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ALICE FIALOWSKI

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We compute the 0-, 1-, and 2-dimensional homology of the vector field Lie algebra L_2 with coefficients in the modules $\mathcal{F}_{\lambda,\mu}$, and conjecture that the higher dimensional homology for any λ and μ is zero. We completely compute the 0- and 1-dimensional homology with coefficients in the more complicated modules $F_{\lambda,\mu}$. We also give a conjecture on this homology in any dimension for generic λ and μ.

Introduction.

Let us consider the infinite dimensional Lie algebra W_1^{pol} of polynomial vector fields $f(x)d/dx$ on $\mathbb{C}$. It is a dense subalgebra of W_1, the Lie algebra of formal vector fields on $\mathbb{C}$. We will compute the homology of the polynomial Lie algebra, and will use the notation $W_1^{\text{pol}} = W_1$. The Lie algebra W_1 has an additive algebraic basis consisting of the vector fields $e_k = x^{k+1}d/dx$, $k \geq -1$, in which the bracket is described by

$$[e_k, e_l] = (l-k)e_{k+l}.$$

Consider the subalgebras L_k, $k \geq 0$ of W_1, consisting of the fields such that they and their first k derivatives vanish at the origin. The Lie algebra L_k is generated by the basis elements $\{e_k, e_{k+1}, \dots\}$. The algebras W_1 and L_k are naturally graded by $\deg e_i = i$. Obviously the infinite dimensional subalgebras L_k of W_1 are nilpotent for $k \geq 1$.

The cohomology rings $H^*(L_k)$, $k \geq 0$ with trivial coefficients are known, there exist several different methods for the computation (see [**G, GFF, FF2, FR, V**]). The result is the following:

$$\dim H^q(L_k) = \binom{q+k-1}{k-1} + \binom{q+k-2}{k-2} \quad for \quad k \geq 1.$$

Not much is known about the cohomology with nontrivial coefficients for the Lie algebra L_k, $k > 1$. Among the known results, we mention the results on L_k, $k \geq 1$ on the cohomology $H^*(L_k; L_s)$ with any $s \geq 1$, see [**F**], and on L_k, $k \leq 3$ on the cohomology with coefficients in highest weight modules over the Virasoro algebra, see [**FF2**] and [**FF3**].

Let F_λ denote the W_1-module of the tensor fields of the form $f(z)dz^{-\lambda}$, where $f(z)$ is a polynomial in z and λ is a complex number; the action of W_1 on F_λ is given by the formula

$$(gd/dx)fdx^{-\lambda} = (gf' - \lambda fg')dx^{-\lambda}.$$

The module F_λ has an additive basis $\{f_j;\ j = 0, 1, \dots\}$ where $f_j = x^j dx^{-\lambda}$ and the action on the basis elements is

$$e_i f_j = (j - (i+1)\lambda) f_{i+j}.$$

Denote by $\mathcal{F}_\lambda$ the W_1-module which is defined in the same way, except that the index j runs over all integers. The W_1-modules F_λ with $\lambda \neq 0$ are irreducible, but as L_0-modules, they are reducible. For getting an L_0-submodule of F_λ, it is enough to take its subspace, generated by f_j, $j \geq \mu$, where μ is a positive integer. Denote the obtained L_0-module by $F_{\lambda,\mu}$.

More general, let us define the L_0-module $F_{\lambda,\mu}$ for arbitrary complex number μ, as the space, generated – like F_λ – by the elements f_j, $j = 0, 1, \dots$, on which L_0 acts by

$$e_i f_j = (j + \mu - (i+1)\lambda) f_{i+j}.$$

Finally define the modules $\mathcal{F}_{\lambda,\mu}$ over W_1 as $F_{\lambda,\mu}$ above, without requiring the positivity of j.

The homology of the Lie algebra L_1 with coefficients in $\mathcal{F}_{\lambda,\mu}$ and $F_{\lambda,\mu}$ are computed in [**FF1**]. We consider everywhere homology rather than cohomology, but the calculations are more or less equivalent. In the case of $\mathcal{F}_{\lambda,\mu}$ one can use the equality

$$(\mathcal{F}_{\lambda,\mu})' = \mathcal{F}_{-1-\lambda,-\mu}$$

which implies that

$$H^q(L_k; \mathcal{F}_{\lambda,\mu})' = H_q(L_k; \mathcal{F}_{-1-\lambda,-\mu}).$$

In the case of $F_{\lambda,\mu}$ one can use the equality

$$(F_{\lambda,\mu})' = (\mathcal{F}_{-1-\lambda,-\mu}) / F_{-1-\lambda,-\mu}$$

(see [**FF1**] for details).

