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**COMMUTING CO-COMMUTING SQUARES AND  
FINITE-DIMENSIONAL KAC ALGEBRAS**

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## COMMUTING CO-COMMUTING SQUARES AND FINITE DIMENSIONAL KAC ALGEBRAS

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**A relationship between finite dimensional Kac algebras and specified commuting co-commuting squares is discussed. The Majid's bicrossproduct Kac algebra is explained in our context.**

### 1. Introduction.

The theory of Kac algebras (Hopf algebras) has been drawing considerable attention (see [6] for the reference), and in fact many intensive studies have been made recently. ([1, 18, 19, 34, 35, 36], etc.) On the other hand, the announcement by A.Ocneanu ([20, 21]) brought us a new aspect in the theory of Kac algebras : it is his claim (proved in [4, 17] and also [28]) that, for an irreducible inclusion of factors  $M \supset N$  with finite index and depth = 2,  $M$  is described as the crossed product algebra of  $N$  by an outer action of a finite dimensional Kac algebra. Hence, we investigate Kac algebras from the Jones index theoretical point of view.

The purpose of this paper is to find a finite dimensional Kac algebra via the index theory : let  $L \supset K$  be an irreducible inclusion of factors with finite index. Suppose that, for an intermediate subfactor  $M$ , both inclusions  $L \supset M$  and  $M \supset K$  are of depth 2. Although the inclusion  $L \supset K$  does not always satisfy the depth 2 condition, it can be proved that this pair is of depth 2 if these factors  $L, M, K$ , and another intermediate subfactor  $N$  form a commuting co-commuting square. Details will be explained in §2 after recalling basic facts on commuting co-commuting squares. Another criterion for the inclusion  $L \supset K$  to be of depth 2 is also obtained. Examples are given in §3.

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2. Main results.

Let

$$\begin{array}{ccc} L & \supset & M \\ \cup & & \cup \\ N & \supset & K \end{array}$$

be a quadruple of type  $\text{II}_1$  factors satisfying  $[L : K] < \infty$ . (For the standard facts on the index theory, see [8, 11, 23, 25, 26].) It is said to be a commuting square if  $E_M^L(N) \subseteq K$ , where  $E_M^L$  is the conditional expectation from  $L$  to  $M$ . (See [8] for other equivalent conditions.) A quadruple  $(L, M, N, K)$  is said to be a co-commuting square if the quadruple

$$\begin{array}{ccc} K' & \supset & M' \\ \cup & & \cup \\ N' & \supset & L', \end{array}$$

or equivalently, that of basic extensions

$$\begin{array}{ccc} \langle L, e_K^L \rangle & \supset & \langle L, e_M^L \rangle \\ \cup & & \cup \\ \langle L, e_N^L \rangle & \supset & L \end{array}$$

on the standard space  $L^2(L)$  is a commuting square. Here,  $e_M^L, e_N^L$ , and  $e_K^L$  are relevant Jones projections (see [27, 30, 31]). For a commuting co-commuting square  $(L, M, N, K)$ , we have  $K = M \cap N$  and  $L = M \vee N$ . (In [27], a quadruple satisfying these equations is called a quadrilateral and, for a quadrilateral  $(L, M, N, K)$ ,  $\text{Ang}(M, N) = \text{Op} - \text{ang}(M, N) = \{\frac{\pi}{2}\}$  corresponds to the commuting co-commuting condition.)

For a commuting square, we have characterization of co-commutativity ([27, Corollary 7.1] and [26, Proposition 1.1.5]).

**Proposition 2.1.** *Let  $(L, M, N, K)$  be a commuting square of type  $\text{II}_1$  factors satisfying  $[L : K] < \infty$ . Then the following are equivalent :*

- (1)  $(L, M, N, K)$  is co-commuting.
- (2)  $L = M \cdot N = \{\sum_{i \in F} m_i n_i; F \text{ is a finite set, } m_i \in M, n_i \in N\}$ .
- (3)  $[L : M] = [N : K]$ .
- (4) A Pimsner-Popa basis for  $N \supset K$  is also that for  $L \supset M$ .

Remark that, in [26], a commuting square satisfying (one of) the above conditions is called “non-degenerate” and (1),(2) of the following proposition are mentioned in [26, Proposition 1.1.6] (see also [9, Proposition 2.3]). We will see them for the completeness of this article.

