

# *Pacific Journal of Mathematics*

**REPRODUCING KERNELS AND COMPOSITION SERIES FOR  
SPACES OF VECTOR-VALUED HOLOMORPHIC FUNCTIONS**

**BENT ØRSTED AND GENKAI ZHANG**

# REPRODUCING KERNELS AND COMPOSITION SERIES FOR SPACES OF VECTOR-VALUED HOLOMORPHIC FUNCTIONS

BENT ØRSTED AND GENKAI ZHANG

We calculate the norm of each  $K$ -type in a vector-valued Hilbert space of holomorphic functions on a tube domain of type I. As a consequence we get composition series of the analytic continuation of certain holomorphic discrete series and an expansion relative to  $K$  of the matrix-valued reproducing kernel.

## Introduction.

The theory of unitary highest weight modules for semisimple Lie groups is by now very well developed. Questions of classification, intertwining operators and primitive ideals have been settled by algebraic means, see [1, 2, 3] and [8] and reference there. However, there remain some open problems on the analytic side, in particular to find analytical proofs and expressions for the unitarity.

The problem we consider here is to calculate by purely analytical means the invariant Hermitian form in a Harish-Chandra module of highest weight. At the same time, we find explicitly the  $K$ -types in a composition series at reducible values of the parameter for the module. This is of interest for example in finding explicit intertwining differential operators giving the various subquotients.

The problem of composition series and expansion of reproducing kernels for analytic continuations of the holomorphic discrete series has been studied extensively. In [15] this is done for the type I domain of tube type and for the scalar-valued Bergman spaces by calculating the norm of each  $K$ -type. Recall the simplest case, where  $G = SU(1, 1)$  and the expansion amounts to the binomial formula

$$(1 - z\bar{w})^{-\lambda} = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} z^k \bar{w}^k.$$

Here the monomials  $z^k$ ,  $k \geq 0$ , of the complex variable  $z$  in the unit disk give the  $K$ -types of the corresponding highest weight module, and the coefficients

in the expansion give the reciprocals of the invariant Hermitian form for the module. Faraut and Koranyi [4] recently solved this problem for a general bounded symmetric domain by using the generalized Gamma-function. The similar problem for vector-valued function spaces of holomorphic functions has been around for some time, see [17]. However, it seems to us that even for the group  $SU(2, 2)$  with the “smallest” representation of  $U(2)$  on  $\mathbb{C}^2$  as fiber no explicit result is known. Generally speaking, this problem involves the explicit decomposition of tensor products of representations of compact groups, which according [20], is *prohibitive*.

In this paper we give the expansion of the reproducing kernel of a vector-valued Bergman space of holomorphic functions on a tube domain of type I. We consider here the first simple nontrivial representation of the compact group  $U(n)$  (or rather  $S(U(n), U(n))$ ), namely its defining representation on  $\mathbb{C}^n$ . The  $K$ -irreducible decomposition of the space of  $\mathbb{C}^n$ -valued polynomials can be then read off abstractly. We find the norm of each  $K$ -space and give the expansion of the matrix-valued reproducing kernel.

As an application we can then read off from our formula the composition series in the analytic continuation and also the unitarizability of the quotients in the composition series. Since the arguments are similar to those in [4] and [15] we only sketch the proofs.

The calculation of the tensor product decompositions in this paper is elementary. We can certainly use the spherical functions on compact groups and Harish-Chandra  $c$ -functions to simplify our result. On the other hand our calculation gives more information about the decomposition. In a subsequent paper we will use the result in this paper to study the vector-valued function spaces of holomorphic functions on some other bounded symmetric domains.

The main results are Theorems 2, 3, and Corollary 1, 2 but also Lemma 1 and Theorem 1 might be of independent interest giving detailed structure of the tensor products with  $\mathbb{C}^n$ .

## §1. Decomposition of tensor products.

In this section we give the decomposition of tensor products of holomorphic representations of  $U(n)$  with its defining representation on  $\mathbb{C}^n$ . The result will be used in the next section to calculate the norm of each  $K$ -type in Hilbert space of vector-valued holomorphic functions on

$$D = SU(n, n)/S(U(n), U(n)).$$

Before going into the calculation we briefly recall here the root system for the compact group  $U(n)$  and fix some notations. See [20].

The Lie algebra  $u(n)$  of  $U(n)$  has complexification  $u(n)_{\mathbb{C}} = gl(n, \mathbb{C})$ , the Lie algebra of  $GL(n, \mathbb{C})$ , which has the standard realization as  $n \times n$  complex matrices. Denote  $E_i$  the diagonal matrix with  $i$ th part being 1 and the rest 0, and  $E_{i,j}$ ,  $i \neq j$ , the matrix with  $(i,j)$  entry being 1 and the rest 0. The elements  $\{E_i, i = 1, \dots, n\}$  span a Cartan algebra of  $u(n)_{\mathbb{C}}$ . We denote  $\varepsilon_i$  the linear functional on the Cartan algebra defined by  $\varepsilon_i(E_j) = \delta_{i,j}$ . The root spaces and roots are  $\mathbb{C}E_{i,j}$  and  $\varepsilon_{i,j} = \varepsilon_i - \varepsilon_j$ , respectively. The positive and positive simple roots are  $\varepsilon_i - \varepsilon_j$ ,  $i < j$  and  $\varepsilon_i - \varepsilon_{i+1}$ ,  $i = 1, \dots, n-1$  respectively.

Let  $(V^{\underline{\mathbf{m}}}, \underline{\mathbf{m}})$  be a holomorphic representation of  $U(n)$  with highest weight  $\underline{\mathbf{m}} = (m_1, \dots, m_n) = m_1\varepsilon_1 + \dots + m_n\varepsilon_n$ , where  $m_1 \geq m_2 \geq \dots \geq m_n \geq 0$ . Denote  $v_{\underline{\mathbf{m}}}$  its highest weight vector with  $\|v_{\underline{\mathbf{m}}}\| = 1$ . When  $\underline{\mathbf{m}} = (1, 0, \dots, 0) = \varepsilon_1$ ,  $V^{\underline{\mathbf{m}}} = \mathbb{C}^n$  is the defining representation of  $U(n)$ . We denote  $\{e_i\}$  its orthonormal basis  $e_1 = (1, 0, \dots, 0)$ , ...,  $e_n = (0, 0, \dots, 1)$ . Hence  $e_1$  is the highest weight vector. We normalize so that  $\|e_1\| = 1$ .

It follows that ([20])

$$(1.1) \quad V^{\underline{\mathbf{m}}} \otimes \mathbb{C}^n = \sum_{i=1}^n V^{\underline{\mathbf{m}}+\varepsilon_i},$$

where the  $i$ th term will not appear if  $\underline{\mathbf{m}} + \varepsilon_i$  violates the condition for highest weight, that is if  $m_{i-1} = m_i$ . We let as above  $v_{\underline{\mathbf{m}}+\varepsilon_i}$  be the highest weight vector of  $V^{\underline{\mathbf{m}}+\varepsilon_i}$ . Our objective in this section is to write explicitly the vectors  $v_{\underline{\mathbf{m}}} \otimes e_i$  as sum of their components in  $V^{\underline{\mathbf{m}}+\varepsilon_j}$  according to the above decomposition.

**Lemma 1.** *We have the following formulae for the highest weight vectors  $v_{\underline{\mathbf{m}}+\varepsilon_i}$  for  $i = 1, 2$*

$$\begin{aligned} v_{\underline{\mathbf{m}}+\varepsilon_1} &= v_{\underline{\mathbf{m}}} \otimes e_1, \\ v_{\underline{\mathbf{m}}+\varepsilon_2} &= f \otimes e_1 - E_{1,2}f \otimes e_2, \end{aligned}$$

where  $f \in E_{2,1}V^{\underline{\mathbf{m}}}$  is such that  $\|f\|^2 + \|E_{1,2}f\|^2 = 1$ . If  $i \geq 3$ , they are of the form

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = f \otimes e_1 - E_{1,2}f \otimes e_2 - \dots - E_{1,i}f \otimes e_i,$$

where  $f \in V^{\underline{\mathbf{m}}}$  is a weight vector of weight  $\underline{\mathbf{m}} - \varepsilon_{1,i}$  and  $E_{2,3}f = \dots = E_{i-1,i}f = 0$ .