Let us recall the results of [**FF1**]. Set $e(t) = (3t^2 + t)/2$ and define the k-th parabola ($k = 0, 1, 2, \dots$) as a curve on the complex plane with the parametric equation

$$\lambda = e(t) - 1$$

$$m - k = e(t) + e(t+k) - 1.$$

For $k_1, k_2 \in \mathbb{Z}$ we set

$$P(k_1, k_2) = (e(k_1) - 1,\ e(k_1) + e(k_1) - 1)$$

and let $\mathbf{P} = \{P(k_1, k_2) :\ k_1, k_2 \in \mathbb{Z}\}$. For a point P of $\mathbf{P}$ let us introduce

$$k(P) = |k_2 - k_1|$$

and

$$K(P) = |k_1| + |k_2|.$$

If $P \in \mathbf{P}$, then $K(P) \geq k(P), K(P) = k(P) \mod 2$ and P lies in the $k(P)$-th parabola. For $k \neq 0$ all the points of the k-th parabola with integer coefficients belong to $\mathbf{P}$. On the 0-th parabola there is one point from $\mathbf{P}$ with $K = 0$, and two points with $K = 2$, two points with $K = 4$, and in general, two points with every even number K. For $k \geq 0$ on the k-th parabola lie $2k+2$ points from $\mathbf{P}$ with $K = k$ and four points with $K = k+2$, four with $k+4$, and in general, four with $K = k + 2i$.

Theorem [**FF1**, Theorem 4.1].

$$\dim H_q^{(m)}(L_1; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 2 & \textit{if} \quad (\lambda, \mu + m) \in \mathbf{P} \textit{ and } K(\lambda, \mu + m) < q \\ 1 & \textit{if} \quad (\lambda, \mu + m) \in \mathbf{P} \textit{ and } K(\lambda, \mu + m) = q \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary. *If λ is not of the form $e(k) - 1$ with $k \in \mathbb{Z}$ and if $\mu \in \mathbb{Z}$, then*

$$H_*(L_1; \mathcal{F}_{\lambda,\mu}) = 0.$$

The homology $H_q(L_1; F_{\lambda,\mu})$ is also computed in [**FF1**]. We will not formulate the result in details, only some important for us facts.

Theorem (Modification of Theorem 4.2, [**FF1**]).

1) *If (λ, μ) is a generic point so that $(\lambda, \mu + m)$ does not lie on any of the parabolas for any integer m, then*

$$H_*(L_1; F_{\lambda,\mu}) = H_*(L_2).$$

2) *If $(\lambda, \mu + j)$ lies on the parabola for some j, then $H_q(L_1; F_{\lambda,\mu})$ is bigger than $H_1(L_2)$ at least for some q.*

3) *In all cases*

$$H_q(L_2) = 2q + 1 \leq \dim H_q(L_1; F_{\lambda,\mu}) \leq 4q + 1$$

and the boundaries are reached.

The next problem is to compute homology of L_2 with coefficients in the modules $\mathcal{F}_{\lambda,\mu}$ and $F_{\lambda,\mu}$. That is the aim of this paper. The results are the following.

Theorem 1.

$$H_0^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} \mathbb{C} & \textit{if} \quad \lambda = -1, m + \mu = -1 \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 2.

$$\dim H_1^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 2 & \textit{if} \quad \lambda = m + \mu = -1 \\ 1 & \textit{if} \quad \lambda = -1, m + \mu = 1, 2, 3 \\ & \quad \textit{or}\ \lambda = 0\ \textit{and}\ m + \mu = 0 \\ & \quad \textit{or}\ \lambda = 1\ \textit{and}\ m + \mu = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

These results are analogous to the ones in [**FF1**] and one can expect that the picture will be similar for higher homology as well. With this in mind, the following result is a surprise.

Theorem 3.

$$\dim H_2^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 1 & \textit{if} \quad \lambda = -1, m + \mu = -1, 1, 2, 3 \\ & \quad \textit{or}\ \lambda = 0\ \textit{and}\ m + \mu = 0 \\ & \quad \textit{or}\ \lambda = 1\ \textit{and}\ m + \mu = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

That means that the singular values of the parameters for the two-dimensional homology are the same, as the ones for the one-dimensional homology, which is not the case for the homology of L_1. Moreover, some partial computational results make the following conjecture plausible.

