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DAVID YANG

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ON COMPLETE METRICS OF NONNEGATIVE CURVATURE ON 2-PLANE BUNDLES

DaGang Yang

This paper is an attempt to understand the 2-plane bundle case for the converse of the soul theorem due to J. Cheeger and D Gromoll. It is shown that there is a class of 2-plane bundles over certain S^2-bundles that carry complete metrics of nonnegative sectional curvature. In particular, every 2-plane bundle and every S^1-bundle over the connected sum $CP^n \# \overline{CP}^n$ of CP^n with a negative CP^n carries a 2-parameter family of complete metrics with nonnegative sectional curvature.

A complete noncompact Riemannian manifold with nonnegative curvature ($K \geq 0$) is diffeomorphic to the normal bundle of a closed totally geodesic submanifold of the noncompact manifold according to a fundamental theorem due to J. Cheeger and D. Gromoll [**4**]. They also proposed the following problem: Which vector bundles over closed manifolds with $K \geq 0$ carry complete metrics with $K \geq 0$?

This paper is an attempt to understand the 2-plane bundle case. For a class of 2-plane bundles and S^1-bundles, we show that there exist families of complete metrics with $K \geq 0$. Our main result is the following

Theorem. *Every 2-plane bundle and every S^1-bundle over the connected sum $\mathbf{C}P^n \# \overline{\mathbf{C}P}^n$ of $\mathbf{C}P^n$ with a negative $\mathbf{C}P^n$ carries a 2-parameter family of complete metrics with $K \geq 0$. More generally, let (B, h) be a Hodge manifold with positive Kahlerian curvature and let M be an associated S^2-bundle over B. Then, there is a two dimensional subspace $\widetilde{H}^2(M)$, generated by two integral cohomology classes, in the cohomology group $H^*(M)$ such that every principal S^1-bundle and every 2-plane bundle over M supported by an Euler class in $\widetilde{H}^2(M)$ carries a 2-parameter family of complete metrics with $K \geq 0$.*

Notice that B can be the product of any number of copies of complex projective spaces with the scaled Fubini-Study metric. When $B = \mathbf{C}P^n$, then $M = \mathbf{C}P^{n+1} \# \overline{\mathbf{C}P}^{n+1}$.

It is worthwhile to point out that examples of complete manifolds with $K \geq 0$ remain scarce. Indeed, all previously known examples of manifolds with $K > 0$ are of group theoretic nature, while there are only 3 types of known examples of closed Riemannian manifolds with $K \geq 0$, that is, all compact Lie groups and their universal covering spaces, the connected sum of 2 symmetric spaces of rank one due to J. Cheeger [**2**], and all those manifolds obtainable from the previous 2 types by taking product and quotient by isometric group actions. The basic tool of constructing manifolds with $K \geq 0$ remains to be O'Neil's Riemannian submersion theorem (c.f. [**3**]).

The metrics constructed in this paper are of a different nature in the sense that they are not group theoretic. Since the associated principal bundle of a 2-plane bundle is an S^1-bundle, we will first study S^1-invariant metrics with $K \geq 0$ on S^1-bundles over closed manifolds with $K \geq 0$. The desired metrics with $K \geq 0$ on the associated 2-plane bundles are then obtained by applying O'Neil's submersion theorem [**3**] to the product of the Euclidean 2-plane with the associated principal S^1-bundles of the 2-plane bundles. This approach has been used in constructing metrics of positive Ricci curvature on S^1-bundles by L.B. Bergery in [**1**].

A connection metric on the total space of a principal fiber bundle over a closed Riemannian manifold is a metric of the form

$$g + s^2 w^2$$

where g is the pullback of the Riemannian metric on the base manifold, $s \neq 0$ is a constant, and w is a principal connection of the fiber bundle.

A harmonic connection on a principal S^1-bundle over a closed Riemannian manifold is a connection such that its curvature form is the pullback of an integral harmonic 2-form on the base manifold. A harmonic connection metric is a connection metric such that w is a harmonic connection. A harmonic connection metric is also called a Kaluza-Klein metric by the physicist as pointed out to the author by professor J.P. Bourguignon.