*Proof.* Since  $v_{\underline{\mathbf{m}}}$  is a highest weight vector of  $V^{\underline{\mathbf{m}}}$  of weight  $\underline{\mathbf{m}}$  we see that the vector  $v_{\underline{\mathbf{m}}} \otimes e_1$  is of weight  $\underline{\mathbf{m}} + \varepsilon_1$ . It is annihilated by the operators  $E_{i,i+1}$ :

$$E_{i,i+1}(v_{\underline{\mathbf{m}}} \otimes e_1) = (E_{i,i+1}v_{\underline{\mathbf{m}}}) \otimes e_1 + v_{\underline{\mathbf{m}}} \otimes (E_{i,i+1}e_1) = 0.$$

Moreover  $\|v_{\underline{\mathbf{m}}} \otimes e_1\| = 1$  so we can take  $v_{\underline{\mathbf{m}}+\varepsilon_1} = v_{\underline{\mathbf{m}}} \otimes e_1$ .

The element  $f \otimes e_1 - E_{1,2}f \otimes e_2$  for  $f \in E_{2,1}V^{\underline{\mathbf{m}}}$  is of weight  $\underline{\mathbf{m}} + \varepsilon_2$ . We need only to prove that it is annihilated by  $E_{i,i+1}$ . Since the weight space with weight  $\underline{\mathbf{m}} + \varepsilon_{1,2}$  is the zero space, we know that  $E_{1,2}E_{1,2}f = 0$ . Clearly

$$E_{1,2}(f \otimes e_1 - E_{1,2}f \otimes e_2) = (E_{1,2}f) \otimes e_1 - E_{1,2}f \otimes (e_1) = 0.$$

If  $j \geq 2$ , then  $E_{j,j+1}f = 0$  since it is of weight  $\underline{\mathbf{m}} - \varepsilon_{1,2} + \varepsilon_{j,j+1}$ , whose weight space is  $\{0\}$  because  $\varepsilon_{1,2}$  and  $\varepsilon_{j,j+1}$  are simple positive. Therefore  $f \otimes e_1 - E_{1,2}f \otimes e_2$  is a highest weight vector and is of norm 1. This proves the formula for  $v_{\underline{\mathbf{m}}+\varepsilon_2}$ .

Now we prove the formula for  $v_{\underline{\mathbf{m}}+\varepsilon_i}$ ,  $i \geq 3$ . Let  $f_1, \dots, f_n$  in  $V^{\underline{\mathbf{m}}}$  be such that

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = f_1 \otimes e_1 + \dots + f_n \otimes e_n$$

if  $V^{\underline{\mathbf{m}}+\varepsilon_i}$  exists. By computing weights in both sides we see that  $f_j$  is of weight  $\underline{\mathbf{m}} - \varepsilon_{j,i}$ . However  $-\varepsilon_{j,i}$  is a positive root if  $j > i$ , whose weight space is  $\{0\}$ . So  $f_j = 0$  if  $j > i$ . That is

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = f_1 \otimes e_1 + \dots + f_i \otimes e_i,$$

and  $f_i$  is a constant multiple of the highest weight vector  $v_{\underline{\mathbf{m}}}$  by the above weight argument.

The condition  $E_{j,j+1}v_{\underline{\mathbf{m}}+\varepsilon_i} = 0$ ,  $j = 1, 2, \dots, i-1$  implies that  $E_{j,j+1}f_1 = 0$  and

$$f_2 = -E_{1,2}f_1, \dots, f_i = -E_{i-1,i}f_{i-1}.$$

Hence

$$f_3 = -E_{2,3}f_2 = E_{2,3}E_{1,2}f_1 = [E_{2,3}, E_{1,2}]f_1 = -E_{1,3}f_1,$$

where the third equality is obtained since  $E_{2,3}f_1 = 0$ . Continuing similar calculation we get  $f_j = -E_{1,j}f_1$ ,  $j = 2, \dots, i$ . This finishes the proof of the Lemma.  $\square$

**Lemma 2.** *The vectors  $v_{\underline{\mathbf{m}}} \otimes e_i$  have the following decomposition according to (1.1)*

$$\begin{aligned} v_{\underline{\mathbf{m}}} \otimes e_1 &= a_{1,1}P_1^{(1)}, \\ v_{\underline{\mathbf{m}}} \otimes e_2 &= a_{2,1}P_1^{(2)} + a_{2,2}P_2^{(2)} \\ &\vdots \\ v_{\underline{\mathbf{m}}} \otimes e_n &= a_{n,1}P_1^{(n)} + a_{n,2}P_2^{(n)} + \dots + a_{n,n-1}P_{n-1}^{(n)} + a_{n,n}P_n^{(n)}, \end{aligned}$$

where  $P_i^{(j)} \in V^{\mathbf{m}+\epsilon_i}$  and  $\|P_i^{(j)}\| = 1$ . That is, each vector  $v_{\mathbf{m}} \otimes e_i$  has no projection in the space  $V^{\mathbf{m}+\epsilon_j}$  if  $j > i$ .

*Proof.* This can be seen by using the similar weight argument as in the proof of Lemma 1.  $\square$

**Theorem 1.** We have the following formula for entries  $a_{k,i}$ ,  $k \geq i$ , in the above (lower triangular) matrix

$$|a_{k,i}|^2 = \frac{\prod_{j=1, j \neq i}^{k-1} (m_i - m_j - i + j + 1)}{\prod_{j=1, j \neq i}^k (m_i - m_j - i + j)},$$

where  $|a_{1,1}|^2 = 1$ .

To prove the Theorem, we need an identity.

**Lemma 3.** The following identity holds

$$\sum_{i=1}^k \frac{\prod_{j=1, j \neq i}^{k-1} (m_i - m_j - i + j + 1)}{\prod_{j=1, j \neq i}^k (m_i - m_j - i + j)} = 1.$$

*Proof.* Consider the following partial fractional expansion

$$(1.2) \quad \prod_{i=1}^{k-1} \left( 1 + \frac{1}{z - n_i} \right) = 1 + \sum_{i=1}^{k-1} \frac{r_i}{z - n_i}.$$

A easy calculation shows that

$$r_i = \frac{\prod_{j=1, j \neq i}^{k-1} (n_i - n_j + 1)}{\prod_{j=1, j \neq i}^{k-1} (n_i - n_j)}.$$

Put  $z = 0$  in (1.2). We get

$$\sum_{i=1}^{k-1} \frac{r_i}{n_i} + \prod_{i=1}^{k-1} \left( 1 - \frac{1}{n_i} \right) = 1.$$

Substituting  $n_j = m_j - m_k - j + k$  we then get the identity in the Lemma.  $\square$

For formulas of this kind see further [14].

*Proof of Theorem 1.* We will prove the Theorem by induction. The idea is simply to use the *branching rule* by restriction of  $U(n)$  to its subgroup  $U(n-1)$ . It is clear that the result is true for  $n = 1$ . Suppose the result is

true for  $n - 1$ , that is the above formula is true for the tensor product of a holomorphic representation of  $U(n - 1)$  with its defining representation on  $\mathbb{C}^{n-1}$ .

