*Pacific Journal of Mathematics*

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Volume 172 No. 2 February 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 172, No. 2, 1996

# IRREDUCIBILITY AND DIMENSION THEOREMS FOR FAMILIES OF HEIGHT 3 GORENSTEIN ALGEBRAS

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**We show that the family of graded Gorenstein Artin algebras of height 3 with a fixed Hilbert function is irreducible, and we prove some dimension theorems about these families.**

## 0. Introduction.

In Chapter 1 we show that when a set $D = (Q, P)$ of generator and relation degrees is given, $Q = \{q_1, \ldots, q_u\}$ and $P = \{p_1, \ldots, p_u\}$, the family $Gor_D$ of Gorenstein algebras $A = R/J$ with $J$ having this set of generator and relation degrees is irreducible. We show that $Gor_D$ is the image of an algebraic map from a dense open set in a product of affine spaces. This depends on Buchsbaum and Eisenbud's structure theorem for height 3 Gorenstein ideals [**BE1**], which is discussed in Chapter 2.

In Chapter 2 we show that when $T$ is fixed, the family $Gor_T$ of all Gorenstein algebras with Hilbert function $T$ is irreducible. We show this by giving an explicit deformation of an ideal with degree set $D$ to an ideal with a smaller degree set $D'$ consistent with $T$. The minimal set $D_{min}$ of generator and relation degrees given $T$ is unique, and we conclude in Theorem 2.7 that $\overline{Gor_D} \supset Gor_{D'}$ for $D' \supset D$, and therefore $Gor_T = \overline{Gor_{D_{min}}}$, the Zariski closure of $Gor_{D_{min}}$ inside $Gor_T$. We give a method for determining the alternating matrix whose pfaffians generate the ideal with the smaller degree set, and show that it is Gorenstein of height 3. We conclude that whenever an ideal $J$ determining $T$ is generated by more than the minimum number needed for $T$, it can be deformed to an ideal with fewer generators.

We again work from the perspective of a fixed Hilbert function $T$ in Chapter 3 to determine the maximum number of generators an ideal determining $T$ may have. This uses the combinatorial data described in [**BE1**] and [**St1**], specifically the conditions on a sequence $\{r_1, r_2, \ldots, r_u\}$ of integers, which we call *diagonal degrees*, that can occur as the differences in a degree set $D$ defining a Gorenstein algebra with Hilbert function $T$. We determine the maximum number of generators possible for a ideal determining a given Hilbert function, and we give an explicit example of a matrix whose pfaffians generate this number for any given $T$. There is a lattice structure we can give to the degree sets that determine $T$, where a vertex of the lattice is a sequence of diagonal degrees that is consistent with $T$, and where two vertices are connected if the sequence at one vertex is a subsequence of the other. There is a unique minimum vertex, corresponding to the difference sequence of $D_{min}$ and a unique maximum vertex, corresponding to the saturated sequence of diagonal degrees defining $T$. We also prove in Proposition 3.12 that there is a one to one correspondence between permissible Hilbert functions $T$ of socle degree $n$ and order $k(T)$ and self-complementary subpartitions of rectangular blocks of size $2k$ by $n-2k+2$.

Chapter 4 investigates various methods for determining the dimension of $Gor_T$ by studying the ranks of the Catalecticant matrices associated to a dual form $f$. For certain Hilbert functions of socle degree $n$, order $k(T)=d$, and bounded by $t$, we make a conjecture for the dimension of the family $Gor_T$, and we show the conjecture to be true for approximately two thirds of the possible cases. We determine the dimension of $Gor(T)$ for certain other $T$.

**0.1. Notation and definitions.** We will use the following notation and definitions throughout this paper.

- $R$ is the ring $k[x,y,z]$, where $k$ is an algebraically closed field. The maximal ideal of $R$ is $m=(x,y,z)$.
- $A$ is a graded, height 3 Gorenstein algebra quotient of $k[x,y,z]$.
- $R_i$ is the space of forms of homogeneous degree $i$ in $R$. $R$ is a graded ring and can be expressed as $\oplus_{i\geq 0}R_i$, with $R_0=k$.
- $R_d(g_i)$ is the subspace $(x^d g_i, x^{d-1}yg_i,\dots,z^d g_i)$ of forms of degree $d+\deg g_i$ in $R$ generated by the homogeneous polynomial $g_i$.
- We let $T=T(A)=(h_0,h_1,\dots,h_n,0,\dots)$ be the Hilbert function of a Gorenstein Artin algebra $A=R/J$, where $J$ is a homogeneous ideal in $k[x,y,z]$. $J$ therefore has a grading $J=\oplus_{i\geq 0}J_i$. The nonnegative integer $h_i$ is the dimension of $R_i/J_i$ as a $k$-vector space.
- The socle of $R/J$ in the set $\{a\in R/J\,|\,a\cdot h\in J\ \forall h\in m\}$. We call $R/J$ a *Gorenstein ring* and $J$ a *Gorenstein ideal* if the dimension of the socle of $R/J$ as a $k$-vector space is 1.
- $D=(\{q_i\},\{p_i\})$ is a set of generator degrees and relation degrees for an ideal $J$ in $R$ corresponding to a given Hilbert function $T$.
- $\mathcal{R}=\mathrm{Hom}_k(R,k)$ is the ring dual to $R$. $R$ acts on $\mathcal{R}$ by contraction; if $X^aY^{c+d}Z^e\in\mathcal{R}$, then $x^ay^d\circ X^aY^{c+d}Z^e=Y^cZ^e$.
- We denote by $f$ a homogeneous polynomial in $\mathcal{R}$ whose annihilator in $R$ is $J$.
- $I_t(M)$ is the ideal of $t+1$ by $t+1$ minors of an $n$ by $m$ matrix $M$. If $t\geq\min\{m,n\}$, then $I_t(M)$ is defined to be the zero ideal $(0)$.

## 1. Variety structure on $Gor_D$.

We assume in this chapter that $k$ is an algebraically closed field, $R = k[x, y, z]$ is a ring, and $J$ is a homogeneous ideal in $R$.

Let $A = R/J$ be a Gorenstein algebra. The set $D = D(J) = (\{q_i, p_i\})$, $i = 1 \dots u$ of generator and relation degrees of $J$ determines the Hilbert function of $R/J$. Buchsbaum and Eisenbud's structure theorem for height 3 Gorenstein ideals proves that all such ideals can be obtained as pfaffians of a suitable alternating matrix $M$ with entries in $R$. The degree matrix $E_M$ of the entries of $M$ is determined by $D$ (see Chapter 2). Denote by $E = E(D)$ the set of entry degrees in $E_M$. The degree matrix determines the number of ways of filling in $M$ with entries chosen generally from $R$. This number $h(E_M)$ is a polynomial in the entry degrees $E$.

Let $\pi$ be the map from the family $\mathbb{A}^{h(E_M)}$ of all alternating matrices with degree matrix equal to $E_M$ to the family of algebras $A = R/J$ having the set $D(J)$ of generator and relation degrees, determined by $E_M$. We will show that when we restrict to a single $D$ and $E_M$, the matrices whose pfaffians form a height 3 ideal is a nonempty dense open set $U_{E_M}$ in $\mathbb{A}^{h(E_M)}$. We say a degree matrix $E_M$ is permissible if $U_{E_M}$ is nonempty. We discuss the conditions for $U_{E_M}$ to be nonempty in Chapter 3.

**1.1. Definition and parametrization of $Gor_D$.** Let $Gor_D$ be the family of Gorenstein algebras having the set $D$ of generator and relation degrees. $D$ determines the Hilbert function of any Artin algebra $R/J$ with $D = D(J)$. We define $Gor_T$ to be the union of all families of Gorenstein algebras $Gor_{D,}$ associated to Hilbert function $T$. $Gor_T$ is a locally closed subset of $G_T$, the family of all graded algebras $R/I$ with Hilbert function equal to $T$. The ideal $I$ has a grading $I = \oplus_{t \geq 0} I_t$. $G_T$ is embedded in a product of Grassmannians $\prod Grass(d_t, R_t)$, with $d_t = |I_t|$, the size of $I_t$ as a $k$-vector space. We give $Gor_T$ the reduced subscheme structure coming from this product.

Define $\pi$ to be the map from an open set $U_{E_M}$ in $\mathbb{A}^{h(E_M)}$ to the Gorenstein algebra $Gor_D$ whose degree set $D$ is determined by the degree matrix $E_M$.

**Theorem 1.1.** *$Gor_D$ is the image of $U_{E_M}$ under the algebraic map $\pi$, and is therefore irreducible.*

*Proof.* Let $D$ be given and let $M$ be an alternating $u$ by $u$ matrix with degree matrix $E_M$ such that the set of pfaffians of $M$ generates a height 3 Gorenstein ideal $J$. Let $M_i$ denote the submatrix of $M$ obtained by eliminating row $i$ and column $i$. The image $\pi(M)$ is the algebra $R/J$ where $J = (g_1, \dots, g_u)$. A generator $g_i$ of $J$ is the square root of the determinant of $M_i$. This generator can also be computed by the formula

$$\sqrt{\det M_i} = Pf(M_i) = \sum_{k=1}^{u-1}(-1)^k \cdot m_{rk} \cdot Pf(M_{ikr}), \tag{1.1}$$

where $r$ is a row by which to expand, $m_{rk}$ is the $(r,k)$ entry of $M_i, M_{ikr}$ is the submatrix of $M$ with rows and columns $i, k$ and $r$ eliminated [**Sa**, p. 71]. This expresses the generators $(g_1, g_2, \dots, g_u)$ of $J$ as polynomials in the entries of $M$, so $\pi$ is an map from $U_{E_M}$ to $Gor_D$. $U_{E_M}$ is an open set in $\mathbb{A}^{h(E_M)}$ since an ideal has height 3 when the determinant of a certain matrix does not vanish (see proof of Theorem 2.3 in Section 2.3). Consequently $U_{E_M}$ is irreducible, and so is its image $Gor_D$. □

**Remark.** The fiber $\pi^{-1}$ over a point $p_J$ parametrizing $J$ includes a product of general linear groups parametrizing different choices of generators for $J$. We get an upper bound for the dimension of $Gor_D$ by subtracting the dimension of this product from the dimension of $\mathbb{A}^{h(E_M)}$. We will use this fact in Chapter 4 in the proof of Theorem 4.4.

If we look at all degree sets $D_1, D_2, \dots$ of ideals $J_1, J_2, \dots$ in $R$ such that the Hilbert function of $R/J_i$ equals $T$ for each $i$, then each $Gor_{D_i}$ is irreducible by Theorem 1.1. There are a finite number of different degree sets $D_i$ for a given Hilbert function, a result of the structure theorem, and we will show in Chapter 2 that the entire family $Gor_T$ of all algebras $A = R/J$ with Hilbert function $T$ is irreducible.

We parametrize the family $Gor_D$ by the product $\prod Grass(t_d, R_d)$ of Grassmannians, which embeds $Gor_D$ as a subspace of a product of projective spaces $\prod \mathbb{P}$ whose coordinates depend polynomially on $(g_1, \dots, g_u)$.