A harmonic connection not only minimizes the L^2-norm of its curvature forms by the Hodge decomposition theorem but also pointwise geometrically optimizes in the sense that the associated harmonic connection metric on the total space of the S^1-bundle is guaranteed to have nonnegative Ricci curvature as long as the base manifold has positive Ricci curvature, as is observed by L.B. Bergery in [**1**]. However, the situation of constructing metrics with $K \geq 0$ on the total space of a principal fiber bundle by connection metrics are much more delicate. We will first derive a necessary and sufficient condition for a connection metric on a principal S^1-bundle to have $K \geq 0$ in Section 2, Lemma 1. This lemma indicates that local geometry on the base manifold plays an essential role in constructing connection metrics with $K \geq 0$. Examples are displayed in Section 5, Remark 6, where harmonic connection metrics on S^1-bundles over a simply connected closed Riemannian manifold with $K \geq 0$ always has negative curvature somewhere. On the other hand, if sufficient informations for the curvature forms are available, then Lemma 1 may be applied to show that certain connection metrics have $K \geq 0$. It is in this thought that harmonic 2-forms are constructed in Section 4 for a class of S^2-bundles. This enables us to show that certain S^1-bundles and their associated 2-plane bundles over the S^2-bundles carry harmonic connection metrics with $K \geq 0$ if a positivity condition is satisfied by the base manifolds of the S^2-bundles. This is carried out in Section 5. The main theorem is then proved in Section 6.

§1. Preliminary.

In this paper, all manifolds will be smooth closed Riemannian manifolds and all functions and tensors will be smooth ones unless otherwise indicated. (M, g) will usually be the base Riemannian manifold of an S^1-bundle or a 2-plane bundle. All metrics over 2-plane bundles will be complete.

Some notations are in the order. ∇ will be the covariant differential operator of the Levi Civita connection of (M, g). $\|\varphi\|$ the pointwise Riemannian norm of a differential form φ induced by the Riemannian metric g on M. i_X the interior product by a vector $X \in TM$ on differential forms. By a nonnegative curvature we will mean nonnegative sectional curvature and this will be denoted by $K \geq 0$. The sectional curvature K of (M, g) will often be considered as a function on the Grassmannian bundle GM of oriented 2-planes of the tangent bundle TM. We will also view a 2-form φ on (M, g) as a function on GM, more specifically, for every oriented 2-plane $\sigma \in GM$, set

$$\varphi(\sigma) = \varphi(e_1, e_2)$$

where e_1, e_2 is an oriented orthonormal basis of σ. It is easily seen that $\varphi(\sigma)$ is independent of the choice of the oriented orthonormal basis e_1, e_2 and φ is a well-defined smooth function on GM.

Consider a principal S^1-bundle

$$S^1 \longrightarrow E \xrightarrow{P} M$$

over a Riemannian manifold (M, g). Let w be a connection on the S^1-bundle E, then dw is the curvature form of the connection w. dw is the pullback of a closed 2-form α on M and α represents the Euler class of the S^1-bundle E, i.e.,

$$dw = P^*\alpha \ .$$

For convenience, we will identify functions and differential forms on the base manifold M with their pullbacks by P^* on E. Thus we will often write α for $P^*\alpha$, g for P^*g, and etc.

Definition 1. A connection metric on E associated with w is a metric of the form

$$g_s = g + s^2 w^2$$

where s is a positive constant parameter. w will be called a harmonic connection if $dw = P^*\alpha$ and α is a harmonic 2-form on (M, g). g_s will be called a harmonic connection metric if w is a harmonic connection.

Remark 1. Given a connection on E, one can always modify it by the pullback of a 1-form on M so that it becomes a harmonic connection on E by the Hodge decomposition theorem. However, harmonic connections on E are not unique although their curvature forms are uniquely determined by the same Hodge theorem. Any two harmonic connections on E defer by a Gauge transformation, which is geometrically insensitive as one can see from Lemma 1.

Remark 2. With the connection metric g_s on E,

$$P : (E, g_s) \to (M, g)$$

is a Riemannian submersion since the principal S^1-action acts on E by isometry. Each fiber of E is a closed geodesic of length $2\pi s$.

§2. A necessary and sufficient condition for $K \geq 0$.

The following lemma gives a necessary and sufficient condition in order for (E, g_s) to have nonnegative sectional curvature $K_s \geq 0$ in terms of the curvature form α of the connection w and the geometry of (M, g).

Lemma 1. *The connection metric g_s on E has nonnegative curvature $K_s \geq 0$ if and only if*
(a) $K(\sigma) \geq \frac{3}{4} s^2 \alpha(\sigma)^2$
(b) $\{(\nabla_e \alpha)(\sigma)\}^2 \leq \|i_e \alpha\|^2 \{K(\sigma) - \frac{3}{4} s^2 \alpha(\sigma)^2\}$
for all unit vectors e in σ and all 2-plane $\sigma \in GM$. Thus (E, g_{s_0}) has nonnegative curvature $K_{s_0} \geq 0$ if and only if $K_s \geq 0$ for all $0 < s \leq s_0$.

Definition 2. The curvature function K on GM is said to be bounded from below by α if there exists a positive number $\varepsilon > 0$ such that $K(\sigma) \geq \varepsilon^2 \alpha(\sigma)^2$ for all $\sigma \in GM$.