**Proposition 2.2.** *Let  $(L, M, N, K)$  be a commuting co-commuting square of type  $\text{II}_1$  factors satisfying  $[L : K] < \infty$ . Then,*

- (1)  $\langle M, e_K^M \rangle \supset M$  is conjugate to  $\langle M, e_N^L \rangle \supset M$ .
- (2) The quadrilateral  $(\langle L, e_N^L \rangle, \langle M, e_N^L \rangle, L, M)$  is also commuting co-commuting.
- (3)  $\langle L, e_K^L \rangle$  is identified with the Jones extension for  $\langle L, e_N^L \rangle \supset \langle M, e_N^L \rangle$ .

*Proof.* (1) While the condition  $\sum_i a_i e_K^M b_i = 0$  ( $a_i, b_i \in M$ ) is equivalent to  $\sum_i a_i E_K^M(b_i c) = 0$  for  $c \in M$  on  $L^2(M)$ , the condition  $\sum_i a_i e_N^L b_i = 0$  ( $a_i, b_i \in M$ ) means  $0 = \sum_i a_i E_N^L(b_i c d) = \sum_i a_i E_N^L(b_i c) d = \sum_i a_i E_K^M(b_i c) d$  for  $c \in M, d \in N$  on  $L^2(L)$  thanks to Proposition 2.1.(2) and the commuting square condition. Hence, we may consider the map  $\phi : \langle M, e_K^M \rangle \rightarrow \langle M, e_N^L \rangle$  defined by  $\phi(\sum_i a_i e_K^M b_i) = \sum_i a_i e_N^L b_i$ . It is easy to see that this map  $\phi$  gives an isomorphism between them and  $\phi|_M = \text{id}$ .

(2) follows from [8, Corollary 4.2.3], [11, Proposition 3.1.7], and Proposition 2.1.

(3) Since the commuting square condition means  $e_M^L e_N^L = e_K^L$ , we have  $\langle \langle L, e_N^L \rangle, e_M^L \rangle = \langle L, e_K^L \rangle$ . We will show that  $\langle L, e_K^L \rangle = \langle \langle L, e_N^L \rangle, e_M^L \rangle$  is the Jones extension for  $\langle L, e_N^L \rangle \supset \langle M, e_N^L \rangle$ . The commuting square condition implies  $[e_M^L, x] = 0$  for  $x \in \langle M, e_N^L \rangle$ . And for the conditional expectation  $E_{\langle L, e_N^L \rangle}^{\langle L, e_K^L \rangle}$ , by [27, Lemma 7.2], we have  $E_{\langle L, e_N^L \rangle}^{\langle L, e_K^L \rangle}(e_M^L) = \frac{1}{[L:M]}$ . Therefore, we get the conclusion by [24, Proposition 1.2.(2)].  $\square$

Thus, we have extensions of a commuting co-commuting square  $(L, M, N, K)$  in compatible ways.

For an irreducible inclusion, we have a refined estimation of the dimension of relative commutant algebras as in [8, Theorem 4.6.3] (cf. [11, Corollary 2.2.3]). We will see this in terms of sectors ([10, 12, 14, 15, 16]).

**Lemma 2.1.** *For an irreducible inclusion  $M \supset N$  of type  $\text{II}_1$  factors satisfying  $[M : N] < \infty$ ,*

$$\dim(M_k \cap N') \leq [M : N]^k,$$

where  $N \subset M = M_0 \subset M_1 \subset \dots$  is the Jones tower.

*Proof.* We only treat the case  $k = 2$  since a similar proof will work for any  $k$ . We may assume that  $M$  and  $N$  are properly infinite and isomorphic (by [16, Lemma 2.3]) and denote  $N$  by  $\rho(M)$  for an endomorphism  $\rho \in \text{End}(M)$ . Consider the irreducible decompositions :  $\bar{\rho}\rho = \sum_j m_j \alpha_j, \bar{\rho}\rho\bar{\rho} = \sum_{j,k} m_j n_{jk} \beta_k$  ( $\alpha_j \bar{\rho} = \sum_k n_{jk} \beta_k$ ), where  $\bar{\rho}$  is the conjugate sector of  $\rho$ . By