## 2. Irreducibility of $Gor_T$.

As a result of Theorem 1.1 in the previous chapter, we know that given $T$ and a set of generator and relation degrees $D$, the family $Gor_D$ is irreducible. However, $T$ may have several different degree sets that correspond to the same Hilbert function.

**2.1. Definition of $Gor_T$.** We have defined $Gor_T$ to be the family of all algebras with Hilbert function $T$. $Gor_T$ is the finite union $\bigcup_i Gor_{D_i}$ over all degree sets $D_i = D(J_i)$ consistent with $T$.

**2.2. Structure theorem for Gorenstein ideals of height 3.** The following is the statement of Buchsbaum and Eisenbud's structure theorem for Gorenstein ideals of height 3 in a local noetherian ring [**BE1**, p. 456].

**Theorem 2.1.** *Let $R$ be a noetherian local ring with maximal ideal $m$.*

1) *Let $n \geq 3$ be an odd integer, and let $F$ be a free $R$-module of rank $n$. Let $f : F^* \mapsto F$ be an alternating map whose image is contained in $mF$. Suppose $Pf_{n-1}(f)$ has grade 3. Then $Pf_{n-1}(f)$ is a Gorenstein ideal, minimally generated by $n$ elements.*

2) *Every Gorenstein ideal of grade 3 arises as in 1).*

Buchsbaum and Eisenbud develop the machinery in [**BE1**] to prove the above theorem. If $J$ is a homogeneous Gorenstein ideal of grade 3 in $R = k[x_0 \dots x_m]$, then a free resolution of $R/J$ has the form

$$(2.1) \qquad E: \ 0 \longrightarrow R(s) \xrightarrow{g^*} \sum_{i=1}^{n} R(p_i) \xrightarrow{f} \sum_{i=1}^{n} R(q_i) \xrightarrow{g} R(0)$$

where the maps $f$ and $g$ are homogeneous of degree 0, the socle of $R/J$ is in degree $s-3$, the integers $\{q_i\}$ are the degrees of the generators of $J$ and integers $\{p_i\}$ are the degrees of the relations among the generators.

Buchsbaum and Eisenbud prove that the matrix representing $f$ in the resolution of $R/J$ will be skew-symmetric, and the matrix representing $g$ (resp. $g^*$) will be the column (resp. row) matrix of pfaffians of the matrix representing $f$. The degree of the $(i,j)$ entry of this middle matrix is $p_j - q_i$. We will always consider the sequence $\{q_i\}$ to be nondecreasing and the sequence $\{p_i\}$ to be nonincreasing. This defines a new sequence $\{r_i\}$, where $r_i = p_i - q_i$. With this ordering on the degrees, $p_i + q_i = s$. The integers $\{r_i\}$ are all even or all odd, since the degree of the $(i,j)$ entry of the alternating matrix representing $f$ is also expressed as $(r_i + r_j)/2$. We have the further relation that $s = \sum r_i$. Therefore the sequence of integers $\{r_i\}$ completely determines the socle degree, generator and relation degrees, and the Hilbert function. We will discuss which sequences of $\{r_i\}$ can occur in Chapter 3. It follows from (2.1) that the Hilbert function $(h_0, h_1, \dots)$ of $R/J$ equals

$$(2.2) \quad h_t = \binom{m+t}{m} - \sum_{i=1}^{n} \binom{m+t-q_i}{m} + \sum_{i=1}^{n} \binom{m+t-p_i}{m} - \binom{m+t-s}{m}.$$

Here the binomial coefficient $\binom{a}{b}$ equals zero if $a$ is less than $b$.

*Example* 2.2. Let $D = (\{3,5,6\},\{11,9,8\})$ be the set of generator and relation degrees of a Gorenstein ideal. The $\{r_i\}$ are equal to $\{8,4,2\}$. We get $s = 8+4+2 = 14$, so the socle degree is 11 and the Hilbert function is

$$(1,3,6,9,12,14,14,12,9,6,3,1,0,\dots).$$

This Hilbert function can also be determined by the sequence

$$D' = (\{3,5,6,6,8\},\{11,9,8,8,6\}).$$

**2.3. Deformation theorem.** We will now show that when the ideal of pfaffians of the matrix representing $f$ has height 3 and is Gorenstein, and the matrix contains non-diagonal degree zero entries, we can allow a pair of these entries to be nonzero constants $c, -c$. The resulting ideal of pfaffians will be Gorenstein, height 3, minimally generated by $n-2$ elements. We start with an existing resolution of a Gorenstein ideal which satisfies all the conditions of the structure theorem. Since $R$ is a polynomial ring it is Cohen-Macaulay, so depth$(I)$ =height$(I)$ for every ideal $I \subset R$. A homogeneous ideal $I$ in $R$ is also *perfect*, so all height 3 ideals have projective resolutions of length 3.

Let $J$ be a Gorenstein ideal of height 3 generated by $v+2$ elements, $v$ odd, with the Hilbert function of $R/J$ equal to $T$. Assume a minimal free resolution of $R/J$ has the form in (2.1). Let the alternating matrix representing the map $f$ be

$$M = \begin{pmatrix} 0 & m_{1,2} & m_{1,3} & \dots & m_{1,v+2} \\ -m_{1,2} & 0 & m_{2,3} & \dots & m_{2,v+2} \\ \vdots & \vdots & \vdots & & \vdots \\ -m_{1,v+2} & -m_{2,v+2} & -m_{3,v+2} & \dots & 0 \end{pmatrix}.$$

Assume that the set $D(J)$ requires that $M$ contain two non-diagonal degree zero entries, which we may assume to be the $(v+1, v+2)$ and $(v+2, v+1)$ entries of $M$. Under the conditions of the structure theorem these must equal zero in order for the image of $f$ to be contained in $mR^{v+2}$. Because the degrees of entries of $M$ are determined by the values of the diagonal degrees $\{r_i\}$, we have $r_{v+1} = -r_{v+2}$.

Let $M'(c)$ be the matrix obtained from $M$ by letting $m'_{v+1,v+2}$ and $m'_{v+2,v+1}$ be nonzero elements $c$ and $-c \in k$ in a neighborhood of zero, and all other entries $m'_{i,j}$ of $M'(c)$ equal to $m_{i,j}$. Let $\pi(M'(c)) = J'(c)$ be the ideal of pfaffians of $M'(c)$.

**Theorem 2.3.** *Let $J$ be a Gorenstein ideal of height 3 with Hilbert function $T$ and minimal free resolution*

$$E: \ 0 \longrightarrow R(s) \xrightarrow{g^*} \sum_{i=1}^{v+2} R(p_i) \xrightarrow{f} \sum_{i=1}^{v+2} R(q_i) \xrightarrow{g} R(0)$$

*where $f$ is represented by the $v+2$ by $v+2$ alternating matrix $M$, $v+2$ odd. Let $M'(c)$ and $J'(c)$ be the matrix and ideal of pfaffians described above. Then:*

(i) *$J'(c)$ has height 3 for all but finitely many $c$, and is generated by $v$ elements;*

(ii) *$R/J'(c)$ has a resolution $E'$ obtained from $E$ by replacing $M$ by $M'(c)$, $g$ by $g'$ and $g^*$ by $g'^*$;*

(iii) *$J'(c)$ is Gorenstein, and $R/J'(c)$ has a minimal resolution explicitly determined by $M$ and $c$;*

(iv) *the Hilbert function of $R/J'(c)$ is $T$.*

We will need the following lemma, which we use without proof:

**Lemma 2.4.** *Let $T = (h_0, h_1, \ldots, h_n, 0, \ldots)$ be the Hilbert function of a Gorenstein algebra $A = R/J$ and let $Q = \{q_i\}$ and $P = \{p_i\}$, $i = 1 \ldots u$, be the sequences of generator and relation degrees of $J$ satisfying*

$$\binom{m+t}{m} - \sum_{i=1}^{u} \binom{m+t-q_i}{m} + \sum_{i=1}^{u} \binom{m+t-p_i}{m} - \binom{m+t-s}{m} = h_t.$$

*Let $I$ be another ideal which defines the same Hilbert function $T$, and $Q' = \{q_j\}$ and $P' = \{p_j\}$ its generator and relation degrees, $j = 1 \ldots u+d$, such that $P'$ contains $P$ and $Q'$ contains $Q$ as subsequences. If we let $P''$ and $Q''$ be the sequences $P' \setminus P$ and $Q' \setminus Q$, both arranged in increasing order, then $p_k = q_k$ for $p_k \in P''$ and $q_k \in Q''$, $k = 1 \ldots d$.*

*Proof of Theorem* 2.3. Let $\{g_1 \ldots g_{v+2}\}$ be the pfaffians of $M$. These are a minimal set of generators for $J$ under the assumptions of Theorem 2.2. The ideal $J'(c)$ will be generated by homogeneous polynomials $\{g'_1 \ldots g'_{v+2}\}$, where $g'_i = g_i + c \cdot h_i(x,y,z)$. Let $q_i = \deg g_i = \deg g'_i$.

i). Assume $J$ has height 3. We will show that the vector space $J'_{n+1}$ of forms of degree $n+1$ in $J'(c)$ has dimension $\binom{n+3}{2}$, and therefore contains everything in degree $n+1$.

Since $J$ is a Gorenstein ideal whose socle is in degree $n$, the vector space $J_{n+1}$ has dimension $\binom{n+3}{2}$. The set of forms in the vector space

$$R_{n+1-q_1}(g_1), \ldots, R_{n+1-q_{v+2}}(g_{v+2})$$

span $J_{n+1}$ and form the row space of a matrix $N$ of size $G$ by $\binom{n+2}{2}$, where $G \geq \binom{n+3}{2}$ is the sum of the dimensions of the vector spaces $R_{n+1-q_i}(g_i)$. Each element of $J_{n+1}$ is expressed in terms of the standard basis of $R_{n+1}$ of monomials $\{x^{n+1}, x^n y, \ldots, z^{n+1}\}$. $N$ has entries in $k$, and the ideal of maximal minors of $N$ must contain at least one nonzero constant $\delta \in k$.

In the same way we take the generators $\{g'_i\}$ of $J'(c)$ and look at the matrix $N(c)$ whose rows are spanned by the forms in the vector spaces $R_{n+1-q_i}(g'_i)$. Since each $g'_i = g_i + c \cdot h_i(x,y,z)$, $N(0) = N$, and the ideal of maximal minors of $N(c)$ contains an element $\delta(c)$ such that $\delta(0) = \delta$.

Since $\delta(c)$ is a polynomial function of the entries of $N(c)$, there are finitely many values of $c$ for which $\delta(c)$ equals zero. Since $k$ is algebraically closed and therefore infinite, we can choose a Zariski open set $U$ containing zero so that when $c \in U$, $\delta(c)$ is nonzero. Since $N(c)$ contains at least one nonzero maximal minor, it has rank $\binom{n+3}{2}$, and therefore the dimension of $J'_{n+1}$ is $\binom{n+3}{2}$.