Corollary 2. *Suppose that the curvature form α is a parallel 2-form on (M, g) and K is bounded from below by α. Then (E, g_s) has nonnegative curvature for s sufficiently small.*

Definition 3. A Kaehler manifold is said to have positive Kaehlerian curvature if its curvature function K is bounded from below by its fundamental Kaehler form Ω.

Positive Kaehlerian curvature is obviously an intermediate positivity condition in between $K \geq 0$ and $K > 0$. It implies both $K \geq 0$ and positive holomorphic sectional curvature since $\Omega(\sigma)^2 = 1$ if σ is a holomorphic 2-plane. (For Kaehler manifolds, we will follow the convention of [WS].) A useful property about this positivity condition is that, unlike positive sectional curvature, the product manifold $(M_1 \times M_2, g_1 + g_2)$ remains to have positive Kaehlerian curvature if both (M_1, g_1) and (M_2, g_2) do. For example, the complex projective space $\mathbf{C}P^n$ equipped with the Fubini-Study metric is a Kaehler manifold with positive sectional curvature $K > 0$ and therefore has positive Kaehlerian curvature. It follows that the product manifold of a number of copies of complex projective spaces with the product metric of the scaled Fubini-Study metrics has positive Kaehlerian curvature while its sectional curvature is only nonnegative.

Corollary 3. *Let (M, g) be a Hodge manifold with positive Kaehlerian curvature. Let*

$$S^1 \longrightarrow E \xrightarrow{P} M$$

be a principal S^1-bundle with an Euler class represented by a constant multiple of the Kaehler form. Then the harmonic connection metrics g_s on E has nonnegative curvature for sufficiently small s.

Both Corollary 2 and Corollary 3 are immediate consequences of Lemma 1. While Lemma 1 may have been known by many geometers, to the author's knowledge, there is no previously known interesting applications available. For completeness, we present a proof here.

Proof of Lemma 1. Let W be the vertical vector field on (E, g_s) such that $sw(W) = 1$. For any point $x \in M$ and any 2-plane σ in T_xM, let e_0, e_1 be any orthonormal basis of σ and $\overline{e}_0$, $\overline{e}_1$ the horizontal lift of e_0, e_1 to $\overline{x} \in P^{-1}(x)$, then for any real number θ,

$$X = \overline{e}_0 \quad \text{and} \quad Y = \operatorname{Sin}\theta\, \overline{e}_1 + \operatorname{Cos}\theta\, W$$

are orthonormal and span a 2-plane $\overline{\sigma}$ in $T_{\overline{x}}E$ since $P : (E, g_s) \to (M, g)$ is a Riemannian submersion. Furthermore, every 2-plane $\overline{\sigma}$ in $T_{\overline{x}}E$ is spanned by orthonormal vectors X and Y of this form.

Let R_s be the Riemannian curvature tensor of type $(0,4)$ for (E, g_s). Then

$$\begin{aligned} K_s(\overline{\sigma}) = {} & \operatorname{Cos}^2\theta\, K_s(\overline{e}_0, W) + \operatorname{Sin}^2\theta\, K_s(\overline{e}_0, \overline{e}_1) \\ & + 2 \operatorname{Sin}\theta \ \operatorname{Cos}\theta\, R_s(\overline{e}_0, \overline{e}_1, \overline{e}_0, W). \end{aligned}$$

By O'Neil's Riemannian submersion theorem,

$$K_s(\overline{e}_0, \overline{e}_1) = K(\sigma) - \frac{3}{4} s^2 \alpha(\sigma)^2$$

since $[\overline{e}_0, \overline{e}_1] = -s\alpha(\sigma)\, W$, where e_0, e_1 have been extended to local commuting vector fields in a neighborhood of x in M and $\overline{e}_0$, $\overline{e}_1$ are the horizontal lift of e_0, e_1 to local vector fields in a neighborhood of $\overline{x}$ in E.

One has by simple computations that

$$K_s(\overline{e}_0, W) = \frac{s^2}{4} \|i_{e_0}\alpha\|^2$$
$$R_s(\overline{e}_0, \overline{e}_1, \overline{e}_0, W) = -\frac{s}{2}(\nabla_{e_0}\alpha)(\sigma) \; .$$

It follows that

$$\begin{aligned} K_s(\overline{\sigma}) = {} & \frac{s^2}{4} \operatorname{Cos}^2\theta \; \|i_{e_0}\alpha\|^2 + \operatorname{Sin}^2\theta \left\{ K(\sigma) - \frac{3}{4}s^2\alpha(\sigma)^2 \right\} \\ & - s \operatorname{Sin}\theta \operatorname{Cos}\theta \, (\nabla_{e_0}\alpha)(\sigma) \end{aligned}$$

and that $K_s(\overline{\sigma}) \geq 0$ for every 2-plane $\overline{\sigma}$ in $T_{\overline{x}}E$ if and only if both (a) and (b) are true for every 2-plane σ in T_xM and every unit vector e_0 in σ. □