Conjecture 1. $H_q(L_2; \mathcal{F}_{\lambda,\mu}) = 0$ *for every* λ, μ *for* $q > 2$.

Let us try to explain the behavior of this homology. The main difference of the L_2 case from the L_1 case is that $H_q(L_1; \mathcal{F}_{\lambda,\mu}) = 0$ for *generic* λ and μ, while $H_q(L_2; \mathcal{F}_{\lambda,\mu}) = 0$ for *all* λ and μ (if $q > 2$). This might have the following explanation. By the Shapiro Lemma (see [**CE**, Ch. XIII/4, Prop. 4.2]),

$$H_q(L_2; \mathcal{F}_{\lambda,\mu}) = H_q\left(L_1; \operatorname{Ind}_{L_2}^{L_1} \mathcal{F}_{\lambda,\mu}\right)$$

and $\operatorname{Ind}_{L_2}^{L_1} \mathcal{F}_{\lambda,\mu}$ may be regarded as a limit case of the tensor product of modules of the type $F_{\lambda',\mu'} \otimes \mathcal{F}_{\lambda,\mu}$. Namely, $\operatorname{Ind}_{L_2}^{L_1} \mathcal{F}_{\lambda,\mu} = F \otimes \mathcal{F}_{\lambda,\mu}$ where F is the L_1-module spanned by $g_j, j \geq 0$, with the L_1-action $e_1 g_j = g_{j+1}$, $e_i g_j = 0$ for $i > 1$; the isomorphism is defined by the formula

$$e_1^k f_j \to \sum_{m=0}^{k} \binom{k}{m} g_m \otimes e_1^{k-m} f_j$$

$\Big($on the left hand side $e_1^k f_j$ means the action of e_1 in $\operatorname{Ind}_{L_2}^{L_1} \mathcal{F}_{\lambda,\mu}$, on the right hand side $e_1^{k-m} f_j$ means the action of e_1 in $\mathcal{F}_{\lambda,\mu}\Big)$. On the other hand, $F = \lim_{\lambda\to\infty} F_{\lambda,a\lambda}$ for any $a \neq 2$: put

$$g_j(\lambda) = (a-2)\lambda((a-2)\lambda+1)\dots((a-2)\lambda+j-1)f_j \in F_{\lambda,a\lambda};$$

then

$$e_i g_j(\lambda) = \frac{((a-i-1)\lambda+j)g_{i+j}(\lambda)}{((a-2)\lambda+j)\dots((a-2)\lambda+j+i-1)}$$

which tends to the action of L_1 in F when $\lambda \to \infty$.

Perhaps the homology

$$H_q(L_1; F_{\lambda',\mu'} \otimes \mathcal{F}_{\lambda,\mu})$$

depending not on two but on four parameters, has singular values for some $\lambda, \mu, \lambda', \mu'$ for each q. The problem of computing the cohomology $H_q(L_2; \mathcal{F}_{\lambda,\mu})$ is the two-parameter limit version of the previous problem, and it is not surprising that the singular solutions of the first problem have effect on the second problem only for small q values.

Our calculation yields also some results for $H_*(L_2; F_{\lambda,\mu})$. We will formulate them in Section 3, Theorem 4 and 5.

From Theorem 4 it follows that for generic λ, μ,

$$\dim H_0(L_2; F_{\lambda,\mu}) = 2,$$

and for singular values of λ, μ, $\dim H_0(L_2; F_{\lambda,\mu}) > 2$.

From Theorem 5 it follows that for generic λ, μ,

$$\dim H_1(L_2; F_{\lambda,\mu}) = 8,$$

and for singular values of λ, μ, $\dim H_1(L_2; F_{\lambda,\mu}) > 8$.

Conjecture 2. *For generic* λ, μ,

$$\dim H_q(L_2; F_{\lambda,\mu}) = 2(q+1)^2$$

or in more details,

$$H_q^{(m)}(L_2; F_{\lambda,\mu}) \simeq H_q^{(m)}(L_3) \oplus H_q^{(m-1)}(L_3).$$

This conjecture is motivated by the following observation. By the Shapiro Lemma,

$$H_q^{(m)}(L_3) = H_q^{(m)}\left(L_2; \operatorname{Ind}_{L_3}^{L_2} \mathbb{C}\right).$$