the Frobenius reciprocity, we have  $\beta_k\rho \geq \sum_j n_{jk}\alpha_j, \alpha_j\bar\rho \geq m_j\bar\rho$ . Combining these, we get

$$\begin{aligned} [M : N]^2 &= ([M : N]_0^2 =) d(\rho)^4 = \sum_{j,k} m_j n_{jk} d(\beta_k\rho) \\ &\geq \sum_{j,k} m_j n_{jk} \sum_{j'} n_{j'k} d(\alpha_{j'}) \\ &\geq \sum_k \left( \sum_j m_j n_{jk} \right)^2 = \dim(M_2 \cap N') \end{aligned}$$

thanks to the additivity and the multiplicativity of the statistical dimension  $d$ . □

As a corollary of [8, Theorem 4.6.3], we have the following ([10, Proposition 4.2]) :

**Corollary 2.1.** *Let  $M \supset N$  be an irreducible inclusion of type  $\text{II}_1$  factors with finite index. Then the following are equivalent :*

- (1) *The inclusion  $M \supset N$  is of depth 2.*
- (2)  $\dim(M_1 \cap N') = [M : N]$ .
- (3)  $M_2 \cap N'$  is a factor.

We give another lemma to prove main results.

**Lemma 2.2.** *Let  $(P, Q, R, \mathbf{C})$  be a commuting square of finite dimensional algebras. Then we have*

$$\dim P \geq \dim Q \cdot \dim R.$$

*Proof.* Let us take a linear basis  $\{x_1, x_2, \cdots, x_m\}$  for  $R$  and a Pimsner-Popa basis  $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$  for  $Q \supset \mathbf{C}$  with respect to the conditional expectation  $E$  from  $P$  to  $R$  ( $m := \dim R, n := \dim Q$ ). Then  $x_i \lambda_j^* (\neq 0)$  are linearly independent ; suppose that  $\sum_{i,j} a_{ij} x_i \lambda_j^* = 0$  for  $a_{ij} \in \mathbf{C}$ . Since  $0 = \left( \sum_{i,j} a_{ij} x_i \lambda_j^* \right) \lambda_k = E \left( \sum_{i,j} a_{ij} x_i \lambda_j^* \lambda_k \right) = \sum_{i,j} a_{ij} x_i E \left( \lambda_j^* \lambda_k \right) = \sum_i a_{ik} x_i$  for any  $k$ , we have  $a_{ik} = 0$ . Hence, we get  $\dim P \geq nm$ . □

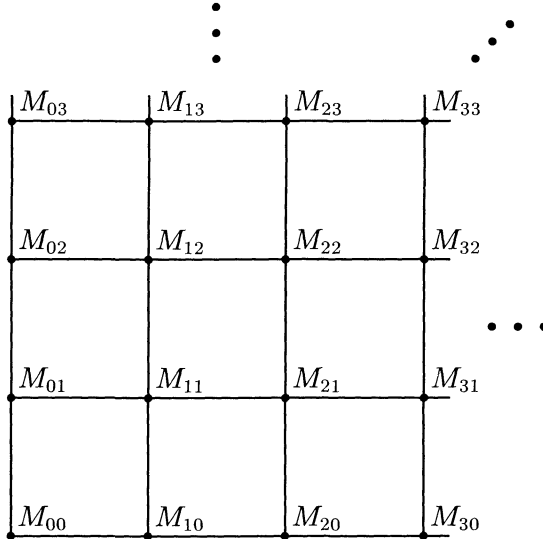
For a given commuting co-commuting square, we can get a kind of tiling by double sequences  $\{M_{ij}\}_{i,j=0,1,2,\cdots}$  of subfactors (see [22, 32]). By looking at a tiling, we have two criteria for an irreducible inclusion  $L \supset K$  to be of depth 2 :

**Theorem 2.1.** *Let  $(L, M, N, K)$  be a commuting co-commuting square of type  $\text{II}_1$  factors satisfying  $[L : K] < \infty$  and  $L \cap K' = \mathbf{C}$ . If both inclusions  $L \supset M$  and  $M \supset K$  are of depth 2, then so is the inclusion  $L \supset K$ .*