Denote by Ric_s and Ric the Ricci curvature of the Riemannian manifolds (E, g_s) and (M, g), respectively. Then averaging the sectional curvature $K_s(\overline{\sigma})$ yields

$$\begin{aligned} \mathrm{Ric}_s(Y) = {} & \operatorname{Sin}^2\theta \left\{ \mathrm{Ric}(e_1) - \frac{1}{2}s^2 \, \|i_{e_1}\alpha\|^2 \right\} \\ & + \frac{s^2}{2} \operatorname{Cos}^2\theta \, \|\alpha\|^2 + s \operatorname{Sin}\theta \operatorname{Cos}\theta \, \delta\alpha(e_1) \end{aligned}$$

where δ is the codifferential operator of (M, g).

Corollary 4. *The connection metric g_s on E has nonnegative Ricci curvature if and only if*
(a′) $\mathrm{Ric}(e) \geq \frac{s^2}{2}\|i_e\alpha\|^2$
(b′) $[\delta\alpha(e)]^2 \leq \|\alpha\|^2 \, \{2\, \mathrm{Ric}(e) - s^2\|i_e\alpha\|^2\}$
for every unit vector e in TM.

In particular, if (M, g) is of positive Ricci curvature, then the harmonic connection metrics on E have nonnegative Ricci curvature for small $s \neq 0$.

§3. S^2-bundles and 2-plane bundles associated with a principal S^1-bundle.

Consider a principal S^1-bundle $S^1 \longrightarrow F \xrightarrow{Q} B$ over a Riemannian manifold (B, h). The S^2-bundle M over B associated with the principal S^1-bundle F

is the quotient space obtained from $F \times S^2$ by the diagonal S^1-action on the product such that it acts on F by the principal S^1-action and on S^2 by the canonical rotation which fixes the north and south poles.

By a (harmonic) connection metric on M, we mean the induced quotient metric of the product metric on $F \times S^2$, where F is equipped with a (harmonic) connection metric and S^2 a rotationally symmetric metric.

It is more convenient to describe the associated S^2-bundle M and its connection metrics in the following way:

Consider the fiber bundle

$$[0,\pi] \times S^1 \longrightarrow [0,\pi] \times F \xrightarrow{Q} B$$

with fiber $[0,\pi] \times S^1$. Then $M = [0,\pi] \times F/\sim$ is the quotient space obtained from $[0,\pi] \times F$ by identifying each component of the boundary of each fiber $[0,\pi] \times S^1$ with a point, furthermore, every smooth connection metric on M is of the form

$$h_t = h + t^2 \ (dr^2 + f^2(r)\theta^2)$$

where r is the variable for the interval $[0,\pi]$, θ is a connection on F, while f is a smooth function on $[0,\pi]$, which is odd at the two end points 0 and π, positive on $(0,\pi)$, and $f'(0) = -f'(\pi) = 1$. In fact, h_t is the quotient metric of the product metric

$$(h + s^2\theta^2) + t^2(dr^2 + \tilde{f}^2(r)d\beta^2)$$

on $F \times S^2$, where

$$\tilde{f}^2(r) = \frac{s^2 f^2(r)}{s^2 - t^2 f^2(r)}$$

and s^2 a constant such that $s^2 > t^2 f^2(r)$ for all $r \in [0,\pi]$.

Again by O'Neil's Riemannian submersion theorem, (M, h_t) will have nonnegative curvature if both $(F, h + s^2\theta^2)$ and $(S^2, dr^2 + \tilde{f}^2(r)d\beta^2)$ do. In particular,

$$h_t = h + t^2(dr^2 + \operatorname{Sin}^2 r\theta^2)$$

has nonnegative curvature for t sufficiently small if $(F, h + s^2\,\theta^2)$ does for some $s \neq 0$.

We will obtain the associated $\mathbf{R}^2$-bundle V over B if S^2 is replaced by $\mathbf{R}^2$ in the above discussion. A smooth connection metric on V is still of the form

$$h_t = h + t^2(dr^2 + f^2(r)\theta^2)$$

where t is a positive constant but f is a smooth function on $[0,\infty)$, f is odd at 0, positive on $(0,\infty)$, and $f'(0) = 1$. Such connection metrics h_t on V are always smooth and complete.