The module $\operatorname{Ind}_{L_3}^{L_2} \mathbb{C}$ is spanned by h_j $(j \geq 0)$ with L_2-action $e_2 h_j = h_{j+1}$, $e_i h_j = 0$ for $i > 2$; the grading in this module is $\deg h_j = 2j$. Hence

$$H_q^{(m)}(L_3) = H_q^{(m)}\left(L_2; \operatorname{Ind}_{L_3}^{L_2} \mathbb{C} + \Sigma \operatorname{Ind}_{L_3}^{L_2} \mathbb{C}\right)$$

where Σ stands for the shift of grading by one. On other words,

$$H_q^{(m)}(L_3) \oplus H_q^{(m-1)}(L_3) = H_q^{(m)}(L_2; F)$$

where F is spanned by g_j, $j \geq 0$, with the L_2-action $e_2 g_j = g_{j+2}$, $e_i g_j = 0$ for $i > 2$. As above, $F = \lim_{\lambda \to \infty} F_{\lambda, a\lambda}$ (now $a \neq 3$), which suggests that

$$H_q^{(m)}(L_2; F) = H_q^{(m)}(L_2; F_{\lambda,\mu})$$

for generic λ, μ.

Similarly one can expect that for generic λ, μ

$$H_q^{(m)}(L_k; F_{\lambda,\mu}) = H_q^{(m)}(L_{k+1}) \oplus H_q^{(m-1)}(L_{k+1}) \oplus \cdots \oplus H_q^{(m-k+1)}(L_{k+1}).$$

Remark, that if it is true that generically $H_q(L_2; \mathcal{F}_{\lambda,\mu}) = 0$ then generically

$$H^q(L_2; \mathcal{F}_{\lambda,\mu}) = H_{q-1}(L_2; F_{-1-\lambda,-\mu})$$

$\Big(H^q(L_2; \mathcal{F}_{\lambda,\mu}) = H_q(L_2; F'_{\lambda,\mu}) = H_q(L_2; \mathcal{F}_{-1-\lambda,-\mu}/F_{-1-\lambda,-\mu})$, and the homology exact sequence associated with the short coefficient exact sequence

$$0 \to F_{-1-\lambda,-\mu} \to \mathcal{F}_{-1-\lambda,-\mu} \to \mathcal{F}_{-1-\lambda,-\mu}/F_{-1-\lambda,-\mu} \to 0$$

provides the above isomorphism$\Big)$. In particular, if the L_2-module $L'_2 = F_{-2,-3}$ is "generic", then Conjecture 2 implies

$$\dim H^2(L_2; L_2) = \dim H_1(L_2; F_{-2,-3}) = 8.$$

Similarly for L_k we have the hypothetical result

$$H^2(L_k; L_k) = k(k+2).$$

The paper by Yu. Kochetkov and G. Post [**KP**] contains the announcement of the equality

$$\dim H^2(L_2; L_2) = 8,$$

as well as some further computations, including explicit formulas for 8 generating cocycles, which imply the description of infinitesimal deformations of the Lie algebra L_2.

I. Spectral sequence.

Let us compute the homology $H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$. Define a spectral sequence with respect to the filtration in the cochain complex $C_*^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$. The space $C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ is generated by the chains

$$e_{i_1} \wedge \ldots \wedge e_{i_q} \otimes f_j$$

where $2 \le i_1 < \ldots < i_q$, $j \in \mathbb{Z}$ and $i_1 + \ldots i_q + j = m$. Define the filtration by $i_1 + \ldots + i_q = p$. Denote by $F_p C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ the subspace of $C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$, generated by monomials of the above form with $i_1 + \ldots + i_q \le p$. Obviously, $\left\{F_p C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})\right\}_p$ is an increasing filtration in the chain complex. The differential acts by the rule

$$\begin{aligned} &d\left(e_{i_1} \wedge \ldots \wedge e_{i_q} \otimes f_j\right) \\ &\quad = d\left(e_{i_1} \wedge \ldots \wedge e_{i_q}\right) \otimes f_j - \sum_{s=1}^{q} (-1)^s e_{i_1} \wedge \ldots \hat{e}_{i_s} \wedge \ldots \wedge \ldots e_{i_q} \otimes e_{i_s} f_j. \end{aligned}$$

As m is fixed, the filtration in bounded.

