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THE POSITIVE-DIMENSIONAL FIBRES OF THE PRYM MAP

JUAN-CARLOS NARANJO

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The fibres of positive dimension of the Prym map are characterized.

Let C be an irreducible complex smooth curve of genus g . Let $\pi : \tilde{C} \rightarrow C$ be a connected unramified double covering of C .

The *Prym variety* associated to the covering is, by definition, the component of the origin of the Kernel of the norm map

$$P(\tilde{C}, C) = \text{Ker}(Nm_\pi)^0 \subset J\tilde{C}.$$

It is a principally polarized abelian variety (p.p.a.v.) of dimension $g(\tilde{C}) - g = g - 1$.

One defines the Prym map

$$P_g : \begin{array}{ccc} \mathcal{R}_g & \longrightarrow & \mathcal{A}_{g-1} \\ (\tilde{C} \xrightarrow{\pi} C) & \longmapsto & P(\tilde{C}, C), \end{array}$$

where \mathcal{R}_g is the coarse moduli space of the coverings π as above and \mathcal{A}_{g-1} stands for the coarse moduli space of p.p.a.v.'s of dimension $g - 1$.

It is well-known that this map is generically injective for $g \geq 7$ (Friedman-Smith, Kanev). On the other hand this map is never injective; this is a consequence of the tetragonal construction due to Donagi (see [Do1] for a description of the construction). This fact is already implicit in the results of Mumford ([M]):

The coarse moduli space \mathcal{RH}_g of unramified double coverings of smooth hyperelliptic curves of genus g has $\lfloor \frac{g-1}{2} \rfloor + 1$ irreducible components $\mathcal{RH}_{g,t}$, $t = 0, \dots, \lfloor \frac{g-1}{2} \rfloor$. For an element $(\tilde{C}, C) \in \mathcal{RH}_{g,t}$ there exist two hyperelliptic curves

$$p_1 : C_1 \rightarrow \mathbb{P}^1, \quad p_2 : C_2 \rightarrow \mathbb{P}^1$$

of genus $g(C_1) = t \leq g - t - 1 = g(C_2)$ such that

- a) $\tilde{C} = C_1 \times_{\mathbb{P}^1} C_2$ and
- b) $C = \tilde{C}/(\sigma_1 \circ \sigma_2)$, where σ_1 (resp. σ_2) is the involution on \tilde{C} attached to the branched covering $\tilde{C} \rightarrow C_1$ (resp. $\tilde{C} \rightarrow C_2$).

Mumford proves (loc. cit. p. 346) that one has an isomorphism of p.p.a.v.

$$P(\tilde{C}, C) \cong JC_1 \times JC_2.$$

Consequently the fibres of the restriction of P_g to \mathcal{RH}_g have positive dimension. In fact $P_g(\mathcal{RH}_{g,t})$ is contained in the product $\mathcal{JH}_t \times \mathcal{JH}_{g-t-1}$, where \mathcal{JH}_s stands for the locus of Jacobians of hyperelliptic curves of genus s . Thus

$$\dim \mathcal{RH}_{g,t} = 2g - 1 > \dim \mathcal{JH}_t \times \mathcal{JH}_{g-t-1} = \begin{cases} 2g - 4 & \text{if } t \neq 0 \\ 2g - 3 & \text{if } t = 0. \end{cases}$$

On the other hand positive dimensional fibres also appear for some coverings of bi-elliptic curves (a curve is called bi-elliptic if it can be represented as a ramified double covering of an elliptic curve).

In this note we characterize the fibres of positive dimension of the Prym map. To state our theorem we need some notation: let \mathcal{RB}_g be the coarse moduli space of the unramified double coverings $\pi : \tilde{C} \rightarrow C$ such that C is a smooth bi-elliptic curve of genus g . This variety has $\lfloor \frac{g-1}{2} \rfloor + 2$ irreducible components

$$\mathcal{RB}_g = \left(\bigcup_{t=0}^{\lfloor \frac{g-1}{2} \rfloor} \mathcal{RB}_{g,t} \right) \cup \mathcal{RB}'_g$$

(see [N] for more details).

We obtain:

Theorem. *Assume $g \geq 13$. A fibre of P_g is positive dimensional at (\tilde{C}, C) if and only if C is either hyperelliptic or*

$$(\tilde{C}, C) \in \bigcup_{t \geq 1} \mathcal{RB}_{g,t}.$$

Proof. If C is hyperelliptic we apply the results of Mumford. On the other hand, all the irreducible components of the fibres of $P_g|_{\mathcal{RB}_{g,t}}$ are positive dimensional for $t \geq 1$ (see [N, §20]). This finishes one implication.

The first step to see the opposite implication is to prove that the curve C is tetragonal (i.e. there exists a g^1_4 on C).

Let $\eta \in JC$ be the two-torsion point characterizing the covering and denote by L the line bundle $\omega_C \otimes \eta$. It is easy to check that L is very ample if C is non-tetragonal. Let Φ_L be the projective embedding of C defined by L .

As in Beauville ([B, p. 379]), we replace \mathcal{R}_g and \mathcal{A}_{g-1} by the corresponding functors. Then, the Prym map defines a morphism of functors Pr_g . Our

hypothesis on the fibre of P_g implies that the cotangent map to Pr_g at (\tilde{C}, C) is not surjective. By loc. cit. Prop. (7.5), this map can be shown as the cup-product map

$$S^2 H^0(C, L) \longrightarrow H^0(C, L^{\otimes 2})$$

followed by the isomorphism induced in cohomology by $L^{\otimes 2} \cong \omega^{\otimes 2}$. Hence, the non-surjectivity implies that $\Phi_L(C)$ is not a projectively normal curve.