Proposition 5. *Let θ be a connection on F such that $d\theta = Q^*\Omega \neq 0$. The associated connection metrics h_t on the S^2-bundle M or the $\mathbf{R}^2$-bundle V is of nonnegative curvature $K_t \geq 0$ if and only if there exists a positive constant $N > 0$ such that*

(a$''$) $K(\sigma) \geq \frac{3}{4}t^2N^2\Omega(\sigma)^2$

(b$''$) $\{(\nabla_e\Omega)(\sigma)\}^2 \leq \|i_e\Omega\|^2\{K(\sigma) - \frac{3}{4}t^2N^2\Omega(\sigma)^2\}$

(c$''$) $(f^2 - N^2)f'' \geq 3f(f')^2$ *and* $f^{-1}f'' < 0$

for all unit vectors e in σ and all 2-plane $\sigma \in GB$, where K and ∇ are the sectional curvature function and the covariant differential operator of the base Riemannian manifold (B, h), respectively.

The proof of Proposition 5 is similar to the one for Lemma 1 and will therefore be omitted.

Remark 3. The projection map Q from (M, h_t) or (V, h_t) onto (B, h) is a Riemannian submersion. All fibers of (M, h_t) or (V, h_t) are totally geodesic submanifolds isometric to

$$(S^2, t^2(dr^2 + f^2(r)d\beta^2)) \quad \text{or} \quad (\mathbf{R}^2, t^2(dr^2 + f^2(r)d\beta^2)).$$

However, the metric $t^2(dr^2 + f^2(r)d\beta^2)$ on S^2 (or $\mathbf{R}^2$) must have positive curvature if (M, h_t) (or (V, h_t)) has nonnegative curvature as a consequence of (c$''$) in Proposition 5. On the other hand, the product metrics $(h + s^2\theta^2) + t^2(dr^2 + r^2d\beta^2)$ on $F \times \mathbf{R}^2$ give rise to connection metrics

$$h + t^2\{dr^2 + r^2(1 + s^{-2}t^2r^2)^{-1}\theta^2\}$$

on V with nonnegative curvature if $(F, h + s^2\theta^2)$ has nonnegative curvature.

§4. Harmonic forms on S^2-bundles.

Although Hodge theory assures the existence of an unique harmonic representative in every De Rham cohomology class of a closed Riemannian manifold, there is no general formulae for harmonic forms available unless the manifold carries a substantial amount of symmetry. A harmonic connection metric on an S^2-bundle or a 2-plane bundle as defined in Section 3 is symmetric enough to write down certain harmonic forms for them. These explicit formulae for harmonic forms will be exploited to show the nonnegativity of sectional curvatures of the harmonic connection metrics on the corresponding S^1-bundles in the next section. However, applications of the construction of harmonic forms for noncompact Riemannian manifolds will be discussed elsewhere.

Let

$$S^2 \longrightarrow M \xrightarrow{Q} B$$

be an S^2-bundle associated with a nontrivial principal S^1-bundle

$$S^1 \longrightarrow F \xrightarrow{Q} B$$

as discussed in Section 3. So M is parametrized as

$$M = [0, \pi] \times F/\sim \ .$$

Let $h_t = h + t^2(dr^2 + f^2(r)\theta^2)$ be a harmonic connection metric on M and let

$$d\theta = nQ^*\Omega \neq 0$$

be the curvature form of the harmonic connection θ, where $\Omega \neq 0$ has been normalized by the nonzero integer factor n so that Ω represents an indivisible integral cohomology class on (B, h). (The trivial case $d\theta = 0$ will be excluded from our discussion.) We will look for harmonic 2-forms on (M, h_t) of the form

$$d(u(r)\theta) = u'\, dr \wedge \theta + ud\theta$$

where u is some smooth function of r on $[0, \pi]$. Notice that $d(u\theta)$ is a smooth closed 2-form on (M, h_t) if u is an even function at both ends 0 and π.

Lemma 6. *There exist harmonic 2-forms of the form $d(u\theta)$ on (M, h_t) if and only if $\|\Omega\|$ is a constant function on B. Assume that $a^2 = n^2\|\Omega\|^2$ is a nonzero constant, then for*

$$u(r) = c_1\, sh\, V(r) + c_2\, ch\, V(r) \ ,$$

$d(u\theta)$ is a smooth harmonic 2-form on (M, h_t) for any constants c_1 and c_2, where $V(r)$ is given by

$$V(r) = at^2 \int_0^r f(r)\, dr \ .$$

Proof. The smoothness of $d(u\theta)$ follows from the fact that h_t is a smooth metric on M and V is an even function at both 0 and π since f is odd at both 0 and π. The rest is a straightforward verification. □

Remark 4. Let $\widetilde{H}^*(M)$ be the subring generated by closed smooth 2-forms of the form $d(u\theta)$ of the De Rham cohomology $H^*(M)$. Then for every cohomology class in $\widetilde{H}^*(M)$, there is a similar formula for its harmonic representatives.