Denote the spectral sequence, corresponding to this filtration by $E(\lambda, \mu, m)$. Then we have

$$E_0^p = C_*^{(p)}(L_2; \mathbb{C})$$

and d_0^p is the differential $\delta_p : C_*^{(p)}(L_2; \mathbb{C}) \to C_{*-1}(L_2; \mathbb{C})$. The first term of the spectral sequence is

$$E_1^p = H_*^{(p)}(L_2; \mathbb{C}).$$

The homology of L_2 with trivial coefficients is known (see [**G**]):

$$H_q^{(p)}(L_2) = \begin{cases} \mathbb{C} & \text{if } \frac{3q^2+q}{2} \le p \le \frac{3(q+1)^2-(q+1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the E_1 term of our spectral sequence looks as follows:

$H_0^{(0)}$		$H_1^{(2)}$	$H_1^{(3)}$	$H_1^{(4)}$			$H_2^{(7)}$	$H_2^{(8)}$	$H_2^{(9)}$	$H_2^{(10)}$	$H_2^{(11)}$				$H_3^{(15)}$	$H_3^{(16)}$	...

where all the spaces $H_q^{(p)}$ shown in this diagram are one dimensional.

The spaces E_1^p do not depend on λ and μ, but the differentials of the spectral sequence do. Let us introduce the notation

$$e_q^{\pm} = \frac{3q^2 \pm q}{2}.$$

The differentials

$$d_{p-r}^p : E_{p-r}^p \to E_{p-r}^r \quad \left(e_q^+ \leq p < e_{q+1}^-,\ e_{q-1}^+ \leq r < e_q^-\right)$$

form a partial multi-valued mapping $\tilde{\delta}_q \ : \ H_q(L_2) \to H_{q-1}(L_2)$. We shall define a usual linear operator $\delta_q \ : \ H_q(L_2) \to H_{q-1}(L_2)$ such that (1) if $\tilde{\delta}_q(\alpha)$ is defined for some $\alpha \in H_q(L_2)$ then $\delta_q(\alpha) \in \tilde{\delta}_q(\alpha)$; (2) $\delta_{q-1} \circ \delta_q = 0$. (Certainly, the mapping δ_q will depend on λ, μ, m.) Then the limit term of the spectral sequence $E(\lambda, \mu, m)$, that is $H_*^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ will coincide with the homology of the complex

$$H_0(L_2) \xleftarrow{\delta_1} H_1(L_2) \xleftarrow{\delta_2} H_2(L_2) \xleftarrow{\delta_3} \dots \quad .$$

To define $\delta_1, \delta_2, \dots$ we fix for any q and any p, $E_q^+ \leq p < e_{q+1}^-$, a cycle $c_q^p \in C_q^{(p)}(L_2)$ which represents the generator of $H_q^{(p)}(L_2)$.

It is evident that for each c_q^p there exist chains

$$\begin{aligned} b_q^{p-u} &\in C_q^{(p-u)}(L_2), \quad && u \geq 1 \\ g_{q-1}^v &\in C_{q-1}^{(v)}(L_2), \quad && v < e_{q-1}^+ \end{aligned}$$

such that

$$d\left(c_q^p \otimes f_{m-p} - \sum_{u \geq 1} b_q^{p-u} \otimes f_{m-p+u}\right) = \sum_{r=e_{q-1}^+}^{e_q^- - 1} \alpha_{p,r} c_{q-1}^r \otimes f_{m-r} + \sum_{v < e_{q-1}^+} g_{q-1}^v \otimes f_{m-v}$$

where $\alpha_{p,r}$ are complex numbers depending on λ, μ, m. These numbers compose the matrix of some linear mapping $H_q(L_2) \to H_{q-1}(L_2)$, and this mapping is our δ_q.

The chains $b_q^{p,u}$ and g_{q-1}^v may be chosen in the following way. Since $dc_q^p = 0$, the differential $d\left(c_q^p \otimes f_{m-p}\right)$ has the form $\sum_{w<p} h_{q-1}^w \otimes f_{m-w}$ with $h_{q-1}^w \in C_{q-1}^{(w)}(L_2)$. Here the leading term h_{q-1}^{p-1} is a cycle, $dh_{q-1}^{p-1} = 0$. Since $H_{q-1}^{p-1}(L_2) = 0$, we have $h_{q-1}^{p-1} = db_q^{p-1}$ with $b_q^{p-1} \in C_q^{(p-1)}(L_2)$. Now, the leading term of $d\left(c_q^p \otimes f_{m-p} - b_q^{p-1} \otimes f_{m-p+1}\right)$. belongs to $C_{q-1}^{(p-1)}(L_2)$ and it is again a cycle. We apply to it the same procedure and do it until the leading term of $d\left(c_q^p \otimes f_{m-p} - \sum b_q^{p-i} \otimes f_{m-p+i}\right)$ belongs to $C_{q-1}^{(e_q^- -1)}(L_2)$. This is still a cycle, but it is not necessarily a boundary, for $H_{q-1}^{e_q^- -1}(L_2) \neq 0$. Now we choose $b_q^{e_q^- -1} \in C_q^{\left(e_q^- -1\right)}(L_2)$ such that $db_q^{e_q^- -1}$ is our leading term up to some multiple of $c_{q-1}^{e_q^- -1}$. Then we do the same for $C_{q-1}^{\left(e_q^- -2\right)}(L_2)$, and so on until we reach $C_{q-1}^{e_{q-1}^+ -1}(L_2)$.