*Proof.* Let us denote extensions by  $\{M_{ij}\}_{i,j=0,1,\dots}$  such that

$$\begin{aligned} (M_{11}, M_{10}, M_{01}, M_{00}) &= (L, M, N, K), M_{22} = \langle M_{11}, e_{00}^{11} \rangle, \\ M_{21} &= \langle M_{11}, e_{01}^{11} \rangle, M_{20} = \langle M_{10}, e_{01}^{11} \rangle, \\ M_{12} &= \langle M_{11}, e_{10}^{11} \rangle, M_{33} = \langle M_{22}, e_{11}^{22} \rangle, \\ M_{32} &= \langle M_{22}, e_{12}^{22} \rangle, M_{31} = \langle M_{21}, e_{12}^{22} \rangle, \\ M_{30} &= \langle M_{20}, e_{12}^{22} \rangle, \quad \text{and so on.} \end{aligned}$$

Here,  $e_{kl}^{ij}$  means the Jones projection for the inclusion  $M_{ij} \supset M_{kl}$ .



**Figure 1.**

Clearly, we have  $M_{22} \cap M'_{00} \supset M_{20} \cap M'_{00}$ ,  $M_{12} \cap M'_{10}$ . But it can be shown that

$$M_{22} \cap M'_{00} = (M_{20} \cap M'_{00}) \cdot (M_{12} \cap M'_{10}).$$

Let us think of the following commuting square (for the conditional expectations of the restriction of the canonical trace on  $M_{22}$ ) :

$$\begin{array}{ccc} M_{22} \cap M'_{00} & \supset & M_{12} \cap M'_{10} \\ E \cup & & \cup \\ M_{20} \cap M'_{00} & \supset & \mathbf{C}. \end{array}$$

Here, we remark that  $(M_{22}, M_{20}, M_{12}, M_{10})$  forms a commuting square. It follows from Corollary 2.1 that  $\dim(M_{20} \cap M'_{00}) = [M : K](=: m)$  and  $\dim(M_{12} \cap M'_{10}) = [L : M](=: n)$ . Applying Lemma 2.2 to this square, we get  $\dim(M_{22} \cap M'_{00}) \geq mn = [L : K]$ . Combining this with Lemma 2.1, we have that  $\dim(M_{22} \cap M'_{00}) = [L : K]$  and  $M_{22} \cap M'_{00} = (M_{20} \cap M'_{00}) \cdot (M_{12} \cap M'_{10})$ . Therefore, we get the conclusion by Corollary 2.1.  $\square$

**Theorem 2.2.** *Let  $(L, M, N, K)$  be a commuting co-commuting square of type  $\text{II}_1$  factors satisfying  $[L : K] < \infty$  and  $L \cap K' = \mathbf{C}$ . If both inclusions  $M \supset K$  and  $N \supset K$  (or  $L \supset M$  and  $L \supset N$ ) are of depth 2, then so is the inclusion  $L \supset K$ .*

*Proof.* Let us keep the same notation as in the proof of Theorem 2.1. It is sufficient to consider the case that  $M \supset K$  and  $N \supset K$  are of depth 2 since another case can be proved by looking at the extension  $(M_{22}, M_{21}, M_{12}, M_{11})$ . For the commuting square  $(M_{22} \cap M'_{00}, M_{22} \cap M'_{20}, M_{20} \cap M'_{00}, \mathbf{C})$ , we remark that

$$M_{22} \cap M'_{20} \cong M_{02} \cap M'_{00}$$

by Proposition 2.2.(3) and Takesaki duality between  $M_{21} \supset M_{20}$  and  $M_{01} \supset M_{00}$ , which follows from a similar argument in [23, Proposition 1.5] about a common Pimsner-Popa basis for  $M_{10} \supset M_{00}$  and  $M_{11} \supset M_{01}$ . Applying Lemma 2.2 and 2.1 to the commuting square  $(M_{22} \cap M'_{00}, M_{22} \cap M'_{20}, M_{20} \cap M'_{00}, \mathbf{C})$ , we get that  $M_{22} \cap M'_{00} = (M_{22} \cap M'_{20}) \cdot (M_{20} \cap M'_{00})$ , and  $\dim(M_{22} \cap M'_{00}) = [L : K]$ . Therefore, we get the theorem by Corollary 2.1.  $\square$

**Remark.** Let  $(L, M, N, K) = (M_{11}, M_{10}, M_{01}, M_{00})$  be a commuting co-commuting square as in Theorem 2.2. The Majid's bicrossproduct method corresponds to looking at the quadruple  $(M_{21}, M_{20}, M_{11}, M_{10})$  and the relative commutant algebra  $M_{32} \cap M'_{10}$ .