Set

$$\begin{aligned} V_1(r) &= at^2 \int_r^{\pi} f(r)\, dr \\ V_2(r) &= at^2 \int_0^{r} f(r)\, dr = V(r) \\ A(t) &= \left[sh\,(2at^2)\right]^{-1} \\ \alpha_1 &= A(t)\, d(sh\, V_1\, \theta) \\ \alpha_2 &= A(t)\, d(sh\, V_2\, \theta)\ . \end{aligned}$$

It is easy to see that both α_1 and α_2 are of integral periods and therefore are smooth integral harmonic 2-forms on (M, h_t). $\frac{1}{n}(\alpha_1 + \alpha_2)$ represents the pull back integral cohomology class $[Q^*\Omega]$ and every integral cohomology class in $\widetilde{H}^2(M)$ is of the form

$$p_1[Q^*\Omega] + q_1[\alpha_2]$$

where p_1, q_1 are integers. They can be represented by a harmonic 2-form

$$\alpha = p\alpha_1 + q\alpha_2$$

where $p = n^{-1}p_1$ and $q = n^{-1}p_1 + q_1$. Thus $|p - q| = |q_1| \geq 1$ unless $p = q$.

Example 1. Suppose that Ω is an indivisible integral parallel 2-form on (B, h). Then $\|\Omega\|$ is a constant and Lemma 6 applies to produce harmonic 2-forms on the associated S^2-bundle (M, h_t). In particular, assume that (B, h) is a Hodge manifold and Ω its fundamental Kahler form so that Ω is an integral differential form on B. Ω is then parallel and Lemma 6 applies.

Example 2. Consider the connected sum $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$, which is the nontrivial S^2-bundle over S^2 associated with the principal S^1-bundle $S^1 \longrightarrow S^3 \xrightarrow{H} S^2$. Thus $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ can be parametrized by $[0, r_0] \times S^3/\sim$. Let ω_1, ω_2, ω_3 be a left invariant orthonormal coframe on S^3 such that $d\omega_1 = -2\omega_2 \wedge \omega_3$, $d\omega_2 = -2\omega_3 \wedge \omega_1$, and $d\omega_3 = -2\omega_1 \wedge \omega_2$. Consider metrics on $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ of the form

$$g(f_1, f_2) = dr^2 + f_1^2(r)\omega_1^2 + f_2^2(r)\omega_2^2 + \omega_3^2$$

where f_1 and f_2 satisfy the following 3 conditions so that $g(f_1, f_2)$ is a smooth metric on $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$:

(1) Both f_1 and f_2 are smooth functions on $[0, r_0]$ and positive on $(0, r_0)$.

(2) f_1 is odd at 0 even at r_0, but f_2 is even at 0 and odd at r_0.

(3) $f_1(r_0) = f_2(0) = f_1'(0) = -f_2'(r_0) = 1$.

Set

$$F_1(r) = \exp\left\{-2\int_{r_0/2}^{r} f_1(\tau)f_2(\tau)^{-1}\,d\tau\right\}$$
$$F_2(r) = \exp\left\{\ 2\int_{r_0/2}^{r} f_2(\tau)f_1(\tau)^{-1}\,d\tau\right\}.$$

Then

$$\Omega_1 = d(F_1(r)\omega_1) \quad \text{and} \quad \Omega_2 = d(F_2(r)\omega_2)$$

are smooth selfdual and antiselfdual harmonic 2-forms on

$$\left(\mathbf{C}P^2\#\overline{\mathbf{C}P^2}, g(f_1,f_2)\right), \quad \text{respectively.}$$

For example, let $r_0 = \pi/2$, $f_1(r) = \operatorname{Sin} r$ and $f_2(r) = \operatorname{Cos} r$, then

$$g(\operatorname{Sin},\operatorname{Cos}) = dr^2 + \operatorname{Sin}^2 r\,\omega_1^2 + \operatorname{Cos}^2 r\,\omega_2^2 + \omega_3^2$$

is a smooth metric on $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ and the two harmonic forms are:

$$\Omega_1 = d(\operatorname{Cos}^2 r\,\omega_1) \quad \text{and} \quad \Omega_2 = d(\operatorname{Sin}^2 r\,\omega_2)\ .$$

Remark 5. None of the metrics $g(f_1,f_2)$ on $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ has $K \geq 0$ if both f_1 and f_2 satisfy the conditions (1), (2), and (3).

§5. Circle bundles with nonnegative curvature.