To see that $J'(c)$ is minimally generated by $v$ elements, note that when we multiply row $i$ of $M'(c)$ by the column matrix $g'$ we get the sum $\sum_{j=1}^{v+2} M'(c)_{ij} \cdot g'_j = 0$. If $i = v+2$, this becomes $\sum_{j=1}^{v+1} M'(c)_{v+2,j} \cdot g'_j = -cg'_{v+2}$, which for nonzero $c$ allows $g'_{v+2}$ to be expressed in terms of previous entries of $g'$. We can express $g_{v+1}$ in the same way in terms of previous generators.

ii). The following is a resolution for $J'(c)$:

$$E' : 0 \longrightarrow R(s) \xrightarrow{g'^*} \sum_{i=1}^{v+2} R(p_i) \xrightarrow{f'} \sum_{i=1}^{v+2} R(q_i) \xrightarrow{g'} R(0)$$

where $f'$ is represented by $M'(c)$ and the pfaffian map $g'$ is represented by $J'(c)$. Note that $E'$ is not a minimal resolution, because the image of $f'$ is no longer contained in $mR^{v+2}$.

To show that $E'$ is a resolution of $J'(c)$, we must show that $E'$ is a complex and that it is exact. Since the matrices representing the maps $g'^*$ and $g'$ are 1 by $v+2$ and $v+2$ by 1 matrices of pfaffians of $M'(c)$, the compositions $g'^* \cdot f'$ and $f' \cdot g' = 0$, so $E'$ is a complex. For any complex of free $R$-modules

$$A : 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

let $J(\phi_k)$ the ideal of minors of size $k$ of the matrix representing $\phi_k$, where $k$ is the size of the largest nonvanishing minor. To show that $\mathbb{A}$ is exact, it is enough to show the following [**BE2**]:

a) rank $\phi_{k+1}+$ rank $\phi_k =$ rank $F_k$;

b) grade $J(\phi_k) \geq k$ or $J(\phi_k) = R$.

a). We need to show $\operatorname{rank} g' + \operatorname{rank} f' = \operatorname{rank} g'^* + \operatorname{rank} f' = v+2$. We know $\operatorname{rank} g' = \operatorname{rank} g'^* = 1$, so we need to show $\operatorname{rank} f' = v+1$. Since $E$ is a resolution, we know $\operatorname{rank} f = v+1$, so $M$ contains a $v+1$ by $v+1$ submatrix whose determinant is nonzero; in particular, the submatrix obtained by eliminating the last row and column has nonzero determinant, since its square root is one of the $v+2$ generators of $J$. This same submatrix occurs in $M'(c)$, so $f' \geq v+1$. On the other hand, $M'(c)$ is skew-symmetric and $v+2$ odd, implying that the determinant of $M'(c)$ equals zero. Therefore rank $f' = v+1$.

b). We already know height $J(g'^*) =$ height $J(g') = 3$. The ideal of $v+1$ by $v+1$ minors of $M'(c)$ contains $J(g')$, therefore $J(M'(c))$ has height $\geq 3$.

iii). To show $J'(c)$ is Gorenstein we will exhibit a *minimal* resolution for $J'(c)$ of the form

$$0 \longrightarrow R \xrightarrow{g'^*} R^v \xrightarrow{\psi} R^v \xrightarrow{g'} R \tag{2.3}$$

where $\psi$ is represented by a $v$ by $v$ alternating matrix $Y_\psi$ whose pfaffians generate $J'(c)$.

Let $W$ be the upper triangular $v+2$ by $v+2$ matrix

$$W = \begin{pmatrix} c\,0\,0\,0 \dots & 0 & -m_{1,v+2} & m_{1,v+1} \\ 0\,c\,0\,0 \dots & 0 & -m_{2,v+2} & m_{2,v+1} \\ \vdots\ \vdots\ \vdots\ \vdots & \vdots & \vdots & \vdots \\ 0\,0\,0\,0 \dots & c & -m_{v,v+2} & m_{v,v+1} \\ 0\,0\,0\,0 \dots & 0 & 1 & 0 \\ 0\,0\,0\,0 \dots & 0 & 0 & 1 \end{pmatrix}.$$

The product $\frac{1}{c}(W \cdot M'(c) \cdot W^T)$ is the matrix $Y =$

$$\begin{pmatrix} 0 & cm_{1,2} - D_{12} & cm_{1,3} - D_{13} & \dots & cm_{1,n} - D_{1n} & 0 & 0 \\ -cm_{1,2} + D_{12} & 0 & cm_{2,3} - D_{23} & \dots & cm_{2,n} - D_{2n} & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ -cm_{1,n} + D_{1n} & cm_{2,n} - D_{2n} & cm_{3,n} - D_{3n} & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}$$

where the term $D_{ij}$ is equal to $m_{i,v+1} \cdot m_{j,v+2} - m_{i,v+2} \cdot m_{j,n+1}$. The pfaffians of this product generate $J'(c)$, and in particular the minors obtained by omitting rows and columns $v+1$ or $v+2$ are equal to zero.

We define the matrix $Y_\psi$ representing $\psi$ in (2.3) to be $Y$ with rows and columns $v+1$ and $v+2$ removed. When $j \le v$, $Pf(Y_j) = Pf((Y_\psi)_j)$, so the pfaffians of $Y_\psi$ generate $J'(c)$. The map $\psi$ now satisfies the condition of the structure theorem that its image is contained in $mR^v$. Since $J'(c)$ has height 3 it is Gorenstein, minimally generated by $v$ elements.

iv). The complex given in (2.3) is exact, since it satisfies the criteria a) and b), so the Hilbert function of $R/J'(c)$ can be computed by (2.2); by Lemma 2.4, this will be the same as the Hilbert function of $R/J$. This completes the proof. $\square$

When we work in 3 variables, the third difference sequence $(d_0, d_1, \dots, d_t, \dots)$ of $T$ gives the net difference between the number of relations and the number of generators in each degree $t$. We denote by the odd number

$\mu(T)$ the minimum number of generators determined by the third difference sequence.

**Corollary 2.6.** *For a Hilbert function $T = (h_0, h_1, \ldots, h_n, 0, \ldots)$ of a height 3 Gorenstein algebra $R/J$, the integer*

$$\mu(T) = 2\left\lceil \frac{-\sum_{d_i<0} d_i}{2} \right\rceil - 1$$

*can be realized as the minimum number of generators for $J$.*

*Proof.* This follows from repeated application of Theorem 2.3, since if an ideal defining $T$ has more generators than $\mu(T)$, then it has generators and relations in the same degree, and its alternating matrix has degree zero entries occuring in pairs. Each pair of entries may be deformed to nonzero constants, causing the number of generators in the ideal to drop by 2. $\square$

**Remark.** When the socle degree $n$ is odd, adding up the negative terms in the third difference sequence may not indicate a minimum number of generators. For example, $T = (1, 3, 6, 6, 3, 1)$ has for its third differences the sequence

$$1, 0, 0, -4, 0, 4, 0, 0, -1,$$

which indicates at least four generators in degree 3 for an ideal which determines $T$. Since $\mu(T)$ must be odd, we need at least 5 generators. $T$ can be generated by the ideal $(x^3, y^3, xz^2, yz^2, x^2y^2 - z^4)$, and it is Gorenstein, with dual polynomial equal to $x^2y^2z + z^5$. The extra generator and relation must be in degree $\frac{n+3}{2}$ to preserve the symmetry of the sums $p_i + q_i = n + 3$. The degree matrix determined by these generator and relation degrees will have no nonzero degree 0 entries, since the only entry whose degree is 0 is on the diagonal, in which case it must be equal to zero.

**Remark.** The entry degrees $E(D)$ of the degree matrix determined by the degree set $D$ need not all be positive. The Hilbert function $(1, 3, 6, 6, 6, 3, 1)$ has for its third differences the sequence

$$1, 0, 0, -4, 3, -3, 4, 0, 0, -1,$$

indicating row degrees $\{3, 3, 3, 3, 5, 5, 5\}$ and column degree $\{6, 6, 6, 6, 4, 4, 4\}$. This means the 7 by 7 degree matrix will have a 3 by 3 block of degree -1 entries, which must be zeros.

Theorem 2.3 and Corollary 2.6 give us the result we want:

**Theorem 2.7.** *$Gor_T$ is irreducible.*

*Proof.* The family of Gorenstein algebras $Gor_T$ with Hilbert function equal to $T$ is equal to the closure $\overline{Gor_{D_{min}}} = \cup Gor_{D'}$ for all $D' \supset D$ by Theorem 2.3, since ideals with degree sets $D'$ can be deformed to the minimal degree set $D_{min}$. Since $Gor_{D_{min}}$ is irreducible by Theorem 1.1, $Gor_T$ is irreducible. $\square$

## 3. Number of generators of height 3 Gorenstein ideals.

We have seen in Corollary 2.6 that when we fix the Hilbert function $T$ of a Gorenstein algebra $A = R/J$ we can determine the minimum number of generators needed for $J$ by taking the third differences of the sequence of integers in $T$. We proved in Theorem 2.3 in the previous chapter that if $J$ has more than $\mu(T)$ generators, we can deform the entries of an alternating matrix whose pfaffians generate $J$ so that $J$ needs two fewer generators.

The degrees of the generators and relations of a height 3 Gorenstein ideal $J$ can be described completely by the integers $\{r_i\}$ defined in Section 2.2 as the differences $p_i - q_i$ of relation and generator degrees of $J$ when arranged in decreasing and increasing order, respectively. These integers can be used to determine the maximum number of generators possible for an ideal $J$ defining a given Hilbert function $T$.

**3.1. Saturated sequences of integers $\{r_i\}$.** Recall that an alternating matrix $M$ can be assigned row degrees $\{q_i\}$ and column degrees $\{p_i\}$ such that the $\{q_i\}$ are nondecreasing and the $\{p_i\}$ are nonincreasing, with integers $\{r_i\}$ defined by $r_i = p_i - q_i$. In order for $M$ to have pfaffians that satisfy Theorem 2.1, the diagonal degrees $\{r_i\}$ of $M$ must satisfy the following conditions:

**Proposition 3.1.** *Let $M$ be an $u$ by $u$ alternating matrix with generic entries, $u$ odd, whose diagonal degrees $\{r_i\}$ are arranged in nonincreasing order. A necessary and sufficient condition for $M$ to have $u$ nonzero pfaffians is*

*the integers $r_i$ are all even or all odd;*

$$r_1 > 0; \tag{3.1}$$

$$r_i + r_{u-i+2} > 0 \textit{ for } i = 2 \dots \frac{(u+1)}{2}.$$

*Proof.* If the condition $r_i + r_{u-i+2} > 0$ is not satisfied, $M$ will contain zeros in all entries $(i,j)$ with $i,j \geq \frac{u+1}{2}$. The submatrix obtained by eliminating row and column $l$ not passing through this block of zeros will have the shape

$$M_l = \begin{pmatrix} \square & \square & A \\ \square & \square & \text{zeros} \\ A^T & \text{zeros} & \text{zeros} \end{pmatrix}$$

where $\square$ indicates a block containing nonzero entries. $A$ will be size $i-2$ by $i-1$, so the determinant of $M_l$ will be zero, contradicting the conditions of the structure theorem that $M$ have $u$ independent pfaffians.