We recall Theorem 1 in [G-L]: If L is very ample and

$$\text{deg}(L) \geq 2g + 1 - 2h^1(L) - \text{Cliff}(C),$$

then $\Phi_L(C)$ is projectively normal (where $\text{Cliff}(C)$ is the Clifford index of C).

Since $h^1(L) = 0$ and $\text{deg}(L) = 2g - 2$ one obtains $\text{Cliff}(C) \leq 2$. By using Clifford's Theorem and [Ma, Propositions 7 and 8], it follows that the curve either possess a g_4^1 or is plane curve of degree six. The second case contradicts $g \geq 13$.

Thus C is tetragonal. Since $g \geq 13$ the results in [De] can be applied: either the fibre is finite (generically, three elements) or we are in one of the following three possibilities: C is either hyperelliptic or bi-elliptic or trigonal.

Assume that C is bi-elliptic. Theorems (9.4), (10.9) and (10.10) in [N] states that $P_g^{-1}(P(\tilde{C}, C))$ consists of two points for every $(\tilde{C}, C) \in \mathcal{RB}_{g,0} \cup \mathcal{RB}'_g$, hence

$$(\tilde{C}, C) \in \bigcup_{t \geq 1} \mathcal{RB}_{g,t}.$$

To finish the proof we have to rule out the case: C trigonal. In [R], Recillas (cf. also [Do2]) establishes an isomorphism

$$\tau : \mathcal{RT}_g \xrightarrow{\cong} \mathcal{M}_{g-1}^{\text{tet},0},$$

where \mathcal{RT}_g is the coarse moduli space of unramified double coverings of trigonal curves and $\mathcal{M}_{g-1}^{\text{tet},0}$ is the moduli space of pairs (X, g_4^1) of tetragonal curves X and a base-point-free tetragonal linear series on X not containing divisors of the form $2x + 2y$. This map satisfies that

$$\tau(\tilde{C}, C) = (X, g_4^1) \implies P(\tilde{C}, C) \cong JX \quad (\text{as p.p.a.v.}).$$

Let us fix (\tilde{C}, C) as above and let $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{D}, D) \cong P(\tilde{C}, C) \cong JX$. Since C is not hyperelliptic, then the singular locus of the theta divisor of $P(\tilde{D}, D)$ has codimension 3 by [M, p. 344]. In loc. cit. a list of the Prym varieties with such property appear. We obtain that D is either trigonal or bi-elliptic. Since $P(\tilde{D}, D)$ is the Jacobian of a curve the bi-elliptic case contradicts [S].

Hence it suffices to prove that all the fibres of the restriction of P_g to \mathcal{RT}_g are zero dimensional. This follows from the bijection τ . Indeed, a curve X of genus $g \geq 12$ has at most one base-point-free g_4^1 without divisors of the form $2x + 2y$; otherwise there exists a map $f : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and then either the genus is ≤ 9 or X is bi-elliptic. By [T, Lemma (4.3)] the linear series of degree 4 and dimension 1 on a bi-elliptic curve come from g_2^1 linear series on the elliptic curve, thus divisors of the forbidden form appear.

Now the classical Torelli Theorem says that

$$\begin{aligned} \mathcal{M}_{g-1}^{tet,0} &\longrightarrow \mathcal{A}_{g-1} \\ (X, g_4^1) &\longmapsto JX \end{aligned}$$

is injective. Composing with τ we are done. \square

Remark. Note that if one drops the hypothesis on the genus, at least one gets that the Clifford index of C is ≤ 2 .

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PACIFIC JOURNAL OF MATHEMATICS

Volume 172 No. 1 January 1996

A class of incomplete non-positively curved manifolds	1
BRIAN BOWDITCH	
The quasi-linearity problem for C^* -algebras	41
L. J. BUNCE and JOHN DAVID MAITLAND WRIGHT	
Distortion of boundary sets under inner functions. II	49
JOSE LUIS FERNANDEZ PEREZ, DOMINGO PESTANA and JOSÉ RODRÍGUEZ	
Irreducible non-dense $A_1^{(1)}$ -modules	83
VJACHESLAV M. FUTORNY	
M -hyperbolic real subsets of complex spaces	101
GIULIANA GIGANTE, GIUSEPPE TOMASSINI and SERGIO VENTURINI	
Values of Bernoulli polynomials	117
ANDREW GRANVILLE and ZHI-WEI SUN	
The uniqueness of compact cores for 3-manifolds	139
LUKE HARRIS and PETER SCOTT	
Estimation of the number of periodic orbits	151
BOJU JIANG	
Factorization of p -completely bounded multilinear maps	187
CHRISTIAN LE MERDY	
Finitely generated cohomology Hopf algebras and torsion	215
JAMES PEICHENG LIN	
The positive-dimensional fibres of the Prym map	223
JUAN-CARLOS NARANJO	
Entropy of a skew product with a Z^2 -action	227
KYEWON KOH PARK	
Commuting co-commuting squares and finite-dimensional Kac algebras	243
TAKASHI SANO	
Second order ordinary differential equations with fully nonlinear two-point boundary conditions. I	255
H. BEVAN THOMPSON	
Second order ordinary differential equations with fully nonlinear two-point boundary conditions. II	279
H. BEVAN THOMPSON	
The flat part of non-flat orbifolds	299
FENG XU	