Now let  $V^{\underline{\mathbf{m}}}$  be a representation of  $U(n)$  with highest weight  $\underline{\mathbf{m}} = (m_1, \dots, m_n)$ ,  $m_1 \geq \dots \geq m_n \geq 0$ . It is easy to see that  $V^{\underline{\mathbf{m}}} = V^{(m_1 - m_n, \dots, m_{n-1} - m_n, 0)} \otimes V^{(m_n, \dots, m_n)}$  and  $V^{(m_n, \dots, m_n)}$  is one-dimensional ([7]). So by tensoring everything for the formulas about  $(m_1 - m_n, \dots, m_{n-1} - m_n, 0)$  by  $V^{(m_n, \dots, m_n)}$  we get the result about  $\underline{\mathbf{m}}$ . So we can assume  $m_n = 0$ . We also notice that the formula for  $|a_{k,j}|^2$  stays the same if we replace  $m_i$  by  $m_i - m_n$ .

We consider the natural imbedding of  $\mathbb{C}^{n-1}$  into  $\mathbb{C}^n$  by identifying the first  $n - 1$  basis vectors  $e_1, \dots, e_{n-1}$  of  $\mathbb{C}^{n-1}$  and of  $\mathbb{C}^n$ . The group  $U(n - 1)$  is then a subgroup of  $U(n)$  under this identification. The space  $V^{\underline{\mathbf{m}}}$ , as a  $U(n - 1)$ -module, is then decomposed into irreducibles, and the multiplicities of each is one. See [20]. One of these is of highest weight  $\underline{\mathbf{m}}' = (m_1, \dots, m_{n-1})$ . With abuse of notation we denote this module by  $V^{\underline{\mathbf{m}}'}$ . Under the  $U(n - 1)$ -action, the vector  $v_{\underline{\mathbf{m}}}$  is of weight  $\underline{\mathbf{m}}'$  and is annihilated by  $E_{i,i+1}$ . By the multiplicity one property we see that  $v_{\underline{\mathbf{m}}} \in V^{\underline{\mathbf{m}}}$  is also in the space  $V^{\underline{\mathbf{m}}'}$  and is the highest weight vector of  $V^{\underline{\mathbf{m}}'}$ .

The tensor product  $V^{\underline{\mathbf{m}}'} \otimes \mathbb{C}^{n-1}$ , as  $U(n - 1)$ -module, is decomposed as follows,

$$V^{\underline{\mathbf{m}}'} \otimes \mathbb{C}^{n-1} = \sum_{i=1}^{n-1} V^{\underline{\mathbf{m}}' + \varepsilon_i},$$

with the same convention as that of (1.1). Let  $v_{\underline{\mathbf{m}}' + \varepsilon_i}$ ,  $i = 1, \dots, n - 1$  be the highest weight vectors of  $V^{\underline{\mathbf{m}}' + \varepsilon_i}$ . We claim that

$$v_{\underline{\mathbf{m}}' + \varepsilon_i} \in V^{\underline{\mathbf{m}} + \varepsilon_i},$$

the subspace in (1.1) of  $U(n)$ -decomposition. In fact, it is clear that  $v_{\underline{\mathbf{m}}' + \varepsilon_i}$  has the weight  $\underline{\mathbf{m}} + \varepsilon_i$  under  $U(n)$ . It is annihilated by  $E_{j,j+1}$ ,  $j = 1, \dots, n - 2$ . We need therefore only to prove that  $E_{n-1,n} v_{\underline{\mathbf{m}}' + \varepsilon_i} = 0$ . From Lemma 1 we know that

$$v_{\underline{\mathbf{m}}' + \varepsilon_i} = f \otimes e_1 - E_{1,2} f \otimes e_2 - \dots - E_{1,i-1} f \otimes e_{i-1} - E_{1,i} f \otimes e_i,$$

with  $f \in V^{\underline{\mathbf{m}}'}$  is of the weight  $\underline{\mathbf{m}}' - \varepsilon_{1,i}$ . Now

$$\begin{aligned} E_{n-1,n} v_{\underline{\mathbf{m}}' + \varepsilon_i} &= E_{n-1,n} f \otimes e_1 - E_{n-1,n} E_{1,2} f \otimes e_2 \dots \\ &\quad - E_{n-1,n} E_{1,i-1} f \otimes e_{i-1} - E_{n-1,n} E_{1,i} f \otimes e_i. \end{aligned}$$

The vector  $E_{n-1,n} f$  under  $U(n)$  is then of weight  $\underline{\mathbf{m}} - \varepsilon_{1,i} + \varepsilon_{n-1,n} = \underline{\mathbf{m}} - \sum_{j=1}^{i-1} \varepsilon_{j,j+1} + \varepsilon_{n-1,n}$ , which is zero by the weight theory. (Every weight appearing in the module  $V^{\underline{\mathbf{m}}}$  is of the form  $\underline{\mathbf{m}} - \sum_{j=1}^{n-1} n_j \varepsilon_{j,j+1}$ . This expression

is unique since  $\varepsilon_{j,j+1}$  are linearly independent.) Similarly, we get

$$E_{n-1,n}E_{1,2}f = 0, \dots, E_{n-1,n}E_{1,i-1}f = 0.$$

The vector  $E_{1,i}f$  is of weight  $\underline{\mathbf{m}}$  therefore  $E_{n-1,n}E_{1,i}f = 0$ . That is  $E_{n-1,n}v_{\underline{\mathbf{m}}'+\varepsilon_i} = 0$ . So  $v_{\underline{\mathbf{m}}'+\varepsilon_i}$  is in  $V^{\underline{\mathbf{m}}+\varepsilon_i}$ . Consequently we have

$$V^{\underline{\mathbf{m}}'+\varepsilon_i} \subset V^{\underline{\mathbf{m}}+\varepsilon_i}.$$

So we have a decomposition table of  $v_{\underline{\mathbf{m}}} \otimes e_i$ ,  $i = 1, \dots, e_{n-1}$ , with the same  $P_j^{(i)}$  as they are viewed as  $U(n)$ -vectors. By our induction assumption, we have

$$(1.3) \qquad v_{\underline{\mathbf{m}}} \otimes e_{n-1} = a_{n-1,1}P_1^{(n-1)} + \dots + a_{n-1,n-1}P_{n-1}^{(n-1)}.$$

Here we have used the simple observation that the formulae of  $|a_{k,i}|^2$ ,  $k, i \leq n-1$  for the tensor product  $V^{\underline{\mathbf{m}}'} \otimes \mathbb{C}^{n-1}$  of  $U(n-1)$  are the same as of  $V^{\underline{\mathbf{m}}} \otimes \mathbb{C}^n$  of  $U(n)$ .

Now we consider the decomposition of  $v_{\underline{\mathbf{m}}} \otimes e_n$ ,

$$(1.4) \qquad v_{\underline{\mathbf{m}}} \otimes e_n = a_{n-1,1}P_1^{(n)} + \dots + a_{n,n-1}P_{n-1}^{(n)} + a_{n,n}P_n^{(n)}.$$

We observe further that

$$a_{n,k}E_{n-1,n}P_k^{(n)} = a_{n-1,k}P_k^{(n-1)}, \quad k = 1, \dots, n-1$$

by operating  $E_{n-1,n}$  on (1.4) and comparing with (1.3). Taking norm in this identity we see that

$$(1.5) \qquad |a_{n,k}|^2 = \|E_{n-1,n}P_k^{(n)}\|^{-2}|a_{n-1,k}|^2.$$

Below we will calculate  $\|E_{n-1,n}P_k^{(n)}\|^{-2}$ . By calculating weights in both sides, we see that  $P_k^{(n)}$  is of weight  $\underline{\mathbf{m}} + \varepsilon_n$ ,  $k = 1, \dots, n-1$  and  $P_n^{(n)}$  is a highest weight vector of  $V^{\underline{\mathbf{m}}+\varepsilon_n}$ . Moreover, all those vectors are annihilated by  $E_{j,j+1}$ ,  $j = 1, \dots, n-2$ . Therefore  $P_k^{(n)} \in V^{\underline{\mathbf{m}}+\varepsilon_k}$  is the highest weight vector for the submodule of  $V^{\underline{\mathbf{m}}+\varepsilon_k}$  with highest weight  $\underline{\mathbf{m}}' + \varepsilon_n$  in the decomposition of  $V^{\underline{\mathbf{m}}+\varepsilon_k}$  under  $U(n-1)$ , again by the multiplicity one result.