To show that these conditions are sufficient, we exhibit a $u$ by $u$ matrix whose pfaffians generate a Gorenstein ideal with $u$ generators:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & y^{\gamma_1} & z^{\beta_1} \\ 0 & 0 & \dots & 0 & y^{\gamma_2} & z^{\beta_2} & x^{\alpha_1} \\ 0 & 0 & \dots & y^{\gamma_3} & z^{\beta_3} & x^{\alpha_2} & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & -y^{\gamma_2} & -z^{\beta_3} & -x^{\alpha_3} & 0 & \dots & 0 \\ -y^{\gamma_1} & -z^{\beta_2} & -x^{\alpha_2} & 0 & 0 & \dots & 0 \\ -z^{\beta_1} & -x^{\alpha_1} & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.2}$$

The integers $\alpha_i, \beta_i$ and $\gamma_i$ are defined as

$$\alpha_i = (r_{i+1} + r_{u+1-i})/2, \; \beta_i = (r_i + r_{u+1-i})/2, \; \gamma_i = (r_i + r_{u-i})/2.$$

If the diagonal degrees satisfy (3.1), the ideal of pfaffians of this matrix contains $x^{\sum \alpha_i}, y^{\sum \gamma_i}$ and $z^{\sum \beta_i}+$ other terms, so it clearly contains pure powers of $x, y$, and $z$ and therefore has height 3. Finally, since the ideal of pfaffians satisfies the conditions of Theorem 2.1 it is Gorenstein, minimally generated by $u$ elements. $\square$

We have learned that J. Herzog, N.V. Trung and G. Valla have arrived independently at the conditions of Proposition 3.1 and give matrix (3.2) as an example for the sufficiency of the conditions.

We say a sequence of integers $R = \{r_i\}$ occuring as the sequence of diagonal degrees of an alternating matrix is *saturated* if it satisfies (3.1) and it is impossible to lengthen the sequence by adding a pair of integers $d, -d$ to $R$ and still satisfy (3.1) without changing the Hilbert function determined by $R$. A sequence $R$ has a unique saturation if whenever $R'$ and $R''$ are two saturations of $R$ with lengths $v$ and $w$ respectively, we have $v = w$ and $r'_i = r''_i$ for all $i$.

**Theorem 3.2.**

(i) *A sequence $R = \{r_i\}$, $i = 1 \dots u$ is saturated if*

$$r_1 > 0,$$

$$r_i \geq r_{i+1} \textit{ for } i = 1 \dots u-1, \tag{3.3}$$

$$\textit{and } r_i + r_{u-i+2} = 2 \textit{ for all } i = 2 \dots \frac{u+1}{2}.$$

(ii) *Every sequence of integers $\{r_i\}$ arising from the generator and relation degrees of a Gorenstein ideal has a unique saturation.*

*Proof.* Let $R = \{r_1, r_2, \dots, r_u\}$ be given and let $R'$ be a saturation of $R$. Denote by $Q, P$ and $Q', P'$ the ordered sequences of generator and relation degrees whose differences are the two sequences $R$ and $R'$. Let $n$ be the socle degree of all ideals defined by these sequences. We will show $R'$ is unique.

As we saw in Chapter 2, the minimum number of generators of a Gorenstein ideal $J$ defining $T$ is $\mu(T)$. Any other ideal $J'$ with a larger number of generators determining the same Hilbert function as $J$ must have additional generators and relations occuring in the same degrees. This implies that $\sum r_i = n+3$ is constant, since adding $d_1$ and $d_2$ to the sequences $\{q_i\}$ and $\{p_i\}$ will add $d_1 - d_2$ and $d_2 - d_1$ to the sequence $\{r_i\}$, leaving $\sum r_i$ unchanged.

The sequences must satisfy $p_i + q_i = p'_i + q'_i = n+3$. Since the sequences determine the same Hilbert function, the smallest generator degree in the ideals they define must be the same; therefore $q_1 = q'_1$ and $p_1 = p'_1$, so $r_1 = r'_1$.

*Proof of* i). To saturate $R$, we begin by adding a pair $d, -d$ to $R$. Clearly $d$ must be less than $r_1$, otherwise we would be adding a generator in degree $q_1$ to the ideal, which would change the Hilbert function. We reorder $R \cup \{d, -d\}$ so that the new sequence is nonincreasing, and continue adding pairs until $r'_i + r'_{v-i+2} = 2$ for $i \geq 2$ in a larger sequence $R' = \{r'_1, \dots, r'_v\}$. $R'$ will be of the form

$$\left(r_1, \dots \text{ positive integers }, \{1's\}, \text{ negative integers }\right)$$

when the integers are all odd, or

$$\left(r_1, \dots \text{ positive integers }, \{2's\}, \{0's\}, \text{ negative integers }\right)$$

when the integers are all even. Once we reach the point where $r'_i + r'_{v-i+2} = 2$ we cannot lengthen $R'$, since the insertion of $d, -d$ into a sequence of length $v$ forces a sum $r'_i + r'_{v+2-i+2} = 0$, where $r'_i = d$ and $r'_{v+2-i+2} = -d$.

If not all sums in $R'$ satisfy $r'_i + r'_{v-i+2} = 2$, there must exist a sum $r'_k + r'_{v+2-k} \geq 4$. We can insert the pair $d, -d$ where $d = r'_k - 2$ to get a new sequence $R' \cup \{d, -d\}$ of the form

$$\left(r_1, \dots, r'_k, d, \dots, r'_{v-k+2}, -d, \dots\right).$$

This new sequence satisfies (3.1).

*Proof of* ii). Now assume $R$ has two different saturations, $R'$ and $R''$, of lengths $v$ and $w$ respectively. Since $\sum r'_i = r'_1 + 2 \cdot \frac{v-1}{2}$ must equal $\sum r''_i = r''_1 + 2 \cdot \frac{w-1}{2}$ and $r'_1 = r''_1$, it follows that $v$ is equal to $w$.

Let $k$ be first position for which $r'_k \neq r''_k$; we may assume $r'_k < r''_k$. Therefore there is at least one more occurrence of the pair $r''_k, -r''_k$ in $R''$. Since $R''$ and $R'$ agree in position 1 through $k-1$ and they are both saturated sequences, they must also agree in position $u-k+3$ through $u$. Suppose $-r''_k$ occurs in position $p$ in $R''$. We must have $p > v-k+2$ to satisfy (3.1). But now $p$ is in the range where both saturations agree, contradicting the fact that $R'$ doesn't contain this occurrence of $-r''_k$. Therefore the two saturations must agree everywhere. $\square$

**3.2. Maximum number of generators of a Gorenstein ideal.** Let $T = (h_0, h_1, \ldots, h_n, 0, \ldots)$ be given, where $T$ is the Hilbert function of $R/J$. Let $k$ be the first position in which $h_k < \binom{k+2}{2}$. We call $k = k(T)$ the *order of* $T$, and it is equal to the smallest degree of the generators of $J$ for any graded ideal $J$ which determines $T$. Clearly, $k$ depends only on $T$.

**Theorem 3.3.** *Let a Hilbert function $T$ of order $k(T)$ be given. The maximum number of generators of a Gorenstein ideal $J$ which determines $T$ is $2 \cdot k(T) + 1$, and it occurs if and only if the sequence of diagonal degrees determined by $J$ is saturated.*

*Proof.* Let $R = \{r_1, r_2, \ldots, r_u\}$ be a saturated sequence of diagonal degrees which satisfies (3.3). Since $\sum_{i=1}^{u} r_i = n+3 = s$, and we assume $r_i + r_{u+2-i} = 2$ for $i = 2 \ldots (u+1)/2$, we know $s = r_1 + u - 1$, and from the definitions of $s$ and $\{r_i\}$, the smallest generator degree equals $k = (s - r_1)/2 = (u-1)/2$, so $u = 2k+1$.

Conversely, if the smallest degree is $k$ and $R$ has length $2k+1$, we get $2k = s - r_1 = \sum_{i=2}^{2k} r_i = (r_2 + r_{2k+1}) + \cdots (r_{(2k+1+1)/2} + r_{(2k+1+3)/2})$, which must all be positive. Therefore they must all be equal to 2, so $R$ is saturated. $\square$

If we set all $\alpha_i$ equal to 1 in the matrix (3.2), we are working with a saturated sequence of $\{r_i\}$. The smallest degree generator is $x^{\sum \alpha_i} = x^k$, and $u = 2k+1$.

**Corollary 3.4.** *Given a permissible Hilbert function $T$, all values between $\mu(T)$ and $2 \cdot k(T) + 1$ can occur as the number of generators of an ideal defining $T$.*

*Proof.* This follows from Theorem 2.3 and Proposition 3.1. $\square$

Stanley has shown [**St1**] that a sequence $T = (h_0, h_1, \ldots, h_n)$ of nonnegative integers with $h_1 \leq 3$ occurs as the Hilbert function of a Gorenstein algebra if and only if it is symmetric and the first difference sequence

$(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_s - h_{s-1})$ is a permissible Hilbert function, where $s = \lfloor n/2 \rfloor$.

Suppose $T$ is a permissible Hilbert function of some graded Gorenstein quotient of $R$. We will consider the family $Gor_T$ of all graded Gorenstein algebra quotients $A = R/J$ having Hilbert function $T$.

*Example* 3.5. $T = (1,3,4,4,\ldots,4,4,3,1)$. The third differences of $T$ are

$$1, 0, -2, 0, 1, 0, 0, \ldots, 0, 0, -1, 0, 2, 0, -1.$$

The ideals having this Hilbert function have at least two generators of degree 2 and one generator of degree $n-1$.

The third difference sequence of $T$ determines diagonal degrees $\{n-1, n-1, 5-n\}$. The sequence is not saturated, so we can lengthen it by adding at least one pair of integers. Since $k(T) = 2$, the only pair we can add is $n-3, 3-n$, getting a new sequence $\{n-1, n-1, n-3, 5-n, 3-n\}$. This sequence is saturated, so the maximum number of generators of an ideal determining $T$ is 5. $T$ can be generated by three generators in degrees $2, 2$ and $n-1$, for example $J = (x^2, y^2, z^{n-1})$, or five generators in degrees $2, 2, 3, n-1$ and $n$, for example $J = (x^2, xy, yz^2, y^{n-1}, z^n)$.