### 3. Examples.

In this section, we will explain two examples. The first one is considered in [33, Proposition].

(1) Let  $G$  be a finite group with two subgroups  $A, B$  satisfying  $G = AB$  and  $A \cap B = \{e\}$ . Let  $\gamma$  be an outer action of  $G$  on a type  $\text{II}_1$  factor  $P$ . Then we have

**Proposition 3.1.** *The inclusion of crossed product algebras*

$$(L :=)(P \otimes l^\infty(G/B)) \rtimes G \supset P \rtimes A(=: K)$$

is irreducible and of depth 2, where the action of  $G$  on  $l^\infty(G/B)$  is induced by the left translation.

*Proof.* Let us consider the commuting co-commuting square

$$((P \otimes l^\infty(G/B)) \rtimes G, (P \otimes l^\infty(G/B)) \rtimes A =: M, P \rtimes G =: N, P \rtimes A).$$

Since  $(P \otimes l^\infty(G/B)) \rtimes G \cap (P \rtimes A)' = l^\infty(G/B)^A$ , the assumption  $G = AB$  corresponds to the irreducibility of the inclusion  $L \supset K$ . Considering Takesaki duality between  $L \supset M$  and  $P \rtimes B \supset P$  as in the proof of Theorem 2.1, and Proposition 2.2.(1) for  $M \supset K(\supset P)$ , we also see that  $L \supset M$  and  $M \supset K$  are of depth 2. Hence, applying Theorem 2.1 to this square  $(L, M, N, K)$ , we get the conclusion.  $\square$

**Remark.** The Jones tower and the tower of relative commutant algebras can be explicitly written down as in [3, 13, 29] ; the Jones tower is

$$\begin{aligned} K &= P \rtimes A \subset (P \otimes l^\infty(G/B)) \rtimes G = L \\ &\subset (P \otimes B(l^2(G/B)) \otimes l^\infty(G/A)) \rtimes G =: L_1 \\ &\subset (P \otimes B(l^2(G/B)) \otimes l^\infty(G/B) \otimes B(l^2(G/A))) \rtimes G =: L_2 \\ &\quad \dots \end{aligned}$$

And the tower of relative commutant algebras is

$$\begin{aligned} \mathbf{C} &= K \cap K' \subset L \cap K' = l^\infty(G/B)^A = \mathbf{C} \\ &\subset L_1 \cap K' = (B(l^2(G/B)) \otimes l^\infty(G/A))^A \\ &\subset L_2 \cap K' = \{B(l^2(G/B)) \otimes l^\infty(G/B) \otimes B(l^2(G/A))\}^A \\ &\cong B(l^2(G/B)) \otimes B(l^2(G/A)) \\ &\quad \dots \end{aligned}$$

Hence, we also see that the depth of  $L \supset K$  is 2. Next we recall the matched pair ([18, 19]) ; because of the uniqueness of the decomposition of an element in  $G = AB = BA$ , we can represent  $ab$  for  $a \in A, b \in B$  as

$$ab = \alpha_a(b)\beta_{b^{-1}}(a^{-1})^{-1} \in BA.$$

The associative law implies

$$\begin{aligned} \alpha_{aa'}(b) &= \alpha_a(\alpha_{a'}(b)), \alpha_a(bb') = \alpha_a(b)\alpha_{\beta_{b^{-1}}(a^{-1})^{-1}}(b'), \\ \beta_{bb'}(a) &= \beta_b(\beta_{b'}(a)), \beta_b(aa') = \beta_b(a)\beta_{\alpha_{a^{-1}}(b^{-1})^{-1}}(a') \end{aligned}$$



for  $a, a' \in A, b, b' \in B$ . Therefore, the matched pair  $(A, B, \alpha, \beta)$  in [18, Theorem 2.3] appears. (Here, we remark that if we write  $ab = \gamma_a(b^{-1})^{-1}\delta_{b^{-1}}(a)(\in BA)$ , then the matched pair of another type  $(A, B, \gamma, \delta)$  in [19] is obtained, but in this article we would like to treat the former one for our purpose.)