We now recall the Gelfand-Cetlin orthonormal basis and tableaux. See [20] for further details. For any  $U(n)$ -module with highest weight  $m_{1,n}\varepsilon_1 + m_{2,n}\varepsilon_2 + \dots + m_{n,n}\varepsilon_n$ , we restrict  $V^{\underline{\mathbf{m}}}$  to its subgroup  $U(n-1)$ . Under  $U(n-1)$  it is decomposed into irreducibles with highest weights  $m_{1,n-1}\varepsilon_1 + m_{2,n-1}\varepsilon_2 + \dots + m_{n-1,n-1}\varepsilon_{n-1}$  as indicated above. We decompose further those spaces



successively under  $U(n-2)$  and so on. In the last step we get one-dimensional spaces of  $U(1)$ . Normalize the nonzero vectors in these spaces we get the so called Gelfand-Cetlin orthonormal basis. Each such vector corresponds to a tableau

$$\begin{pmatrix} m_{1,n} & \cdots & m_{n,n} \\ m_{1,n-1} & \cdots & m_{n-1,n-1} \\ \cdots & \cdots & \cdots \\ & m_{1,2} m_{2,2} & \\ & m_{1,1} & \end{pmatrix}.$$

It follows from our observation above that  $P_k^{(n)}$  is a normalized Gelfand-Cetlin basis vector with the corresponding weight tableaux with entries  $m_{k,n} = m_k + k$  and  $m_{i,j} = m_i$  if  $i \neq k$ . It follows further from the Theorem 7 in [20, p. 205] that

$$\begin{aligned} \|E_{n-1,n} P_k^{(n)}\|^2 &= \frac{\prod_{i=1, i \neq k}^n (m_i - i - (m_k - k))}{\prod_{i=1}^{n-1} (m_i - i - (m_k - k) - 1)} \frac{\prod_{i=1}^{n-2} (m_i - i - (m_k - k) - 1)}{\prod_{i=1, i \neq k}^{n-1} (m_i - i - (m_k - k))} \\ &= \frac{m_n - m_k - n + k}{m_{n-1} - m_k - n + k} \\ &= \frac{m_k - m_n - k + n}{m_k - m_{n-1} - k + n}. \end{aligned}$$

(Notice that only the term  $j = k$  is nonzero in the formula in Theorem 7 there.)

From this, (1.5) and the induction assumption we see that

$$\begin{aligned} |a_{n,k}|^2 &= \frac{(m_k - m_{n-1} - k + n)}{(m_k - m_n - k + n)} |a_{n-1,k}|^2 \\ &= \frac{(m_k - m_{n-1} - k + n)}{(m_k - m_n - k + n)} \frac{\prod_{j=1, j \neq k}^{n-2} (m_k - m_j - k + j + 1)}{\prod_{j=1, j \neq k}^{n-1} (m_k - m_j - k + j)} \\ &= \frac{\prod_{j=1, j \neq k}^{n-1} (m_k - m_j - k + j + 1)}{\prod_{j=1, j \neq k}^n (m_k - m_j - k + j)} \end{aligned}$$

for  $i = 1, \dots, n-1$ . So the theorem is true for  $a_{k,i}$ ,  $k, i \leq n-1$ .

Since  $\sum_{i=1}^n |a_{n,i}|^2 = 1$ , by Lemma 3 we know that the Theorem is true for  $a_{n,n}$ . This finishes the proof.  $\square$

**Lemma 4.** *Let  $a_{k,i}$  be the entries in Theorem 1. The following identity holds*

$$\sum_{k=i}^n |a_{k,i}|^2 = \frac{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j + 1)}{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j)}.$$

*Proof.* We prove this by induction on the number  $n$ . If  $n = 1$  there is nothing to prove. If  $n = 2$  the above claim reads

$$1 + \frac{1}{m_1 - m_2 - 1 + 2} = \frac{m_1 - m_2 - 1 + 2 + 1}{m_1 - m_2 - 1 + 2}$$

for  $i = 1$ , which is clearly true; for  $i = 2$  it is just an obvious identity.

Suppose this is true for all  $n - 1$  tuples of integers  $m_1 \geq \dots \geq m_{n-1}$ . We prove this for  $n$ . For  $i \leq n$  we have

$$\begin{aligned} & \sum_{k=i}^n |a_{k,i}|^2 \\ &= \sum_{k=i}^{n-1} |a_{k,i}|^2 + |a_{n,i}|^2 \\ &= \frac{\prod_{j=1, j \neq i}^{n-1} (m_i - m_j - i + j + 1)}{\prod_{j=1, j \neq i}^{n-1} (m_i - m_j - i + j)} + \frac{\prod_{j=1, j \neq i}^{n-1} (m_i - m_j - i + j + 1)}{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j)} \\ &= \frac{\prod_{j=1, j \neq i}^{n-1} (m_i - m_j - i + j + 1)(m_i - m_n - i + n + 1)}{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j)} \\ &= \frac{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j + 1)}{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j)}. \end{aligned}$$

□

## §2. The norm of $K$ -types.

Let  $D$  be the tube domain of type I, that is,

$$D = \{Z \in M(n, \mathbb{C}) : 1 - Z^*Z > 0\}.$$

The domain  $D$  is the bounded realization of  $G/K$ , where  $G = SU(n, n)$  and  $K = S(U(n), U(n))$ .

Let  $(\mathbb{C}^n, \varepsilon_1)$  be the representation space of  $U(n)$  with highest weight  $\varepsilon_1$  as in §1. For  $\lambda > 2n - 1$  we consider the Hilbert space  $H_\lambda$  of  $\mathbb{C}^n$ -valued holomorphic functions  $f(Z)$  on  $D$  with the norm

(2.1)

$$\|f\|_\lambda^2 = d(\lambda) \int_D \left\langle (1 - Z^*Z)f(Z), f(Z) \right\rangle \det(1 - Z^*Z)^{\lambda-2n} dZ < \infty.$$

Here  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{C}^n$ ,  $dZ$  is the Lebesgue measure and

$$d(\lambda) = \pi^{-n^2} \frac{\lambda}{\lambda - n} \frac{\prod_{j=1}^n \Gamma(\lambda - j + 1)}{\prod_{j=1}^n \Gamma(\lambda - n - j + 1)}.$$

This normalization corresponds to the constant vector-valued function  $e_1$  has norm 1, i.e.  $\|e_1\|_\lambda = 1$ .