*Example* 3.6. Let $T = (1,3,6,7,6,3,1)$. The third differences of $T$ indicate at least 3 generators of degree 3. Since $k(T) = 3$, the maximum number of generators an ideal $J$ determining $T$ can have is 7. A 5-generator ideal having Hilbert function $T$ is $J_5 = (x^3, x^2z, xy^2 - z^3, y^3z, y^5)$, corresponding to $R = \{3,3,3,1,-1\}$, and a 3-generator ideal is $J_3 = (x^3, y^3, z^3)$, corresponding to $\{3,3,3\}$. The 7-generator ideal

$$J_7 = (x^2y, x^2z, xyz, y^3z - xz^3, xy^3 - yz^3, x^5 - z^5, y^5 - z^5)$$

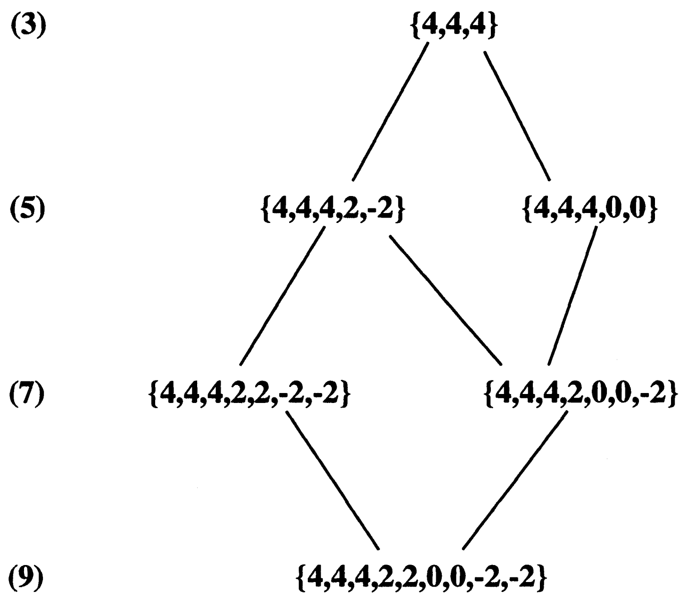
corresponding to the saturated sequence $\{3,3,3,1,1,-1,-1\}$, determines the same Hilbert function. The Hilbert function of $R/J_7$ was computed using *Macaulay.*

In summary, if a Hilbert function $T$ with socle in degree $n$ is given, and an ideal $J$ determining $T$ has $k$ for its smallest generator degree, then the upper bound on the number of generators that can generate $J$ is $2k+1$. This upper bound can be achieved for all permissible Hilbert functions in 3 variables.

**3.3. The lattice $\mathcal{L}(T)$ of $T$.** Define $\mathcal{L}(T)$ to be a lattice associated to the Hilbert function $T$. Its vertices are all the sequences of integers $\{r_i\}$ satisfying (3.1) which determine $T$. Two vertices are connected if the sequence at one vertex is a subsequence of the vertex below it. We call the vertex corresponding to the smallest subset the *minimal* vertex.

*Example* 3.7. Let $T = (1, 3, 6, 10, 12, 12, 10, 6, 3, 1)$. The minimum and maximum number of generators for ideals which determine $T$ are 3 and 9, respectively.

Number of generators




The sequence at the bottom level is saturated.

We have seen that if Hilbert function $T$ is fixed, we can exhibit ideals which achieve the minimum and maximum number of generators possible among all ideals which determine $T$. By Theorem 2.3, we can deform an ideal with $v+2$ generators into one with $v$ generators without changing $T$ as long as $v$ is at least $\mu(T)$. We are able to deform an ideal with $v+2$ generators into one with $v$ generators only when the matrix $M$ contains degree zero entries, which happens only when the sequence of integers $\{r_i\}$ for $M$ contains a pair $r_i$ and $r_j$, $i \neq j$ such that $r_i + r_j = 0$. Therefore there is a one to one correspondence between degree sets $D = (\{Q_i\}, \{P_i\})$ for ideals defining $T$ and the sequence of diagonal degrees for the corresponding alternating matrix.

The lattice structure associated to $T$ illustrates the irreducibility of certain subfamilies of $Gor_T$. A vertex $\mathcal{V}$ in the lattice represents a family $Gor_D$, with the degree sets $D$ specified by $\mathcal{V}$. $\mathcal{L}(T)$ is a geometric lattice, since any two vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ representing $Gor_{D_1}$ and $Gor_{D_2}$ determine unique vertices corresponding to $D_1 \cap D_2$ and $D_1 \cup D_2$. $Gor_{D_{min}}$ corresponds to the minimal vertex of $\mathcal{L}(T)$.

**Theorem 3.8.** *The closure $\overline{Gor_D}$ is equal to $\bigcup Gor_{D'}$ for all $D' \supset D$ and is irreducible.*

*Proof.* As seen in Chapter 2, ideals with degree sets $D'$ can be deformed to an ideal with degree set $D$ when $D' \supset D$. This shows in the lattice representation as $\overline{\mathcal{V}}$ = sublattice descending from $\mathcal{V}$. The minimal vertex is unique as a result of the unique third difference sequence of $T$. Thus $Gor_T = \overline{Gor_{D_{min}}}$, where $D_{min}$ is the degree set specified by the minimal vertex. □

**3.4. Saturated sequences of $\{r_i\}$ and partitions.** There is a convenient pairing between saturated sequences of $\{r_i\}$ and partitions as follows. Let the socle degree $n$ of a Gorenstein ideal $J$ and the order $k(T)$ of the Hilbert function it defines be given. We will look at all possible sequences $\{r_i\}$ of diagonal degrees for $J$.

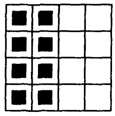
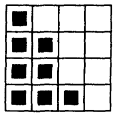
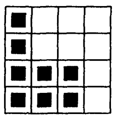
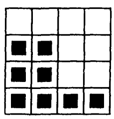
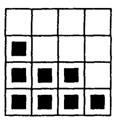
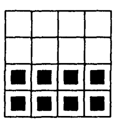
The maximum number of generators for $J$ is $2 \cdot k(T) + 1$. Construct a partition of a rectangle of size $2k+1$ by $s = n+3$ by dividing the $s$ blocks of row $i$ into $q_i$ and $p_i$ blocks, such that $p_i - q_i = r_i$. Eliminate the first row, $k$ columns from the left and $k+1$ columns from the right to get a $2k$ by $n-2k+2$ rectangle. The resulting partition of this rectangle will be self-complementary, and it retains the original information needed to reconstruct the ideal and the Hilbert function it determines.

**Proposition 3.9.** *When the socle degree $n$ of a height 3 Gorenstein ideal $J$ is fixed and the order $k = k(T)$ of the Hilbert function defined by $J$ is given, there is a one to one correspondence between Hilbert functions $T$ and self-complementary partitions of $2k$ by $n-2k+2$ blocks.*

*Proof.* The partition constructed above for any given Hilbert function will be self-complementary, since the $n-2k+2$ blocks of row $i$ are partitioned into $q_{i+1}-k$ and $p_{i+1}-k-1$ blocks, whose difference is $r_{i+1}-1$. If $j = 2k+1-i$, the blocks of row $j$ are partitioned into two parts whose difference is $r_{j+1}-1$. Since $r_{i+1}+r_{j+1} = 2$, we know $r_{i+1} = -r_{j+1}$, so rows $i$ and $j$ are complementary.

If we are given a self-complementary partition of a $2k$ by $n-2k+2$ rectangle, where row $i$ is divided into $a_i$ and $b_i$ blocks, $a_i + b_i = n-2k+2$, then we can recover the saturated sequence of diagonal degrees from the partition by letting $r_1 = n+3-2k$ and $r_{i+1} = b_i - a_i + 1$ for $i = 1 \ldots 2k$. Since a partition defines a unique sequence of diagonal degrees, it defines a unique Hilbert function. □

*Example* 3.10. Let $n = 6$ and $k(T) = 2$. There are 6 self-complementary partitions of a 4 by 4 rectangle, corresponding to the 6 different Hilbert functions with order 2 and socle degree 6:

| $T$ | *partition* | $\{r_i\}$ | *generator degrees* | *colength* |
|---|---|---|---|---|
| 1 3 5 7 5 3 1 |  | 5 1 1 1 1 | 2 4 4 4 4 | 25 |
| 1 3 5 6 5 3 1 |  | 5 3 1 1 -1 | 2 3 4 4 5 | 24 |
| 1 3 5 5 5 3 1 |  | 5 3 3 -1 -1 | 2 3 3 5 5 | 23 |
| 1 3 4 5 4 3 1 |  | 5 5 1 1 -3 | 2 2 4 4 6 | 21 |
| 1 3 4 4 4 3 1 |  | 5 5 3 -1 -3 | 2 2 3 5 6 | 20 |
| 1 3 3 3 3 3 1 |  | 5 5 5 -3 -3 | 2 2 2 6 6 | 17 |

Obviously $T$ is no longer fixed; the constants are now the socle degree and the order $k$. By counting the number of self-complementary partitions of a given size we are counting the number of Hilbert functions with the given socle degree and order. Since the partition is self-complementary, it is determined by the partition of a subrectangle with $k$ rows and $\lfloor n/2 \rfloor - k + 1$ columns into nondecreasing rows. There are $\binom{\lfloor n/2 \rfloor + 1}{k}$ such partitions.

Therefore the number of permissible Hilbert functions of a given socle degree equals

$$\sum_{k=0}^{\lfloor n/2\rfloor+1} \binom{\lfloor n/2\rfloor+1}{k} = 2^{\lfloor n/2\rfloor+1}.$$

See [**St2**] for a more general discussion of generating functions for plane partitions with varying degrees of symmetry.

## 4. Dimension of $Gor_T$.

We define a different parametrization in Section 4.2 for $Gor_T$ than that used in earlier chapters. With this parametrization the closure $\overline{Gor_T}$ includes Gorenstein algebras with different Hilbert functions.

**4.1. Matlis Duality and the dual polynomial $f$.** Let $k$ be an algebraically closed field of characteristic zero, $R = k[x,y,z]$ with maximal ideal $m = (x,y,z)$. Emsalem [**Em**] states that the dual $\mathcal{A} = \mathrm{Hom}_R(A,k)$ of a Gorenstein algebra $A = R/J$ with socle in degree $n$ can be obtained by the procedure of taking the vector space generated by a homogeneous degree $n$ polynomial $f$ and its partial derivatives of all orders. We use the divided powers of the derivatives of $f$ and write $f = \sum_{\alpha=1}^{\binom{n+2}{2}} b_\alpha x^\alpha$, $b_\alpha \in k$, where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ satisfies $|\alpha| = n$.

An isomorphism exists between $R$-closed subspaces

$$\mathcal{J} \subset \mathcal{R} \qquad \text{and} \qquad \mathrm{Hom}_R(R/J, E),$$

where $J$ is an ideal of $R$ and $E$ is the injective envelope of $R/m$. This isomorphism is shown in the theorem proved by Matlis [**Ma**] and discussed by Miri [**Mi**]. We assume $R$ is a commutative, Noetherian, complete local ring, with $E = \mathcal{R}$. From the exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

we derive the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(R/J,E) \arrow[r] \arrow[d, "\phi"] & \mathrm{Hom}_R(R,E) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(J,E) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{J} \arrow[r] & E \arrow[r] & E/\mathcal{J} \arrow[r] & 0.
\end{tikzcd}$$

Since $E$ is injective, the top row is exact. The bottom row is clearly exact, and the vertical map $\phi$ is an isomorphism defined as follows [**Ma**, p. 526]:

Let $R/J$ be generated by the element $e$, so that $g \in J$ if and only if $ge = 0$. Let $h \in \mathrm{Hom}_R(R/J,E)$ and define $\phi : \mathrm{Hom}_R(R/J,E) \to \mathcal{J}$ by $\phi(h) = (h \cdot e)$. Clearly $\phi$ is a well-defined $R$-homomorphism. If $\phi(h) = 0$, then $he = 0$, so $h = 0$. This shows $\phi$ is one-to-one. If we let $x \in \mathcal{J}$, define $h : R/J \to E$ by $he = x$. Then $\phi(h) = x$, so $\phi$ is onto, and therefore $\mathrm{Hom}_R(R/J, E) \cong \mathcal{J}$. Since the first two vertical maps are isomorphisms, so is the third, so $\mathrm{Hom}_R(J, E) \cong E/\mathcal{J}$.