For the matched pair  $(A, B, \alpha, \beta)$ , we have a finite dimensional Kac algebra of Majid's type ([19]) ; the bicrossproduct Kac algebra consists of the crossed product algebra  $Q := l^\infty(B) \rtimes_\alpha A$  on  $l^2(B) \otimes l^2(A)$  (and others, see below) generated by  $m_f \otimes 1$  (simply denoted by  $f \otimes 1 = f$ ) for  $f \in l^\infty(B)$  and  $u_a \otimes \lambda_a$  (simply denoted by  $\lambda_a$ ) for  $a \in A$ , where  $m_f$  is the pointwise multiplication operator on  $l^2(B)$ , the action  $\alpha$  of  $A$  on  $l^\infty(B)$  is induced by the action  $\alpha$  of  $A$  on  $B$  ;  $\alpha_a(f)(b) = f(\alpha_{a^{-1}}(b))$  for  $f \in l^\infty(B)$ ,  $u_a$  is the implementing unitary on  $l^2(B)$  such that  $(u_a \xi)(b) = \xi(\alpha_{a^{-1}}(b))$  for  $\xi \in l^2(B)$ , and  $\lambda_a$  is the left regular translation;  $(\lambda_a \xi)(a') = \xi(a^{-1}a')$  for  $\xi \in l^2(A)$ .

We know the Kac algebra structure of this crossed product algebra and its dual Kac algebra ([19]) ; for the crossed product algebra  $Q = l^\infty(B) \rtimes_\alpha A$ , the comultiplication  $\Gamma$ , the antipode  $\kappa$ , and the Haar weight  $\psi$  are described by :

$$\begin{aligned}\Gamma(\chi_b) &= \sum_{b'b''=b} \chi_{b'} \otimes \chi_{b''}, \\ \Gamma(\lambda_a) &= \sum_b \chi_b \lambda_a \otimes \lambda_{\beta_{b^{-1}}(a)}, \\ \kappa(\chi_b) &= \chi_{b^{-1}}, \\ \kappa(\lambda_a) &= \sum_b \chi_b \lambda_{\beta_b(a^{-1})}, \\ \psi\left(\sum_a f_a \lambda_a\right) &= \frac{1}{|B|} \sum_b f_e(b)\end{aligned}$$

for  $f_a \in l^\infty(B)$ , and  $\chi_b$  is the characteristic function on  $b \in B$ . And we have

$$(l^\infty(B) \rtimes_\alpha A)^\wedge = B_\beta \ltimes l^\infty(A),$$

where the right-hand side is generated by  $1 \otimes f (f \in l^\infty(A))$  and  $\lambda_b \otimes v_b (b \in B)$  on  $l^2(B) \otimes l^2(A) ((v_b \xi)(a) = \xi(\beta_{b^{-1}}(a)), \xi \in l^2(A))$ .

The above Kac algebra  $\mathbf{K} = (l^\infty(B) \rtimes_\alpha A (= Q), \Gamma, \kappa, \psi)$  has a left action ([5]) on the factor  $K = P \rtimes_\gamma A$  ; let us write two generators  $\pi(p) (p \in P)$  and  $\lambda'_a (a \in A)$  of  $P \rtimes_\gamma A$  :

$$(\pi(p)\xi)(a') = \gamma_{a'^{-1}}(p)\xi(a'), (\lambda'_a \xi)(a') = \xi(a^{-1}a')$$

for  $\xi \in l^2(A, L^2(P))$ .

**Lemma 3.1.** *The following map  $\delta_K : K \rightarrow K \otimes Q$  gives a left action of the Kac algebra  $\mathbf{K}$  on  $K = P \rtimes A$  :*

$$\begin{aligned}\delta_K(\pi(p)) &= \sum_b \pi(\gamma_{b^{-1}}(p)) \otimes \chi_b, \\ \delta_K(\lambda'_a) &= \sum_b \lambda'_{\beta_{b^{-1}}(a)} \otimes \chi_b \lambda_a.\end{aligned}$$

This lemma follows from direct computation, hence the author leaves its proof to the reader.

So far, we are now ready to give the theorem.