Now fix a harmonic connection metric

$$h_t = h + t^2\left(dr^2 + \operatorname{Sin}^2 r\,\theta^2\right)$$

on an associated S^2-bundle

$$S^2 \longrightarrow M \xrightarrow{Q} B\ .$$

Assume in this section that

$$d\theta = nQ^*\Omega \neq 0, \quad a = n\|\Omega\| \quad \text{is a constant}\ ,$$

Ω is a parallel 2-form on (B,h). Then

$$V_1(r) = 2at^2\operatorname{Cos}^2\frac{r}{2}\ ,\ V_2(r) = 2at^2\operatorname{Sin}^2\frac{r}{2}\ .$$

According to Lemma 6, both

$$\alpha_1 = A(t)d(sh\,V_1\,\theta) \quad \text{and} \quad \alpha_2 = A(t)d(sh\,V_2\,\theta)$$

are smooth integral harmonic 2-forms on (M, h_t).

Theorem 7. *Assume that B is 1-connected and the sectional curvature of (B, h) is bounded from below by Ω. Then every principal S^1-bundle and every 2-plane bundle over M carries a two parameter family of connection metrics $h_{t,s}$ with nonnegative curvature if the bundle is supported by an Euler class in $\widetilde{H}^2(M)$, that is, it can be represented by a harmonic form of the form*

$$\alpha = p\alpha_1 + q\alpha_2 \ .$$

Proof. Notice that the 2-plane bundle part of the theorem follows from the S^1-bundle part of the theorem and O'Neil's Riemannian submersion theorem.

For $p = q$, $\alpha = p(\alpha_1 + \alpha_2)$ represents the integral cohomology class $Q^*[np\Omega]$ and the S^1-bundle is bundle isomorphic to the product of S^2 with the S^1-bundle over B with Euler class $[np\Omega]$, which carries a two parameter (scaling along the S^2 factor and scaling along the fiber of the S^1-bundle over B) family of metrics and the nonnegativity of the curvatures follows from Corollary 2.

Set $g = h_t$, let $g_s = h_{t,s}$ be a harmonic connection metric on the S^1-bundle over (M, g) with an Euler class represented by $\alpha = p\alpha_1 + q\alpha_2$. Let ∇ and K be the covariant differential operator and the curvature function of (M, g), respectively.

For $p \neq q$, we will demonstrate that there exist $t_0 = t_0(p, q) > 0$ and $s_0 = s_0(t) > 0$ such that for all $0 < t \leq t_0$ and all $0 < s \leq s_0$, both conditions (a) and (b) in Lemma 1 are satisfied.

Since (M, g) is an S^2-bundle with a connection metric, its tangent bundle splits into the direct sum of a 2-plane bundle $T'M$ tangential to the fiber and its orthogonal complement $T''M$ which is the bundle of horizontal vectors, i.e.,

$$TM = T'M \oplus T''M \ .$$

Let $x_0 \in M$ and σ an oriented 2-plane in $T_{x_0}M$. Let X, Y be an oriented orthonormal basis of σ, then

$$X = X_1 + X_2 \quad \text{and} \quad Y = Y_1 + Y_2$$

split into the sums of a vertical vector and a horizontal vector. Let

$$e_1 = t^{-1}\frac{\partial}{\partial r}, \quad e_2 = \csc r\, W$$

where W is the vector field on M tangent to the fiber of M and $t\theta(W) = 1$, $dr(W) = 0$. Set $a_i = g(e_i, X_1)$, $b_i = g(e_i, Y_1)$, $i = 1, 2$.

A routine computation gives

$$\begin{aligned}
K(\sigma) &= \widetilde{R}(X_2, Y_2, X_2, Y_2) - \frac{3}{4} n^2 t^2 \operatorname{Sin}^2 r \Omega(X_2, Y_2)^2 \\
&\quad + \frac{n^2}{4} t^2 \operatorname{Sin}^2 r \left\| i_{(a_2 Y_2 - b_2 X_2)} \Omega \right\|^2 \\
&\quad - 3n \operatorname{Cos} r (a_1 b_2 - a_2 b_1) \Omega(X_2, Y_2) \\
&\quad + t^{-2} (a_1 b_2 - a_2 b_1)^2 \\
\|i_X \alpha\|^2 &= u_1^2(r) \|i_{X_2} \Omega\|^2 + u_2^2(r) |X_1|^2 \\
\alpha(\sigma) &= u_1 \Omega(X_2, Y_2) + u_2 (a_1 b_2 - a_2 b_1) \\
(\nabla_X \alpha)(X, Y) &= \frac{n}{2} t \operatorname{Sin} r \Big\{ u_1(r) g\left(i_{X_2} \Omega, i_{a_2 Y_2 - b_2 X_2} \Omega\right) \\
&\quad + 3 a_1 u_2(r) \Omega(X_2, Y_2) + 2 a_1 n^{-2} a^2 u_1(r) (a_1 b_2 - a_2 b_1) \Big\}
\end{aligned}$$

where $\widetilde{R}$ is the curvature tensor of type $(0, 4)$ for (B, h) and

$$\begin{aligned}
u_1(r) &= n\, A(t) \{p\, sh\, V_1(r) + q\, sh\, V_2(r)\} \\
u_2(r) &= a\, A(t) \{-p\, ch\, V_1(r) + q\, ch\, V_2(r)\}
\end{aligned}$$