**4.2. Catalecticant matrices associated to $f$.** We parametrize $Gor_T$ by using the coefficients of the dual polynomial $f$ described by Emsalem up to nonzero constant multiple. To specify a Hilbert function $(h_0, h_1, \ldots, h_n)$, we require that a degree $n$ polynomial $f$ have $h_d$ linearly independent partial derivatives of order $d$. The permissible Hilbert functions are those for which such a polynomial exists. This is the intersection of an open and closed condition on the $\binom{n+2}{2}$ coefficients of $f$.

When $f$ is a homogeneous polynomial in 3 variables, the $rth$ partial derivatives of $f$ form the row space of a $\binom{r+2}{2}$ by $\binom{n-r+2}{2}$ matrix. We denote this matrix $M_{r,n-r}(f)$, called the $r$th *Catalecticant* matrix associated to $f$. When $n = 2d$ is even, the Catalecticant $M_{d,d}(f)$ is square and symmetric.

Let $\mathcal{A} = k[\{b_\alpha\}][x, y, z]/J$, where $J$ is the annihilator of $f$ in the Matlis duality. Let $I_t(M_{d,d}(f))$ be the ideal in $k[\{b_\alpha\}] = k[B]$ of all determinantal minors of size $t+1$ of $M_{d,d}(f)$. When $t = \binom{d+2}{2}$ we set $I_t(M_{d,d}(f)) = (0)$, the zero ideal. For each $t$ from 0 to $\binom{d+2}{2}$ we get a different Hilbert function $T$, equal to $(1, 3, 6, \ldots, t, t, \ldots, t, \ldots, 6, 3, 1)$, the largest possible given $t$. The codimension in $k[B]$ of $I_t(M_{d,d}(f))$ will be the dimension of $k[B]/I_t(M_{d,d}(f))$.

**4.3. Codimension of $I_t(M_{2,2}(f))$.**

*Example* 4.1. The Hilbert functions of $k[B]/I_t(M_{2,2}(f))$ were computed for values of $t$ from 0 to 6 using the commutative algebra computer program *Macaulay*. In the case $n = 4$ the results are summarized below, where $H$ denotes the Hilbert function of a minimal reduction of $k[B]/I_t(M_{2,2}(f))$.

| $t$ | *codimension* | *dimension* | $H$ | *degree* |
|---|---|---|---|---|
| 0 | 15 | 0 | 1 | 1 |
| 1 | 12 | 3 | 1 12 3 | 16 |
| 2 | 9 | 6 | 1 9 45 17 3 | 75 |
| 3 | 6 | 9 | 1 6 21 56 21 6 1 | 112 |
| 4 | 3 | 12 | 1 3 6 10 15 | 35 |
| 5 | 1 | 14 | 1 1 1 1 1 1 | 6 |
| 6 | 0 | 15 | 0 | 1 |

This information was computed using the commutative algebra computer program Macaulay.

All values of $t$ can occur in the 3-variable case [**BE1**], and in the general case for any number of variables [**I1**].

The pattern of codimensions of $I_t(M_{2,2}(f))$ at first exhibits behavior similar to that of a generic symmetric matrix: the codimensions follow the pattern $1,3,6$, for corank $1,2,3$; after that corank they jump by 3's. This suggests examining other even values of $n$ to see if this pattern is sustained. When $n=3$, $M_{3,3}(f)$ will be size 10 by 10. If the same pattern evolves, we expect to see the values in the following table.

**Table 4.2**

| $t$ | *codimension* of $I_t(M_{3,3}(f))$ | *dimension* of $I_t(M_{3,3}(f))$ |
|---|---|---|
| 1 | 25 | 3 |
| 2 | 22 | 6 |
| 3 | (19) | $\geq 9$ |
| 4 | (16) | $\geq 12$ |
| 5 | (13) | $\geq 15$ |
| 6 | (10) | $\geq 18$ |
| 7 | 6 | 22 |
| 8 | 3 | 25 |
| 9 | 1 | 27 |
| 10 | 0 | 28 |

Numbers in () have not been verified. Dimension and codimension for $t=7,8$ and 9 follow from Theorem 4.4. Dimension for $t=1$ and 10 have been shown independently. The dimension for $t=2$ has been verified by Macaulay. The lower bounds of dimension of $I_t(M_{3,3}(f))$ for $3 \leq t \leq 6$ follow from Lemmas 4.8 and 4.9 below.

**4.4. Codimension of $I_t(M_{d,d}(f))$.** Let $f = \sum b_\alpha x^\alpha$, $|\alpha| = 2d = n$. The dimension of $k[B]$ equals $\binom{n+2}{2}$. The size of the matrix $M_{d,d}(f)$ is $\binom{d+2}{2}$ by $\binom{d+2}{2}$. Let $t$ equal the rank of $M_{d,d}(f)$, determined by the vanishing of the minors of size $t+1$ of $M_{d,d}(f)$, and let $a$ equal $\binom{d+2}{2}-t$, the corank of $M_{d,d}(f)$.

**Conjecture 4.3.** *The codimension of $I_t(M_{d,d}(f))$ in $k[B]$ is $\binom{a+1}{2}$ if $t \geq \binom{d+1}{2}$, or $\binom{d+2}{2} - 3t$ if $t \leq \binom{d+1}{2}$.*

**Remark.** This pattern is numerically consistent. Suppose the sequence of codimensions of $I_t(M_{d,d}(f))$ is $0,1,3,6,10,\ldots$ for $a = 0,1,2,\ldots$, and persists to the $j$th term. We find that the equation $\binom{n+2}{2} = \binom{j+1}{2} + 3\left(\binom{d+2}{2} - j\right)$ has solution $j = d+1$ for each $d$.

If $X = \{x_{ij}\}$ is a symmetric $\binom{d+2}{2}$ by $\binom{d+2}{2}$ matrix of indeterminants in $S = k[x_{ij}]$, then $I_t(X)$ will have codimension $\binom{d+2}{2} - \binom{t+1}{2}$ in $S$. The resolution structure for ideals $I_t(X)$ is given in [**JPW**]. We obtain $I_t(M_{d,d}(f))$ by a change of rings, defining $\phi : S \to R$ by $\phi(x_{ij}) =$ the $(i,j)$ entry of $M_{d,d}(f)$. Since the ideals $I_t(X)$ are generically perfect [**EN**], if $\mathbb{F}$ is a resolution of

$I_t(X)$, then $\mathbb{F} \otimes_S R$ will be a resolution of $I_t(M_{d,d}(f))$. It follows from [**EN**] that the codimension of $I_t(M_{d,d}(f))$ in $k[B]$ is less than or equal to the codimension of $I_t(X)$ in $S$. Conjecture 4.3 says that this is an equality for $\binom{d+1}{2} \leq t \leq \binom{d+2}{2}$, or equivalently, for $0 \leq a \leq d+1$.

To verify Conjecture 4.3 for large values of $t$, we will use Theorem 2.1 to determine an upper bound for the dimension of the family of algebras having a given Hilbert function. In part 2 of the proof of the theorem, Buchsbaum and Eisenbud show that if a Gorenstein ideal $I$ has a resolution

$$\mathbb{F} : 0 \longrightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} R,$$

then $f_2$ may be chosen to be an alternating map for any appropriate map $f_1$, so that all sets of generators for $I$ occur as pfaffians of an alternating matrix.

In the discussion that follows we add 1 to the computation of the projective dimension of $\mathbb{A}^{h(E_M)}$ in order to compare it with the affine count for the dimension of $Gor_D$.

Let $f$ be a degree $n = 2d$ homogeneous polynomial in $x, y$ and $z$. Let $I_t(M_{d,d}(f))$be the ideal of size $t+1$ minors of $M_{d,d}(f)$, imposing the condition that the rank of $M_{d,d}(f)$ is less than or equal to $t = \binom{d+2}{2} - a$. If we restrict $a$ so that $a \leq d+1$, then we determine the Hilbert function

$$T = 1, 3, 6, 10, \ldots, t, \ldots, 10, 6, 3, 1, \tag{4.1}$$

where all the matrices $M_{r,n-r}(f)$ have maximal rank except $M_{d,d}(f)$. The third difference sequence of $T$ is

$$1, 0, 0, \ldots, 0, -a, 3a-3-n, -3a+3+n, a, 0, \ldots, 0, 0, -1.$$

**Theorem 4.4.** *If $3a \leq n+3$, then there is an irreducible component $Gor(T)$ of $I_t(M_{d,d}(f))$ with codimension equal to $\binom{n+2}{2} - \binom{a+1}{2}$.*

*Proof.* We can break up the proof into 2 cases:
**Case 1.** $3a = n+3$. The sequence of third differences is

$$1, 0, 0, \ldots, 0, -a, 0, 0, -a, 0, \ldots, 0, 0, -1.$$

We get $a$ generators in degree $d$ and $a$ relations in degree $d+3$ in the ideal $J$ with the smallest possible number of generators defining a Gorenstein algebra $A$ with Hilbert function $T$. All entries in the $a$ by $a$ alternating matrix $M$ will have degree 3, meaning a choice of 10 coefficients possible for each, since we are choosing generic entries, so $h(E_M) = 10\binom{a}{2}$. For $J$, there are $a$ generators to choose, each of which is a linear combination of a fixed set of $a$ forms in $J$ of degree $d$, giving dimension $a^2$. We subtract this from the number of parameters we found for $M$, since each of these ideals determines the same algebra. Therefore, the maximum dimension of the family equals

$$10\binom{a}{2} + 1 - a^2 = 4a^2 - 5a + 1.$$

Since the codimension of the family cannot be larger than that for a generic symmetric matrix, this dimension is also a minimum, so it is equal to the dimension given in the conjecture.

**Case 2.** $3a < n+3$. This gives $a$ generators in degree $d$ and $-3a+3+n$ generators in degree $d+1$, $-3a+3+n$ relations in degree $d+2$ and $a$ relations in degree $d+3$. The alternating matrix $M$ will have this shape:

$$\begin{pmatrix} 0 & \text{Degree 3 entries} & \text{Degree 2 entries} \\ \text{Degree 3 entries} & 0 & \text{Degree 1 entries} \\ \text{Degree 2 entries} & \text{Degree 1 entries} & 0 \end{pmatrix}.$$

The maximum affine dimension of the family equals

$$\begin{aligned} 10\binom{a}{2} &+ 6a(-3a+3+n) + 3\binom{-3a+3+n}{2} + 1 \\ &\quad - (a^2 + (2n+3)(-3a+3+n)) \\ &= \left(\frac{a^2}{2} - 6da + 6d^2 - 19\frac{a}{2} + 15d + 10\right) \\ &\quad - (a^2 - 6ad + 12d + 4d^2 - 9a + 9) \\ &= -\frac{a^2}{2} + 2d^2 - \frac{a}{2} + 3d + 1, \end{aligned}$$

and this is equal to $\binom{n+2}{2} - \binom{a+1}{2}$ as claimed. □

**Remark.** This $Gor(T)$ is not in the closure of another, larger $Gor(T')$, because $T$ is maximal given $t$, and by Theorem 3.8.