**Theorem 3.1.** *The factor  $(P \otimes l^\infty(G/B)) \rtimes G$  is described as the crossed product algebra of  $P \rtimes A$  by the left action  $\delta_K$  in Lemma 3.1 of the Majid's bicrossproduct algebra  $\mathbf{K} = (l^\infty(B) \rtimes A, \Gamma, \kappa, \psi)$ .*

*Proof.* We may think that three kinds of generators  $\tilde{\pi}(p)$  ( $p \in P$ ),  $\tilde{\pi}(f)$  ( $f \in l^\infty(G/B)$ ), and  $\tilde{\lambda}_g$  ( $g \in G$ ) of the crossed product algebra  $(P \otimes l^\infty(G/B)) \rtimes G$  look like

$$\begin{aligned}(\tilde{\pi}(p)\xi)(aB, g') &= \gamma_{g'^{-1}}(p)\xi(aB, g'), \\ (\tilde{\pi}(f)\xi)(aB, g') &= f(aB)\xi(aB, g'), \\ (\tilde{\lambda}_g\xi)(aB, g') &= \xi(g^{-1}aB, g^{-1}g')\end{aligned}$$

for  $\xi \in l^2(G/B \times G, L^2(P))$ . Identifying  $G/B \times G$  with  $A \times B \times A$  by

$$(a'B, g = ba) \leftrightarrow (a, b, a'),$$

we may write these generators acting on  $L^2(P) \otimes l^2(A) \otimes l^2(B) \otimes l^2(A)$  such as

$$\begin{aligned}(\tilde{\pi}(p)\xi)(a, b, a') &= \gamma_{(ba)^{-1}}(p)\xi(a, b, a'), \\ (\tilde{\pi}(f)\xi)(a, b, a') &= f(a')\xi(a, b, a'), \\ (\tilde{\lambda}_{\tilde{a}}\xi)(a, b, a') &= \xi(\beta_{b^{-1}}(\tilde{a})^{-1}a, \alpha_{\tilde{a}^{-1}}(b), \tilde{a}^{-1}a'), \\ (\tilde{\lambda}_{\tilde{b}}\xi)(a, b, a') &= \xi(a, \tilde{b}^{-1}b, \beta_{\tilde{b}^{-1}}(a'))\end{aligned}$$

for  $f \in l^\infty(A)$ ,  $\tilde{a} \in A$ ,  $\tilde{b} \in B$  and  $\xi \in l^2(A \times B \times A, L^2(P))$ . On the other hand, the crossed product algebra of  $N$  by the (outer) action  $\delta_K$  of  $l^\infty(B) \rtimes A$  is generated by  $\delta_K(K) \vee 1 \otimes (l^\infty(B) \rtimes A)^\wedge = \delta_K(K) \vee 1 \otimes (B \rtimes l^\infty(A))$ . (See [5].) It is easy to see that  $\delta_K(\pi(p)) = \tilde{\pi}(p)$ ,  $\delta_K(\lambda'_a) = \tilde{\lambda}_a$ ,  $1 \otimes \lambda_b = \tilde{\lambda}_b$ , and  $1 \otimes f = \tilde{\pi}(f)$ . Therefore, we are done.  $\square$

(2) Let  $M \supset N$  be an irreducible inclusion of type  $\text{II}_1$  factors satisfying  $[M : N] < \infty$  and depth 2, and  $G$  be a finite group with an outer action  $\gamma$  on both  $M$  and  $N$ . Moreover, suppose that  $(M \rtimes G) \cap N' = \mathbf{C}$ . (This condition is equivalent to strong outerness of the action  $\gamma$  for  $M \supset N$ .) Then we have the depth 2 inclusion  $(M \otimes l^\infty(G)) \rtimes G \supset N \rtimes G$ . In fact, this inclusion is contained in the commuting co-commutative square  $((M \otimes l^\infty(G)) \rtimes G, (N \otimes l^\infty(G)) \rtimes G, M \rtimes G, N \rtimes G)$ . Similar argument as in Proposition 3.1 implies that the assumption in Theorem 2.1 for the inclusions  $(M \otimes l^\infty(G)) \rtimes G \supset (N \otimes l^\infty(G)) \rtimes G (\cong M \supset N \text{ by Takesaki duality})$  and  $(N \otimes l^\infty(G)) \rtimes G \supset N \rtimes G$  holds, hence we have the conclusion. (Cf. the orbifold construction [3, 7] and also [2].)

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