so

$$\begin{aligned}
&\|(\nabla_X \alpha)(X, Y)\|^2 \\
&\quad \le \frac{n^2 t^2}{4} \operatorname{Sin}^2 r \left\{ u_1^2 \, \|i_{X_2} \Omega\|^2 + \frac{1}{2} u_2^2 |X_1|^2 + \frac{1}{2} u_1^2 |X_1|^2 \right\} \\
&\quad\quad \times \left\{ \left\| i_{(a_2 Y_2 - b_2 X_2)} \Omega \right\|^2 + 18 \Omega(X_2, Y_2)^2 + 8 n^{-4} a^4 (a_1 b_2 - a_2 b_1)^2 \right\}.
\end{aligned}$$

Since the sectional curvature of (B, h) is bounded from below by Ω, there exists a positive constant $\varepsilon > 0$ such that

$$\widetilde{R}(X_2, Y_2, X_2, Y_2) \ge \varepsilon^2 \Omega(X_2, Y_2)^2 .$$

If one chooses

$$t^2 \le \min \left\{ \frac{1}{2a}, \quad \frac{|p - q|}{4a|q|}, \quad \frac{n \varepsilon^2}{9n^3 + 4a^2 \varepsilon^2} \right\}$$

$$s^2 \le \frac{t^2}{12(|p| + |q|)^2}$$

then

$$|u_1| \le |u_2|$$

and

$$\|(\nabla_X\alpha)(\sigma)\|^2 \leq \|i_X\alpha\|^2 \left\{K(\sigma) - \frac{3}{4}s^2\alpha(\sigma)^2\right\} .$$

The theorem follows from Lemma 1. □

Remark 6. For $p = q \neq 0$, the connection metrics $g_s = h_{t,s}$ on the corresponding S^1-bundle over M cannot have nonnegative curvature. For any $t, s > 0$, for example, let σ be the 2-plane spanned by e_1, e_2, let $X = e_1$, then $(\nabla_{e_1}\alpha)(\sigma) = ta^2n^{-1}u_1(r) \neq 0$, while

$$\|u_{e_1}\alpha\| = 0 .$$

Thus $K_{t,s}(\overline{\sigma}) < 0$, where $\overline{\sigma}$ is a 2-plane in the tangent bundle of the S^1-bundle spanned by $\overline{e}_1$ and $\operatorname{Sin}\beta\,\overline{e}_2 + \operatorname{Cos}\beta\, V$ for some β, where $\overline{e}_1, \overline{e}_2$ are the horizontal lift of e_1, e_2, and V is the unit tangent to the fiber of the S^1-bundle.

§6. Proof of the main result.

The connected sum $\mathbf{C}P^{n+1}\#\,\overline{\mathbf{C}P}^{n+1}$ is an S^2-bundle over $\mathbf{C}P^n$ associated with the Hopf fibration $S^1 \longrightarrow S^{2n+1} \xrightarrow{H} \mathbf{C}P^n$. The fundamental Kaehler form of $\mathbf{C}P^n$ with the Fubini-Study metric is a curvature form of this S^1-bundle. Since every De Rham cohomology class in $H^2(\mathbf{C}P^{n+1}\#\,\overline{\mathbf{C}P}^{n+1})$ can be represented by harmonic forms of the form $\alpha = p\alpha_1 + q\alpha_2$ as constructed in Section 4, it follows from Theorem 7 that every S^1-bundle and every 2-plane bundle over $\mathbf{C}P^{n+1}\#\,\overline{\mathbf{C}P}^{n+1}$ carries a complete smooth metric with $K \geq 0$. Indeed, if M is any S^2-bundle over $\mathbf{C}P^n$ with structure group the abelian group S^1, it follows that all S^1-bundles and all 2-plane bundles over M carry complete smooth metrics with $K \geq 0$.

In general, let (B, h) be a Hodge manifold with positive Kahlerian curvature, Ω its fundamental Kahler form.

Let

$$S^2 \longrightarrow M \xrightarrow{Q} B$$

be the associated S^2-bundle over B. It follows from Theorem 7 that every principal S^1-bundle and every 2-plane bundle over M carries a 2-parameter family of metrics $h_{t,s}$ with nonnegative curvature if the bundle is supported by an Euler class in $\widetilde{H}^2(M)$, that is, it can be represented by harmonic forms of the form

$$\alpha = p\alpha_1 + q\alpha_2$$

as constructed in Section 4. The proof is completed.

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TULANE UNIVERSITY
NEW ORLEANS, LA 70118 USA
Email address: dgy@math.tulane.edu

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