**Remark.** When $3a > n+3$, the third differences of (4.1) show that an ideal with this Hilbert function needs a minimum of $a$ generators in degree $d$ and $3a-3-n$ generators in degree $3, 1$, and $-1$ in the following shape:

$$\begin{pmatrix} 0 & \text{Degree 3 entries} & \text{Degree 1 entries} \\ \text{Degree 3 entries} & 0 & \text{Degree -1 entries} \\ \text{Degree 1 entries} & \text{Degree -1 entries} & 0 \end{pmatrix}.$$

If we assume the degree $-1$ entries are all zeros, we get

$$10\binom{a}{2} + 3a(3a-3-n) - (a^2 + (4d+6)(3a-3-n))$$
$$= 13a^2 - 32a + 4da + 24d + 8d^2 + 19$$

for the maximum dimension of $Gor_T$ and

$$2d^2 + 3d + 1 - \frac{a^2}{2} - \frac{a}{2}$$

for the conjectured dimension. The difference between these is

$$3\binom{3a-3-n}{2},$$

which is three times the number of -1's in the degree matrix. This suggests that the fiber $\pi^{-1}$ over an ideal is larger than we have accounted for. It remains an interesting problem to justify subtracting $3\binom{3a-3-n}{2}$ from our count of the dimension of $U_{E_M}$ by explaining this difference.

Since we have shown the result for $a \leq 2d/3+1$, and the only values of $a$ that are possible are $0 \leq a \leq d+1$, we have proven the conjecture true for roughly two thirds of the range where we expect codimension $I_t(M_{d,d}(f))$ to be the same as codimension $I_t(X)$.

*Example* 4.5. Let us look at the Hilbert function $T = (1,3,4,3,1)$. We get the sequence of third differences $1, 0, -2, -1, 1, 2, 0, -1$, indicating a smallest possible minimal resolution of 2 generators in degree 2, one generator in degree 3, one relation in degree 4, and two relations in degree 5. This translates to the following alternating matrix pattern:

$$\begin{pmatrix} - & 3 & 2 \\ 3 & - & 2 \\ 2 & 2 & - \end{pmatrix}.$$

When we count dimensions for the entries we get $1\cdot 10 + 2\cdot 6 + 1 = 22 + 1 = 23$. There are $2\cdot 2 + 1\cdot 7 = 11$ choice of generators for the ideal. Therefore the maximum dimension of the family is $23 - 11 = 12$. The conjecture predicts 12 as well, and this information is shown in Example 4.1.

One such matrix $M$ is

$$\begin{pmatrix} 0 & z^3 & y^2 \\ -z^3 & 0 & x^2 \\ -y^2 & -x^2 & 0 \end{pmatrix}.$$

It can be easily checked that the ideal of pfaffians $(z^3, y^2, x^2)$ determines the Hilbert function $(1,3,4,3,1)$.

**Remark.** We can also use this same method of counting to find the dimension of a variety in some cases when the Hilbert function of an algebra is determined by a rank condition on one of the nonsquare Catalecticant matrices. The result is that when a corank 1 condition is imposed on $M_{r,n-r}(f)$, the codimension of $I_t(M_{r,n-r}(f))$ for $t = \binom{r+2}{2} - 1$ is $\frac{1}{2}(n+3)(n-2r)+2$, the same as the codimension of $I_t(G)$ when $G$ is a generic $\binom{r+2}{2}$ by $\binom{n-r+2}{2}$ matrix. This is the case no matter how "nonsquare" $M_{r,n-r}(f)$ is.

**4.5. A lower bound for the dimension of $Gor_T$.** We can determine a lower bound on the dimension of $M_{d,d}(f)$ by looking at sums of powers of linear forms. Let $l_1, \dots, l_s$ be linear forms in the variables $x, y$ and $z$. We look at the tangent space of the image of the map $P : (k^3)^s$ to $k^N$

$$P : l_1, \dots, l_s \mapsto l_1^n + \cdots + l_s^n,$$

which we denote $\mathcal{T}_n(P)$.

*Example* 4.6. Take $l_1 = ax + by + cz$ and $l_2 = dx + ey + fz$, $n = 4$.

$$\begin{aligned} l_1^4 + l_2^4 = {} & (ax)^4 + 4(ax)^3 by + \cdots + (cz)^4 \\ & + (dx)^4 + 4(dx)^3 ey + \cdots + (fz)^4, \end{aligned}$$

so the points $(a, b, c)$ and $(d, e, f)$ get mapped to

$$(a^4, 4a^3b, 4a^3c, \dots, c^4) + (d^4, 4d^3e, 4d^3f, \dots, f^4),$$

a 15-dimensional space. If we let $(a', b', c')$ and $(d', e', f')$ be tangent vectors at the points $(a, b, c)$ and $(d, e, f)$, then $\mathcal{T}_4(P)$ is

$$\begin{aligned} & ((a+a')^4, 4(a+a')^3(b+b'), \dots, (c+c')^4) \\ & \quad - (a^4, 4a^3b, \dots, c^4) \\ & + ((d+d')^4, 4(d+d')^3(e+e'), \dots, (f+f')^4) \\ & \quad - (d^4, 4a^3e, \dots, d^4). \end{aligned}$$

If we choose $a', \dots, f'$ small, then the quadratic terms and those of higher degree are approximately zero, so we only need to look at the linear terms in $a', \dots, f'$.

The dimension of $\mathcal{T}_4(P)$ will be equal to the rank of the following 15 by 6 matrix:

$$\begin{pmatrix} 4a^3 & 0 & 0 & 4d^3 & 0 & 0 \\ 12a^2b & 4a^3 & 0 & 12d^2e & 4d^3 & 0 \\ 12a^2c & 0 & 4a^3 & 12d^2f & 0 & 4d^3 \\ 12ab^2 & 12a^2b & 0 & 12de^2 & 12d^2e & 0 \\ 24abc & 12a^2c & 12a^2b & 24def & 12d^2f & 12d^2e \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 4c^3 & 0 & 0 & 4f^3 \end{pmatrix}.$$

This matrix has rank 6, computed by *Macaulay.*

When we add more linear forms we add columns to this matrix. Let $l_3 = gx + hy + zi$, $l_4 = jx + ky + lz$, $l_5 = mx + ny + oz$. The dimension of $\mathcal{T}_4(P)$ is equal to the rank of the 15 by 15 matrix

$$\begin{pmatrix} 4a^3 & 0 & 0 & \dots & 4m^3 & 0 & 0 \\ 12a^2b & 4a^3 & 0 & \dots & 12m^2n & 4m^3 & 0 \\ 12a^2c & 0 & 4a^3 & \dots & 12m^2o & 0 & 4m^3 \\ 12ab^2 & 12a^2b & 0 & \dots & 12mn^2 & 12m^2n & 0 \\ 24abc & 12a^2c & 12a^2b & \dots & 24mno & 12m^2o & 12m^2n \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 4c^3 & \dots & 0 & 0 & 4o^3 \end{pmatrix}$$

which has rank 14, not 15 as we might expect. This information is contained in the table in Example 4.1, and is a classical result by Sylvester [**El**, pp. 293-295].

We would like to be able to say that when $n > 4$ we can find $r$ linear forms so that the dimension of $\mathcal{T}_n(P)$ is $3r$, providing $3r \leq \binom{n+2}{2}$. Since the dimension of the tangent space is given by the condition that a certain matrix has maximal rank, which is an open condition, we will be assured of being able to find $r$ linear forms whenever the matrix has maximal rank.

**Conjecture 4.7.** *When $n > 4$, there exist $s = \left\lfloor \frac{\binom{n+2}{2}}{3} \right\rfloor$ linear forms $l_1, \dots, l_s$ in $x, y, z$ which the map $P$ injective (that is, for which the tangent map $T_n(P)$ has rank $3s$).*

**Lemma 4.8.** *If $l_1, \dots, l_s \mapsto l_1^n + \dots + l_s^n$ is injective, then $l_1, \dots, l_s \mapsto l_1^m + \dots + l_s^m$ is injective for $m \geq n$.*

*Proof.* Assume $l_1, \dots, l_s \mapsto l_1^m + \dots + l_s^m$ is not injective; then there exist coefficients $c_1, \dots, c_s$ in $R$ not all equal to zero such that $\sum c_i l_i^{m-1} = 0$. If we differentiate this sum $m - n$ times, we will get a nontrivial linear relation $\sum c_i' l_i^{n-1} = 0$. But this gives a nontrivial linear relation among the $l_i^n$, which means the map $l_1, \dots, l_s \mapsto l_1^n + \dots + l_s^n$ is not injective, a contradiction. □

**Lemma 4.9.** *Conjecture* 4.7 *is true for* $n = 5$.

*Proof.* Choose linear forms $l_1 = x$, $l_2 = y$, $l_3 = z$, $l_4 = ax+y$, $l_5 = dx+z$, $l_6 = y + cz$, and $l_7 = x + y + z$, and look at the tangent space of the image of these forms under the map $P$ at the tangent vectors

$$(a_1, b_1, c_1),\ (a_2, b_2, c_2), \dots, (a_7, b_7, c_7).$$

The dimension of $\mathcal{T}_5(P)$ is the rank of this 21 by 21 matrix:

$$\begin{pmatrix} 5\,0\,0\ldots & 5d^4 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0\,5\,0\ldots & 0 & 5d^4 & 0 & 0 & 0 & 0 & 20 & 5 & 0 \\ 0\,0\,5\ldots & 0 & 0 & 5d^4 & 0 & 0 & 0 & 20 & 0 & 5 \\ \vdots\,\vdots\,\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\,0\,0\ldots & 0 & 0 & 0 & 0 & 20c^3 & 30c^2 & 0 & 20 & 30 \\ 0\,0\,0\ldots & 0 & 20 & 0 & 0 & 5c^4 & 20c^3 & 0 & 5 & 20 \\ 0\,0\,0\ldots & 0 & 0 & 5 & 0 & 0 & 5c^4 & 0 & 0 & 5 \end{pmatrix}$$

The determinant was computed to be nonzero using *Macaulay*. □

The following table shows the degree $n$, value of $\left\lfloor \frac{\binom{n+2}{2}}{3} \right\rfloor$, and number $s$ of forms such that the dimension of $\mathcal{T}_n(P) = 3s$. Unverified values are in parentheses.

