p > \dfrac{d}{d+1} = \dfrac{n-1}{n}$. *Then we have*

$$\frac{\partial \mathcal{S}^+(f)}{\partial \nu^+} = \frac{1}{2} f + \mathcal{K}^*(f), \qquad f \in H^p(\partial D)$$
$$\frac{\partial \mathcal{S}^-(f)}{\partial \nu^-} = \frac{1}{2} f - \mathcal{K}^*(f), \qquad f \in \tilde{H}^p(\partial D)$$

$$\left\|\left(\frac{1}{2}I+\mathcal{K}^*\right)(f)\right\|_{H^p(\partial D)}+\|M(\nabla\mathcal{S}^+(f))\|_{L^p(\partial D)}+\|\mathcal{S}(f)\|_{H^p_1(\partial D)} \leq C\|f\|_{H^p(\partial D)}$$

$$\left\|\left(\frac{1}{2}I-\mathcal{K}^*\right)(f)\right\|_{\tilde{H}^p(\partial D)}+\|M(\nabla\mathcal{S}^-(f))\|_{L^p(\partial D)}+\|\mathcal{S}(f)\|_{H^p_1(\partial D)} \leq C\|f\|_{\tilde{H}^p(\partial D)}.$$

The normal derivatives and the tangential derivatives exist at the boundary in the sense described in Section 1.

Theorem 3.2. *Let* $0<\alpha<1$ *and suppose that* $f\in\tilde{C}^\alpha(\partial D)$, *then we have that*

$$\mathcal{D}^+(f)|_{\partial D}=\frac{1}{2}f+\mathcal{K}(f) \text{ a.e.}$$
$$\mathcal{D}^-(f)|_{\partial D}=-\frac{1}{2}f+\mathcal{K}(f) \text{ a.e.}$$

and we may redefine $\mathcal{K}(f)$ *so that these equalities hold everywhere. Furthermore, we have*

$$\|\mathcal{D}^+(f)\|_{\tilde{C}^\alpha(\partial D)}\leq C\|f\|_{\tilde{C}^\alpha(\partial D)}$$

and

$$\|\mathcal{D}^-(f)\|_{C^\alpha(\partial D)}\leq C\|f\|_{C^\alpha(\partial D)}.$$

Finally, we have that $\frac{1}{2}I+\mathcal{K}:\ \tilde{C}^\alpha(\partial D)\to\tilde{C}^\alpha(\partial D)$ *is the adjoint of* $\frac{1}{2}I+\mathcal{K}^*:\ \tilde{H}^p(\partial D)\to\tilde{H}^p(\partial D)$ *and that* $-\frac{1}{2}I+\mathcal{K}:\ C^\alpha(\partial D)\to C^\alpha(\partial D)$ *is the adjoint of* $-\frac{1}{2}I+\mathcal{K}^*:\ H^p(\partial D)\to H^p(\partial D)$ *when* α *and* p *are related by* $\alpha=d\left(\frac{1}{p}-1\right)$.

The next result uses the ideas of G. Verchota [**14**, **15**] to establish our main estimate.

Proposition 3.3. *Let* D *be a starshaped Lipschitz domain, then we have*

$$\|f\|_{H^p(\partial D)}\leq C\|\left(\frac{1}{2}I+\mathcal{K}^*\right)(f)\|_{H^p(\partial D)}$$

and

$$\|f\|_{\tilde{H}^p(\partial D)}\leq C\|\left(\frac{1}{2}I-\mathcal{K}^*\right)(f)\|_{\tilde{H}^p(\partial D)}.$$

Proof. We consider the first estimate. From Theorem 3.1, and the uniqueness results of Theorems 2.8 and 2.10, we see that the estimates of Theorem C and D apply to $\mathcal{S}(f)$ in D and D^*. Thus

$$\begin{aligned}\|f\|_{H^p(\partial D)} &\le \left\|\left(\frac{1}{2}I+\mathcal{K}^*\right)(f)\right\|_{H^p(\partial D)} + \left\|\left(\frac{1}{2}I-\mathcal{K}^*\right)(f)\right\|_{H^p(\partial D)} \\ &\le \left\|\left(\frac{1}{2}I+\mathcal{K}^*\right)(f)\right\|_{H^p(\partial D)} + C\|\mathcal{S}(f)\|_{H^p_1(\partial D)} \\ &\le C'\left\|\left(\frac{1}{2}I+\mathcal{K}^*\right)(f)\right\|_{H^p(\partial D)}.\end{aligned}$$

The first inequality is the triangle inequality, the second is Theorem C and the third is Theorem D. The proof of the second estimate of our theorem is similar. □

We are now ready to give our representation theorem for solutions of the Dirichlet problem with C^α data.

Theorem 3.4. *Let D be a starshaped Lipschitz domain and let $1-\epsilon < p < 1$. Then the maps*

$$\frac{1}{2}I+\mathcal{K}^* : \ H^p(\partial D) \to H^p(\partial D)$$

$$\frac{1}{2}I-\mathcal{K}^* : \ \tilde{H}^p(\partial D) \to \tilde{H}^p(\partial D)$$

are invertible.

Proof. The estimate of Proposition 3.3 implies that $\frac{1}{2}I+\mathcal{K}^*$ is injective and has closed image. Thus, to establish the invertibility, we only need show that the image of $\frac{1}{2}I+\mathcal{K}^*$ is dense in $H^p(\partial D)$. But this is easy since it is known [**14**, **15**] that $\frac{1}{2}I+\mathcal{K}^*$ is invertible on $L^2_0(\partial D) \equiv L^2(\partial D) \cap \{f : \int f = 0\}$. □

Corollary 3.5. *The maps*

$$+\frac{1}{2}I+\mathcal{K} : C^\alpha(\partial D) \to C^\alpha(\partial D)$$

and

$$-\frac{1}{2}I+\mathcal{K} : \tilde{C}^\alpha(\partial D) \to \tilde{C}^\alpha(\partial D)$$

are invertible for $0 < \alpha < d\epsilon/(1-\epsilon)$ where ϵ is as in Theorem 3.4.

Proof. This follows immediately from Theorem 3.4 and the duality relations stated in Theorem 3.2. □

Acknowledgments.

I thank E.B. Fabes for gently encouraging me to finish this note and I thank C.E. Kenig for suggesting that Wilson's work [**16**] would be helpful for obtaining the atomic decomposition of Theorem 1.2.

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Received June 24, 1993. Supported in part by the NSF and the Commonwealth of Kentucky through the Kentucky EPSCoR program.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,
Berkeley, CA 94720, A NON-PROFIT CORPORATION
This publication was typeset using AMS-LATEX,
the American Mathematical Society's TEX macro system.

PACIFIC JOURNAL OF MATHEMATICS

Volume 171 No. 2 December 1995