The group  $G$  acts on  $H_\lambda$  via the following

$$U_\lambda(g) : f(Z) \mapsto (\det(CZ + D))^{-\lambda} (CZ + D)^{-1} f(g^{-1}Z), g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where  $g^{-1}Z = (AZ + B)(CZ + D)^{-1}$ . Note that  $\mathbb{C}^n$  here is also viewed as a representation space of  $K$ . So for  $P(Z) \otimes v$ , with  $v \in \mathbb{C}^n$  and  $P(Z)$  a polynomial, viewed as a  $\mathbb{C}^n$ -valued polynomial, we have

$$(2.2) \quad U_\lambda(g) : P(Z) \otimes v \mapsto (\det(CZ + D))^{-\lambda} P(g^{-1}Z) \otimes (CZ + D)^{-1}v.$$

Denote  $K_{\lambda,w}(Z) = K_\lambda(Z, W)$  the  $\mathbb{C}^n \otimes \mathbb{C}^n$ -valued reproducing kernel of  $H_\lambda$  in the sense that

$$f(W) = (f, K_{\lambda,w}(\cdot))_\lambda$$

for any polynomial  $f$  in  $H_\lambda$ . It follows from the transformation property of  $K_\lambda$  under the group  $G$  that

$$K_\lambda(Z, W) = (\det(1 - W^*Z))^{-\lambda} (1 - W^*Z)^{-1}.$$

Let  $\mathcal{P}$  be the space of holomorphic polynomials on  $D$ . The space of  $K$ -finite functions in  $H_\lambda$  is then  $M = \mathcal{P} \otimes \mathbb{C}^n$ , as representations of  $K$ . As is well-known ([15]),

$$\mathcal{P} \cong \sum_{\mathbf{m} \geq 0} V^{\mathbf{m}} \otimes (V^{\mathbf{m}})^*,$$

where  $(V^{\mathbf{m}})^*$  is the contragredient representation of  $V^{\mathbf{m}}$ .

From this and (1.1) we have that

$$M = \mathcal{P} \otimes \mathbb{C}^n \cong \sum_{i=1}^n \sum_{\mathbf{m} \geq 0} V^{\mathbf{m}+\varepsilon_i} \otimes (V^{\mathbf{m}})^*,$$

where we use the same convention of the summation as in (1.1). The intertwining operator from the right hand side to the left in the above is

$$\left( \sum_{i=1}^n g_i \otimes e_i \right) \otimes f \in V^{\mathbf{m}+\varepsilon_i} \otimes (V^{\mathbf{m}})^* \mapsto \sum_{i=1}^n (D^{\mathbf{m}}(Z)g_i, f)e_i, \quad Z \in D.$$

Here  $D^{\mathbf{m}}$  denotes the action in  $V^{\mathbf{m}}$  of  $U(n)$  and its holomorphic continuation to the complexification. We will therefore identify the vectors in  $V^{\mathbf{m}+\varepsilon_i} \otimes (V^{\mathbf{m}})^*$  with the corresponding  $\mathbb{C}^n$ -valued polynomials on  $D$ . Note that here we are identifying the space  $(V^{\mathbf{m}})^*$  with  $V^{\mathbf{m}}$ .

**Theorem 2.** *The norm square of the highest weight vector  $v_{\underline{\mathbf{m}}+\varepsilon_i} \otimes v_{\underline{\mathbf{m}}}$  in the  $K$ -type  $(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*$  is given by*

$$\frac{\prod_{j=1}^n (m_j + n - j)!}{\prod_{j < k} (m_j - m_k - j + k)} \frac{\lambda \prod_{j=1}^n \Gamma(\lambda - j + 1)}{\Gamma(m_i + \lambda - i + 2) \prod_{j=1, j \neq i}^n \Gamma(m_j + \lambda - j + 1)}.$$

*Proof.*

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = \sum_{j=1}^n f_j \otimes e_j,$$

with  $\sum_{j=1}^n \|f_j\|^2 = 1$ . We have

$$\begin{aligned} & \left\langle (1 - Z^* Z) v_{\underline{\mathbf{m}}+\varepsilon_i} \otimes v_{\underline{\mathbf{m}}}, v_{\underline{\mathbf{m}}+\varepsilon_i} \otimes v_{\underline{\mathbf{m}}} \right\rangle \\ &= \left\langle (1 - Z^* Z) \sum_{j=1}^n (D^{\underline{\mathbf{m}}}(Z) f_j, v_{\underline{\mathbf{m}}}) e_j, \sum_{j=1}^n (D^{\underline{\mathbf{m}}}(Z) f_j, v_{\underline{\mathbf{m}}}) e_j \right\rangle \\ &= \sum_{j=1}^n |(D^{\underline{\mathbf{m}}}(Z) f_j, v_{\underline{\mathbf{m}}})|^2 - \left\| \sum_{j=1}^n (D^{\underline{\mathbf{m}}}(Z) f_j, v_{\underline{\mathbf{m}}}) Z e_j \right\|^2 \\ &= I_1 - I_2. \end{aligned}$$

The integral of  $I_1$  in the above formula over  $D$  with respect to the measure in (2.1), by the Schur lemma, is

$$\int I_1 = d(\lambda) \int_D |(D^{\underline{\mathbf{m}}}(Z) v_{\underline{\mathbf{m}}}, v_{\underline{\mathbf{m}}})|^2 \det(1 - Z^* Z)^{\lambda-2n} dZ,$$

which furthermore by the formula (13) in [15, p. 568], is

$$\frac{\lambda}{\lambda - n} \frac{\prod_{j=1}^n (m_j + n - j)!}{\prod_{j < k} (m_j - m_k - j + k)} \frac{\prod_{j=1}^n \Gamma(\lambda - j + 1)}{\prod_{j=1}^n \Gamma(m_j + \lambda - j + 1)}.$$

The second term  $I_2$  is

$$\begin{aligned} I_2 &= \left\| \sum_{j=1}^n (D^{\underline{\mathbf{m}}}(Z) f_j, v_{\underline{\mathbf{m}}}) Z e_j \right\|^2 \\ &= \sum_{k=1}^n \left| \sum_{j=1}^n (D^{\underline{\mathbf{m}}}(Z) f_j, v_{\underline{\mathbf{m}}}) \langle Z e_j, e_k \rangle \right|^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^n \left| \sum_{j=1}^n (D^{\underline{\mathbf{m}} \otimes \varepsilon_1} (Z) f_j \otimes e_j, v_{\underline{\mathbf{m}}} \otimes e_k) \right|^2 \\
&= \sum_{k=1}^n |(D^{\underline{\mathbf{m}} \otimes \varepsilon_1} (Z) v_{\underline{\mathbf{m}} + \varepsilon_i}, v_{\underline{\mathbf{m}}} \otimes e_k)|^2 \\
&= \sum_{k=1}^n |(D^{\underline{\mathbf{m}} + \varepsilon_i} (Z) v_{\underline{\mathbf{m}} + \varepsilon_i}, v_{\underline{\mathbf{m}}} \otimes e_k)|^2 \\
&= \sum_{k=i}^n |(D^{\underline{\mathbf{m}} + \varepsilon_i} (Z) v_{\underline{\mathbf{m}} + \varepsilon_i}, v_{\underline{\mathbf{m}}} \otimes e_k)|^2,
\end{aligned}$$

where in the last equality we use the fact that  $v_{\underline{\mathbf{m}}} \otimes e_k$  only has projections in  $V^{\underline{\mathbf{m}} + \varepsilon_i}$  for  $k \geq i$  as proved in Lemma 2.

Let  $v_{\underline{\mathbf{m}}} \otimes e_k = a_{k,1} P_1^{(k)} + \cdots + a_{k,k} P_k^{(k)}$  as in Lemma 2. So

$$I_2 = \sum_{k=i}^n |a_{k,i}|^2 \left| (D^{\underline{\mathbf{m}} + \varepsilon_i} (Z) v_{\underline{\mathbf{m}} + \varepsilon_i}, P_i^{(k)}) \right|^2.$$

Its integral is,

$$\begin{aligned}
\int I_2 &= \sum_{k=i}^n |a_{k,i}|^2 \int \left| (D^{\underline{\mathbf{m}} + \varepsilon_i} (Z) v_{\underline{\mathbf{m}} + \varepsilon_i}, P_i^{(k)}) \right|^2 \\
&= \sum_{k=i}^n |a_{k,i}|^2 \int |(D^{\underline{\mathbf{m}} + \varepsilon_i} (Z) v_{\underline{\mathbf{m}} + \varepsilon_i}, v_{\underline{\mathbf{m}} + \varepsilon_i})|.
\end{aligned}$$