**Table 4.10.**

| n | $\binom{n+2}{2}$ | $s = \left\lfloor \frac{\binom{n+2}{2}}{3} \right\rfloor$ |
|---|---|---|
| 1 | 3 | 1 |
| 2 | 6 | 1 |
| 3 | 10 | 3 |
| 4 | 15 | 4 |
| 5 | 21 | 7 |
| 6 | 28 | (9) |
| 7 | 36 | (12) |
| 8 | 45 | (15) |
| 9 | 55 | (18) |
| $\vdots$ | $\vdots$ | $\vdots$ |

Verifying Conjecture 4.7 for $n > 5$ requires finding $s$ linear forms so that the dimension of $\mathcal{T}_n(P)$ is $3s$. Once we have $s$ linear forms where the dimension of $T_n(P)$ is $3s$, by Lemma 4.8 those forms will still "spread out" to fill up dimension $3s$ when we increase $n$. Since computing determinants of large matrices is cumbersome, it would be nice to be able to choose the $s$ simplest linear forms and show that they give dimension $\mathcal{T}_n(P) = 3s$, but this is not always possible. For the proof of Lemma 4.9 we could not have chosen $l_7 = ex + fz$ instead of $l_7 = x + y + z$. The linear forms $x, y$ and $z$ will have nonzero coefficients on the 9 monomials $x^n, x^{n-1}y, x^{n-1}z, xy^{n-1}, y^n, y^{n-1}z, xz^{n-1}$, $yz^{n-1}$, and $z^n$. The image of $ax+y$ will be nonzero on monomials with power of $z$ at most one. The image of

$dx + z$ will have nonzero coefficients on $2n + 1$ monomials whose power of $y$ is at most one; the same for $ex + fz$. However, when $n = 5$ this means the 4 linear forms $x, z, ax + y$, and $dx + z$ are nonzero on the 11 monomials whose power of $y$ is at most one; therefore the dimension of $\mathcal{T}_5(P) \leq 20$ for this choice of 7 forms.

**4.6. Dimension of a family of complete intersections.** A height 3 Gorenstein ideal $I$ defines a complete intersection when it can be generated by 3 elements. For example, the Hilbert function

$$(1, 3, 6, 7, 6, 3, 1)$$

can be determined by 3 generators in degree 3.

If an ideal $I$ defines a complete intersection with the Hilbert function of $R/I$ equal to $T = (h_0, h_1, \dots, h_t, \dots)$, the dimension of $Gor_T = \sum e_i h_i$, where $e_i$ is the number of generators in degree $i$ in a minimal generating set for $I$ [**I2**]. We can also give the projective dimension of a complete intersection ideal strictly in terms of the generator degrees by using the pfaffian method of Section 4.4. There are several cases to consider when all generator degrees are less than all relation degrees:

**Case 1.** $q_1 \neq q_2 \neq q_3$. In degree $q_1$ the size of the remaining space is 1, in degree $q_2$ it is $1 + \binom{q_2 - q_1 + 2}{2}$, and in degree $q_3$ it is $1 + \binom{q_3 - q_1 + 2}{2} + \binom{q_3 - q_2 + 2}{2}$. The sum equals the number of ways of choosing the generators for $J$. By subtracting the sum from the number of choices for $M$, we find the dimension of the complete intersection to be

$$\frac{9q_1}{2} + \frac{3q_2}{2} - \frac{3q_3}{2} + q_1 q_2 + q_1 q_3 + q_2 q_3 - \frac{q_1^2}{2} - \frac{q_2^2}{2} - \frac{q_3^2}{2} - 3.$$

**Case 2.** $q_1 = q_2 \neq q_3$. The dimension of the complete intersection will be

$$6q_1 - \frac{3q_3}{2} + 2q_1 q_3 - \frac{q_3^2}{2} - 4.$$

**Case 3.** $q_1 \neq q_2 = q_3$. The dimension is

$$\frac{9q_1}{2} + 2q_1 q_2 - \frac{q_1^2}{2} - 4.$$

**Case 4.** $q_1 = q_2 = q_3$. The size of the remaining space is $\binom{q_1 + 2}{2} - 3$, so the dimension of the complete intersection is

$$\frac{q_1^2 + 3q_1 + 2}{2} - 9.$$

Note that the projective dimension of the complete intersection above is computed to be 21 using this formula, which agrees with the affine dimension 22 shown in Table 4.2.

When a relation occurs between the two generators of degrees $q_1$ and $q_2$ before the generator in degree $q_3$, the size of the remaining space in degree $q_3$ is independent of $q_3$, so it doesn't appear in the formula for the dimension.
**Case 5.** $q_1 \neq q_2, p_3 < q_3$. The dimension is

$$3q_1 + 2q_1q_2 - 2.$$

**Case 6.** $q_1 = q_2, p_3 < q_3$. We get

$$2q_1^2 + 3q_1 - 3.$$

**4.7. Dimensions of $Gor_{T(1,k,n)}, Gor_{T(2,k,n)}, Gor_{T(3,k,n)}$.**
Let $T(2, k, n)$ denote the symmetric Hilbert function with socle in degree $n$ which follows this pattern:

$$1, 3, 6, \dots, \binom{k+1}{2}, \binom{k+2}{2} - 2, \binom{k+3}{2} - 5, \dots,$$
$$\binom{k+3}{2} - 5, \binom{k+2}{2} - 2, \dots, 1.$$

An ideal determining this Hilbert function has 2 generators in degree $k$ with a relation in degree $k+1$, and no further generators until degree $\lceil \frac{n+1}{2} \rceil$. We assume $k < \lfloor \frac{n}{2} \rfloor$.

**Proposition 4.11.** *The projective dimension of*

$$Gor_{T(2,k,n)} = \binom{n+2}{2} - \binom{n-k+3}{2} + \frac{k^2+k+2}{2} = k(n+3) - (n+1).$$

*Proof.* Denote by $g_1$ and $g_2$ the two generators of $J$ that occur in degree $k$. Since they have a linear relation in degree $k+1$, they must share a common degree $k-1$ factor; denote this by $g$. Then we can express the generators as $g_1 = g \cdot l_1$ and $g_2 = g \cdot l_2$, where $l_1$ and $l_2$ span a 2-dimensional subspace $V$ of the vector space with basis $\langle x, y, z\rangle$. The number of parameters for $g_1$ and $g_2$ is counted by first choosing $g$ in dim $Grass\left(1, \binom{k+1}{2}\right)$ ways, then choosing $V$ in dim $Grass(1, 3) = 2$ ways, giving a total of $\frac{k^2+k+2}{2}$ parameters for the generators.

If we let $J$ stand for the ideal generated by $g_1$ and $g_2$, then the dimension of $J_n$ as a vector space equals $\binom{n-k+3}{2} - 1$, so that the dual form in degree $n$

must be chosen from the $\binom{n+2}{2} - \binom{n-k+3}{2} + 1$ forms in $\mathbb{P}(J_n)^{\perp}$. Therefore the number of parameters for a dual form $f$ up to nonzero multiple equals

$$\binom{n+2}{2} - \binom{n-k+3}{2}.$$

The projective dimension of the variety equals $\binom{n+2}{2} - \binom{n-k+3}{2} + \frac{k^2+k+2}{2} = k(n+3) - (n+1)$. □

*Example* 4.12. Consider the variety $Gor_{T(2,3,8)}$ of Gorenstein algebras having Hilbert function

$$T(2,3,8) = (1,3,6,8,10,8,6,3,1).$$

This can be determined by requiring $M_{3,5}(f)$ to have rank 8 and making no additional conditions on $M_{4,4}(f)$, allowing it to have the largest rank possible. The projective dimension of $I_9(M_{3,5}(f))$, the ideal of 9 by 9 minors of $M_{3,5}(f)$, is therefore $3 \cdot (8+3) - (8+1) = 24$.

In the same way we define $Gor_{T(3,k,n)}$ to be the variety of Gorenstein algebras where ideals determining $Gor_{T(3,k,n)}$ have 3 generators in degree $k$, 2 relations in degree $k+1$ and no further generators until degree $\lceil \frac{n+1}{2} \rceil$. We define $Gor_{T(1,k,n)}$ to be the variety of algebras whose ideals have one generator in degree $k$ and no further generators until degree $\lceil \frac{n+1}{2} \rceil$.

**Proposition 4.13.** *The projective dimension of $Gor_{T(3,k,n)}$ is $k(n+3) - (2n-2)$. The projective dimension of $Gor_{T(1,k,n)}$ is $k(n+3) - 1$.*

*Proof.* We get these formulas by following the same arguments as in Proposition 4.11. □

The dimension of $Gor_{T(1,k,n)}$ is the same as the dimension of the generic and Catalecticant matrices of size $\binom{k+2}{2}$ by $\binom{n-k+2}{2}$ with corank 1.

The following is a table of dimensions for Hilbert functions with socle in degree 6 and order 2.

**Table 4.14**

| $T$ | *projective dimension* | *comments* |
|---|---|---|
| (1, 3, 5, 7, 5, 3, 1) | 17 | corank 1 condition on 6 by 15 catalecticant matrix |
| (1, 3, 5, 6, 5, 3, 1) | 16 | complete intersection |
| (1, 3, 5, 5, 5, 3, 1) | 14 | sum of 5 powers of linear forms |
| (1, 3, 4, 5, 4, 3, 1) | 11 | T(2, 2, 6) |
| (1, 3, 4, 4, 4, 3, 1) | 11 | sum of 4 powers of linear forms; complete intersection |
| (1, 3, 3, 3, 3, 3, 1) | 8 | sum of 3 powers of linear forms; T(3, 2, 6) |

The above dimensions are for $Gor(T)$, not necessarily for the subset prametrizing ideals needing 5 generators. For example, the dimension of the 5-generator subsets $(1,3,4,4,4,3,1)$ and $(1,3,5,6,5,3,1)$ will be less than the dimension of $Gor(T)$. When fixing the rank of Catalecticant matrices does not uniquely specify $T$, then the determinantal variety defined by these ranks may be reducible.

*Example* 4.15. Let $n = 8$ and consider the determinantal variety $V$ associated to a dual form $f$ such that the rank of $M_{3,5}(f)$ equals 7. The valid Hilbert functions satisfying rank $M_{3,5}(f) = 7$ with socle in degree 8 are

$$T_1 = (1,3,6,7,8,7,6,3,1)$$
$$T_2 = (1,3,6,7,7,7,6,3,1)$$
$$T_3 = (1,3,5,7,7,7,5,3,1)$$
$$T_4 = (1,3,5,7,8,7,5,3,1)$$
$$T_5 = (1,3,5,7,9,7,5,3,1)$$

$V = \cup Gor_{T_i}$ and each $Gor_{T_i}$ is an irreducible subvariety of $V$. We cannot specialize $Gor_{T_1}$ to $Gor_{T_2}$, since the projective dimension of $Gor_{T_2}$ is 20, while the dimension of $Gor_{T_1}$ is 19. We also cannot specialize $Gor_{T_2}$ to $Gor_{T_1}$, since $Gor_{T_2}$ requires generators in degree 6 but $Gor_{T_1}$ does not. Therefore $\overline{Gor_{T_1}} \not\subset \overline{GorT_2}$ and $\overline{Gor_{T_2}} \not\subset \overline{GorT_1}$, so $V$ is not irreducible.

**Remark.** If we let $n$ vary and parametrize $V$ by a dual polynomial $f$ where the rank of $M_{2,n-2}(f) = 3$, we do get an irreducible variety, since we have fixed the Hilbert function to be $(1, 3, 3, \ldots, 3, 3, 1)$. In general, whenever the parametrization of $f$ is a rank condition on catalecticant which fixes all the other ranks, it fixes $T$. Since $I_t(M_{d,d}(f))$ fixes a Hilbert function, it is an irreducible ideal.

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Received February 10, 1993. The contents of this paper form part of the author's 1992 Ph.D. dissertation directed by Anthony Iarrobino at Northeastern University.

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# PACIFIC JOURNAL OF MATHEMATICS

Volume 172 No. 2 February 1996