Here we have used the Schur Lemma since  $\|P_i^{(k)}\| = \|v_{\underline{\mathbf{m}} + \varepsilon_i}\| = 1$ . Now by Lemma 4 above and Proposition 1 in [15] we have

$$\begin{aligned}
\int I_2 &= \sum_{k=i}^n |a_{k,i}|^2 d(\lambda) \int_D |(D^{\underline{\mathbf{m}} + \varepsilon_i} (Z) v_{\underline{\mathbf{m}} + \varepsilon_i}, v_{\underline{\mathbf{m}} + \varepsilon_i})|^2 \det(1 - Z^* Z)^{\lambda - 2n} dZ \\
&= \frac{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j + 1)}{\prod_{j=1, j \neq i}^n (m_i - m_j - i + j)} \times \\
&\times \frac{\lambda}{\lambda - n} \frac{\prod_{j=1}^n (m_j + n - j)!}{\prod_{j < k} (m_j - m_k - j + k)} \frac{\prod_{j=1}^n \Gamma(\lambda - j + 1)}{\prod_{j \neq i} \Gamma(m_j + \lambda - j + 1) \Gamma(m_i + \lambda - i + 2)}.
\end{aligned}$$

Finally we get the norm of  $v_{\underline{\mathbf{m}}} \otimes v_{\underline{\mathbf{m}} + \varepsilon_i}$ , after a long computation, from

$$\begin{aligned}
&\int I_1 - \int I_2 \\
&= \frac{\prod_{j=1}^n (m_j + n - j)!}{\prod_{j < k} (m_j - m_k - j + k)} \frac{\lambda \prod_{j=1}^n \Gamma(\lambda - j + 1)}{\Gamma(m_i + \lambda - i + 2) \prod_{j=1, j \neq i}^n \Gamma(m_j + \lambda - j + 1)}.
\end{aligned}$$

This completes the proof.  $\square$

**Remark 1.** To write the above expression further in a compact form we recall briefly here the Gindikin Gamma-function, see [4]. For a bounded symmetric space  $D = G/K$  with the corresponding strongly orthogonal roots  $\gamma_1, \dots, \gamma_r$ . Let  $a$  be the (common) multiplicity of the root space in  $\mathfrak{k}^*$  of  $\frac{\gamma_i - \gamma_j}{2}$ ,  $i \neq j$ . (See [4] for precise definition.) The Gindikin Gamma-function is defined by

$$\Gamma(\underline{m}) = \prod_{i=1}^r \Gamma\left(m_i - \frac{a}{2}(i-1)\right).$$

One defines similarly then the Pochhammer's symbol

$$(\lambda)_{\underline{m}} = \frac{\prod_{i=1}^r \Gamma(\lambda + m_i - \frac{a}{2}(i-1))}{\prod_{i=1}^r \Gamma(\lambda - \frac{a}{2}(i-1))}.$$

In our case  $a = 2$ . In terms of the Gindikin's Gamma-function (see [4]) the formula in Theorem 2 can be written as

$$\frac{\Gamma(\underline{m} + n)}{\prod_{j < k} (m_j - m_k - j + k)} \frac{\lambda}{(\lambda)_{\underline{m} + \varepsilon}}.$$

Note that we may write  $\prod_{j < k} (m_j - m_k - j + k)$  as  $(2n-2)^{\frac{n(n-1)}{2}} \prod_{\alpha} (\underline{m} + \underline{\nu}, \alpha)$ , where the product is over all the positive roots of  $u(n)_{\mathbb{C}}$  and  $\underline{\nu} = (-1, -2, \dots, -n)$  and  $(\cdot, \cdot)$  is the Killing form. This expression of norm is quite analogous to the corresponding formula for the  $K$ -types in the scalar-valued case ([4, Theorem 3.6]). This indicates that presumably there is a general formula for the vector-valued case. See our forthcoming preprint on type II and III domains.

Let  $K_{\underline{m}}^{(i)}(Z, W)$  be the reproducing kernel of each  $K$ -space  $V^{\underline{m} + \varepsilon_i} \otimes (V^{\underline{m}})^*$ , normalized as above so that  $v_{\underline{m}}$  and  $e_1$  have norm 1. Denote  $d_{\underline{m}, i}$  the dimension of the space  $V^{\underline{m} + \varepsilon_i} \otimes (V^{\underline{m}})^*$ , see [7] and [18].

**Corollary 1.** For  $\lambda \in \mathbb{C}$  we have the following expansion

$$(2.3) \quad K_{\lambda}(Z, W) = \sum_{i=1}^n \sum_{\underline{m} \geq 0} d_{\underline{m}, i}^{-1} \frac{\prod_{j < k} (m_j - m_k - j + k)}{\Gamma(\underline{m} + n)} \frac{(\lambda)_{\underline{m} + \varepsilon_i}}{\lambda} K_{\underline{m}}^{(i)}(Z, W).$$

This series converges uniformly on compact sets of  $D \times D$ .

*Proof.* The case when  $\lambda > 2n - 1$  follows from the standard argument about reproducing kernels, see [5] for the scalar case. The case for general  $\lambda \in \mathbb{C}$  follows from similar arguments as in [4].  $\square$

**Example.** We take  $n = 2$ . Let  $W = I$  and  $Z = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$  in the above expansion. The  $(1, 1)$  entry of this matrix equality is the following

$$\begin{aligned} & (1-s)^{-\lambda-1}(1-t)^{-\lambda} \\ &= \sum_{l=0, m=0}^{\infty} \frac{\Gamma(m+1)\Gamma(l+m+2)}{(\lambda-1)_m(\lambda+1)_{l+m}} (st)^m \left( \sum_{i=0}^l (l+1-i)s^{l-i}t^i \right) + \\ &+ \sum_{l=1, m=0}^{\infty} \frac{\Gamma(m+1)\Gamma(l+m+2)}{(\lambda-1)_{m+1}(\lambda+1)_{m+l-1}} (st)^m \left( \sum_{i=1}^l i s^{l-i}t^i \right). \end{aligned}$$

This might be of interest from the combinatorics point of view.

### §3. The composition series.

In this section we will study the analytic continuation in the parameter  $\lambda$  of the expansion of the reproducing kernel obtained above. In particular, we will find the Wallach set, namely the set of those  $\lambda$ 's at which we get a unitary representation on the module generated by the reproducing kernel, and we will find the composition series when  $M$  is reducible. We will only briefly indicate the results since the argument is the much same as in [4] and [15].

**Lemma 5.** *For  $\lambda \in \mathbb{C}$  the function  $K_\lambda(Z, W)$  is positive definite on  $D \times D$  if and only if  $\lambda > n - 1$  or  $\lambda = n - 1, \dots, 1$ .*

*Proof.* It can be proved similarly as Lemma 5.1 in [4] that  $K_\lambda(Z, W)$  is positive definite if and only if the coefficients in the expansion of  $K_\lambda(Z, W)$  are positive. The Lemma follows then by simple calculation.  $\square$

**Remark 2.** The Wallach set in this case is therefore  $(n-1, \infty) \cup \{1, \dots, n-1\}$ . We note here that at the last point in the Wallach set, i.e., at  $\lambda = 1$  the module  $M_0$  is infinite dimensional, whereas in the scalar case the last point of the Wallach set is  $\lambda = 0$  and it corresponds to the trivial representation.

For  $\lambda \in \mathbb{C}$  let  $q = q(\lambda)$  be the number of nonnegative integers among the  $n$  complex numbers  $\lambda + 1, \lambda - 1, \dots, \lambda - n + 1$ . For fixed  $0 \leq j \leq q$  let

$$M_j = \sum_i \sum_{\underline{m}} V^{\underline{m} + \varepsilon_i} \otimes (V^{\underline{m}})^*$$

where the summation is over the set of  $\underline{m}$  and  $i$  for which the function  $\lambda \rightarrow \frac{(\lambda)_{\underline{m} + \varepsilon_i}}{\lambda}$  has zero of at most multiplicity  $j$ , and with the same convention

as in (1.1). We can therefore read off from our Theorem 2 the zeros of this function and get

**Theorem 3.** *For  $\lambda \in \mathbb{C}$ ,  $M$  has a composition series*

$$M_0 \subset M_1 \subset \cdots \subset M_q = M$$

*of length  $q$ .  $q = q(\lambda) > 0$  if and only if  $\lambda \leq n-1$  and is an integer. Explicitly those modules are given by (to minimize the notations we use highest weight to indicate a module)*

(1) *if  $\lambda = n - k$ ,  $k = 1, \dots, n - 1$  then  $q(\lambda) = k$  and*

$$M_l = \sum_{i \neq n-k+l+1}^n \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{n-k+l+1} \leq l\} \\ \oplus \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_{n-k+l+1}) \otimes \underline{\mathbf{m}}^*, m_{n-k+l+1} \leq l-1\},$$

*for  $0 \leq l \leq k$ ;*

(2) *if  $\lambda = 0$ , then  $q = n - 1$  and*

$$M_l = \sum_{i \neq l+2}^n \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{l+2} \leq l+1\} \\ \oplus \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_{l+2}) \otimes \underline{\mathbf{m}}^*, m_{l+2} \leq l\}$$

*for  $0 \leq l \leq n - 1$ ;*

(3) *if  $\lambda$  is a negative integer then  $q = n$  and*

$$M_l = \sum_{i \neq l+1}^n \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{l+1} \leq -\lambda + l\} \\ \oplus \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_{l+1}) \otimes \underline{\mathbf{m}}^*, m_{l+1} \leq -\lambda + l - 1\}$$

*for  $0 \leq \lambda \leq n$ .*

Note here  $M_0$ , the most interesting part, has the following form at  $\lambda = n - k$ ,  $k = 1, \dots, n - 1$ ,

$$M_0 = \sum_{i=1}^{n-k} \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{n-k+1} = \cdots = m_n = 0\}.$$

**Corollary 2.**  *$M_0$  is unitarizable precisely when  $\lambda > n-1$  or  $\lambda = 1, \dots, n-1$ .  $M_0$  is finite dimensional if and only if  $\lambda$  is a negative integer.  $M_j/M_{j-1}$ ,*



$j \geq 1$ , is unitarizable if and only if  $j = q$  and  $\lambda \leq n - 1$  is an integer. In that case it is isomorphic to the Harish-Chandra module of polynomials in  $H_{2n-\lambda}$ .

*Proof.* The first two parts of the Corollary are direct consequences of Lemma 5 and Theorem 3 and some computations, keeping track of the signs of the coefficients in the expansion, see [15] and [4]. We now prove the third part. We consider here the case  $\lambda = n - k$  the general case is just the same.

In the realization of the the group  $G$  as above we let  $F_i$  be the elements of  $2n \times 2n$  diagonal matrices

$$F_i = \begin{pmatrix} E_i & 0 \\ 0 & -E_i \end{pmatrix}$$

where  $E_i$  is the  $n \times n$  matrix we used in §1. We choose a maximal abelian subspace of the Lie algebra  $\mathfrak{k}_{\mathbb{C}}$  containing  $F_i$ , where  $\mathfrak{k}_{\mathbb{C}}$  is the complexification of the Lie algebra of  $K$ . We define linear functional  $\gamma_i$  on the Cartan algebra by the relation  $\gamma_i(F_j) = 2\delta_{i,j}$  and  $\gamma_i$  is 0 on the orthogonal component (respect to the Killing form) of the linear span of  $F_j$ 's. Those are the Harish-Chandra strongly orthogonal roots, as in [18] for example. We choose a compatible ordering for the compact roots so that the space  $\mathbb{C}^n$  is a representation of  $K$  with highest weight  $-\frac{\gamma_1}{2}$  and  $e_1$  is a highest weight vector. The positive and negative noncompact root spaces of  $\mathfrak{g}_{\mathbb{C}}$ , the complexification of the Lie algebra of  $G$ , will be denoted by  $\mathfrak{p}_+$ ,  $\mathfrak{p}_-$  respectively.  $\mathfrak{p}_+$  is simply the spaces of the matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

where  $x$  is a  $n \times n$  matrix.

Now the module  $H_{\lambda}$  for  $\lambda \in \mathbb{C}$  is a highest weight module of  $\mathfrak{g}_{\mathbb{C}}$  with highest weight  $-(\lambda)(\sum_{i=1}^n \gamma_i) - \frac{1}{2}\gamma_1$  with highest weight vector  $1 \otimes e_1$ , i.e., the constant vector-valued function. See [19] for the general case and [4] for the scalar case.

We fix  $\lambda = n - k$ , so  $q = k$ . We claim that  $M_k/M_{k-1}$  is a highest weight module with highest weight  $-(\lambda+2k) \sum_{i=1}^n \gamma_i - \frac{1}{2}\gamma_1$  and highest weight vector  $(\det Z)^k \otimes e_1$ . In fact it is clear that  $(\det Z)^k \otimes e_1$  is of the stated weight. Moreover this vector is annihilated by the compact positive root vectors, since generally, a tensor product of highest weight vectors is a highest weight vector. Thus we need only to prove that it is annihilated by  $\mathfrak{p}_+$ . Let  $X \in \mathfrak{p}_{\mp}$ , and by (2.2) calculate

$$\begin{aligned} X((\det Z)^k \otimes e_1) &= X(\det Z)^k \otimes e_1 + (\det Z)^k \otimes X e_1 \\ &= X(\det Z)^k \otimes e_1, \end{aligned}$$

since  $X$  is upper triangular. However  $X(\det Z)^k \otimes e_1$  is a polynomial of lower degree ([4]) so by Theorem 3 we know that  $X(\det Z)^k \otimes e_1$  is in  $M_{k-1}$ . That is  $X((\det Z)^k \otimes e_1) = 0$  modulo  $M_{k-1}$ .  $\square$

The argument above is as in [4]. One can also use the argument as in [15] to compare the invariant Hermitian form on  $M_q/M_{q-1}$  with that on  $H_{2n-\lambda}$  to prove the Corollary.

From the above results we can conclude

**Proposition.** *The space  $M$  as a  $\mathfrak{g}_{\mathbb{C}}$ -module is reducible precisely when  $\lambda \leq n-1$  is an integer.*

**Remark 3.** At  $\lambda = -1$  in the analytic continuation of our expansion (2.3) we see that the left hand side is then  $\text{ad}(1 - W^*Z)$ , the adjoint matrix of  $(1 - W^*Z)$ . The coefficients of  $K_{\underline{m}}^{(i)}(Z, W)$  in the right hand side are nonzero only for  $\underline{m} = \varepsilon_1 + \cdots + \varepsilon_{i-1} = (1, \dots, 1, 0, \dots, 0)$  with the first  $i-1$  components being 1 and the rest 0. Therefore the formula (2.3), after a direct calculation, is

$$\text{ad}(1 - W^*Z) = \sum_{i=1}^n (-1)^{i-1} i d_{\varepsilon_1 + \cdots + \varepsilon_{i-1}, i}^{-1} K_{\underline{m}}^{(i)}(Z, W).$$

Here  $d_{\varepsilon_1 + \cdots + \varepsilon_{i-1}, i}$  as we defined in §2 is the dimension of the representation with highest weight  $(\varepsilon_1 + \cdots + \varepsilon_{i-1}) \otimes (\varepsilon_1 + \cdots + \varepsilon_{i-1} + \varepsilon_i)$ . We thus obtain a representation theoretic interpretation of adjoint.

Consider finally the example  $n = 2$ , i.e., the conformal group. Here the last Wallach point is the well-known positive-spinor solutions to Dirac's equation. Indeed, one checks that the  $K$ -types satisfying this are exactly  $M_o$  above at  $\lambda = 1$  with  $K$ -types  $(\underline{m} + \varepsilon_i) \otimes \underline{m}^*$  with  $i = 1$ ,  $m_2 = 0$ , and  $m_1 \geq 0$ . This of course, is in agreement with [9].

## References

- [1] T.J. Enright, *Analogues of Kostant's  $u$ -cohomology formulas for unitary highest weight modules*, J. reine angew. Math., **392** (1988), 27-36.
- [2] T.J. Enright, R. Howe and N.R. Wallach, *A classification of unitary highest weight modules*, Representation Theory of Reductive Groups, Boston 1983, 97-143.
- [3] T.J. Enright and A. Joseph, *An intrinsic analysis of unitarizable highest weight modules*, preprint.
- [4] J. Faraut and A. Koranyi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Func. Anal., **89** (1990), 64-89.
- [5] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, London, 1978.
- [6] ———, *Groups and geometric analysis*, Academic Press, London, 1984.

- [7] L.K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Amer. Math. Soc., Providence, Rhode Island, 1963.
- [8] H.P. Jakobsen, *Hermitian symmetric spaces and their unitary highest weight modules*, J. Funct. Anal., **52** (1983), 385-412.
- [9] H.P. Jakobsen and M. Vergne, *Wave and Dirac operators, and representations of the conformal group*, J. Funct. Anal., **24** (1977), 52-106.
- [10] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil Representations and Harmonic Polynomials*, Invent. Math., **44** (1978), 1-47.
- [11] ———, *Functions on the Shilov boundary of the generalized half space*, Noncommutative Harmonic Analysis, Lectures Notes in Math., **728**, Springer-Verlag, Berlin - Heidelberg - New York, 1978, 136-176.
- [12] A. W. Knap, *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton, New Jersey 1986.
- [13] O. Loos, *Bounded Symmetric Domains and Jordan Pairs*, University of California, Irvine, 1977.
- [14] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, New York, 1979.
- [15] B. Ørsted, *Composition Series for Analytic Continuations of Holomorphic Discrete Series Representations of  $SU(n, n)$* , Trans. Amer. Math. Soc., **260** (1980), 563-573.
- [16] H. Rossi and M. Vergne, *Analytic continuation of the holomorphic discrete series of a semisimple Lie group*, Acta Math., **136** (1976), 1-59.
- [17] W. Rühl, *Field representations of the conformal group with continuous mass spectrum*, Commun. Math. Phys., **30** (1973), 287-302.
- [18] H. Upmeyer, *Jordan algebras and harmonic analysis on symmetric spaces*, Amer. J. Math., **108** (1986), 1-25.
- [19] N. Wallach, *The analytic continuation of the discrete series*, I, II, Trans. Amer. Math. Soc., **251** (1979), 1-17; 19-37.
- [20] D.P. Zelobenko, *Compact Lie groups and their representations*, Transl. Math. Monographs, **40**, Amer. Math. Soc., Providence, Rhode Island, 1973.

Received July 16, 1993 and revised November 3, 1993.

ODENSE UNIVERSITET  
 DK-5230 ODENSE M  
 DENMARK

*E-mail address:* orsted@imada.ou.dk, genkai@imada.ou.dk

## PACIFIC JOURNAL OF MATHEMATICS

Founded by E. F. Beckenbach (1906-1982) and F. Wolf (1904-1989)

### EDITORS

Sun-Yung Alice Chang (Managing Editor)

University of California  
Los Angeles, CA 90095-1555  
pacific@math.ucla.edu

F. Michael Christ  
University of California  
Los Angeles, CA 90095-1555  
christ@math.ucla.edu

Thomas Enright  
University of California  
San Diego, La Jolla, CA 92093  
tenright@ucsd.edu

Nicholas Ercolani  
University of Arizona  
Tucson, AZ 85721  
ercolani@math.arizona.edu

Robert Finn  
Stanford University  
Stanford, CA 94305  
finn@gauss.stanford.edu

Vaughan F. R. Jones  
University of California  
Berkeley, CA 94720  
vfr@math.berkeley.edu

Steven Kerckhoff  
Stanford University  
Stanford, CA 94305  
spk@gauss.stanford.edu

Martin Scharlemann  
University of California  
Santa Barbara, CA 93106  
mgscharl@math.ucsb.edu

Gang Tian  
Courant Institute  
New York University  
New York, NY 10012-1100  
tiang@taotao.cims.nyu.edu

V. S. Varadarajan  
University of California  
Los Angeles, CA 90095-1555  
vsv@math.ucla.edu

### SUPPORTING INSTITUTIONS

CALIFORNIA INSTITUTE OF TECHNOLOGY  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
STANFORD UNIVERSITY  
UNIVERSITY OF ARIZONA  
UNIVERSITY OF BRITISH COLUMBIA  
UNIVERSITY OF CALIFORNIA  
UNIVERSITY OF HAWAII

UNIVERSITY OF MONTANA  
UNIVERSITY OF NEVADA, RENO  
UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA  
UNIVERSITY OF UTAH  
UNIVERSITY OF WASHINGTON  
WASHINGTON STATE UNIVERSITY

The supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

Manuscripts must be prepared in accordance with the instructions provided on the inside back cover.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$215.00 a year (10 issues). Special rate: \$108.00 a year to individual members of supporting institutions.

Subscriptions, orders for back issues published within the last three years, and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

---

The Pacific Journal of Mathematics at the University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 6143, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,  
Berkeley, CA 94720, A NON-PROFIT CORPORATION

This publication was typeset using AMS-LATEX,  
the American Mathematical Society's TEX macro system.  
Copyright © 1995 by Pacific Journal of Mathematics

# PACIFIC JOURNAL OF MATHEMATICS

Volume 171      No. 2      December 1995

---

On $H^p$ -solutions of the Bezout equation	297
ERIC AMAR, JOAQUIM BRUNA FLORIS and ARTUR NICOLAU	
Amenable correspondences and approximation properties for von Neumann algebras	309
CLAIRE ANANTHARAMAN-DELAROCHE	
On moduli of instanton bundles on $\mathbb{P}^{2n+1}$	343
VINCENZO ANCONA and GIORGIO MARIA OTTAVIANI	
Minimal surfaces with catenoid ends	353
JORGEN BERGLUND and WAYNE ROSSMAN	
Permutation model for semi-circular systems and quantum random walks	373
PHILIPPE BIANE	
The Neumann problem on Lipschitz domains in Hardy spaces of order less than one	389
RUSSELL M. BROWN	
Matching theorems for twisted orbital integrals	409
REBECCA A. HERB	
Uniform algebras generated by holomorphic and pluriharmonic functions on strictly pseudoconvex domains	429
ALEXANDER IZZO	
Quantum Weyl algebras and deformations of $U(g)$	437
NAIHUAN JING and JAMES ZHANG	
Calcul du nombre de classes des corps de nombres	455
STÉPHANE LOUBOUTIN	
On geometric properties of harmonic $\text{Lip}_1$ -capacity	469
PERTTI MATTILA and P. V. PARAMONOV	
Reproducing kernels and composition series for spaces of vector-valued holomorphic functions	493
BENT ØRSTED and GENKAI ZHANG	
Iterated loop modules and a filtration for vertex representation of toroidal Lie algebras	511
S. ESWARA RAO	
The intrinsic mountain pass	529
MARTIN SCHECHTER	
A Frobenius problem on the knot space	545
RON G. WANG	
On complete metrics of nonnegative curvature on 2-plane bundles	569
DAVID YANG	
Correction to: "Free Banach-Lie algebras, couniversal Banach-Lie groups, and more"	585
VLADIMIR G. PESTOV	
Correction to: "Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc"	589
KAI SENG (KAISING) CHOU (TSO) and TOM YAU-HENG WAN	