The matrix $|\alpha_{p,r}|$ depends on the choice of the cycles c_q^p. It depends also on the particular choice of the chains b_q^{p-u}, but only up to a triangular transformation. In particular, the kernels and the images of the mappings δ_q, and hence the homology $\operatorname{Ker}\delta_q/\operatorname{Im}\delta_{q+1}$, are determined by the cycles c_q^p.

Remark that $\dim H_q(L_2) = 2q+1$ and hence the matrix of δ_q is a $(2q-1)\times(2q+1)$-matrix depending on λ, μ, m. We get

$$(*) \qquad \dim H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = 2q + 1 - \operatorname{rank}\delta_q - \operatorname{rank}\delta_{q-1}.$$

II. Computations of $H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$.

1. The space $H_0^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$.

As the action of W_1 on $\mathcal{F}_{\lambda,\mu}$ is

$$e_i \otimes f_j \to [j + \mu - \lambda(i+1)] f_{i+j}$$

and the nontrivial cycles of $H_1(L_2)$ are $c_1^2 = e_2$, $c_1^3 = e_3$, $c_1^4 = e_4$, the differentials are the following:

$$e_2 \otimes f_{m-2} \to (m - 2 + \mu - 3\lambda) f_m,$$
$$e_3 \otimes f_{m-3} \to (m - 3 + \mu - 4\lambda) f_m,$$
$$e_4 \otimes f_{m-4} \to (m - 4 + \mu - 5\lambda) f_m.$$

The coefficients in the right hand sides depend on λ and $m + \mu$, which is natural, because the whole complex $C_*^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ depends only on λ and $m+\mu$. On the other hand, there is an isomorphism $\mathcal{F}_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu+1}$, $f_j \to f_{j+1}$ with the shift of grading by 1. Therefore we may put $m = 0$ and the differential matrix $\delta_1 : H_1(L_2) \to H_0(L_2)$ has the form

$$(\mu - 2 - 3\lambda \mid \mu - 3 - 4\lambda \mid \mu - 4 - 5\lambda).$$

The rank of the matrix is 0 if $\lambda = m = -1$ and 1 in all the other cases. From this it follows

Theorem 1.

$$\dim H_0^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 1 & \textit{if } \lambda = -1, m + \mu = -1 \\ 0 & \textit{otherwise.} \end{cases}$$

2. The space $H_1^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$.

The nontrivial cycles of $C_2(L_2; \mathbb{C})$ are

$$\begin{aligned} c_2^7 &= e_2 \wedge e_5 - 3e_3 \wedge e_4 \\ c_2^8 &= e_2 \wedge e_6 - 2e_3 \wedge e_5 \\ c_2^9 &= 3e_2 \wedge e_7 - 5e_3 \wedge e_6 \\ c_2^{10} &= e_2 \wedge e_8 - 3e_4 \wedge e_6 \\ c_2^{11} &= 5e_2 \wedge e_9 - 7e_3 \wedge e_8 \end{aligned}$$

of weight $7, 8, 9, 10, 11$.

Let us put $\mu - k\lambda - 1 = A(k, 1)$. Direct calculation shows that

$$\begin{aligned} d\,((e_2 \wedge e_5 - 3e_3 \wedge e_4) \otimes f_{-7} - A(3,7) e_2 \wedge e_3 \otimes f_{-5}) \\ = -3A(4,7) e_4 \otimes f_{-4} \\ + [3A(5,7) - A(3,7)A(3,5)] e_3 \otimes f_{-3} \\ + [-A(6,7) + A(3,7)A(4,5)] e_2 \otimes f_{-2}, \end{aligned}$$

hence

$$\begin{aligned} \delta_2\,(c_2^7) = [-A(6,7) + A(3,7)A(4,5)] c_1 1^2 \\ + [3A(5,7) - A(3,7)A(3,5)] c_1^3 - 3A(4,7) c_1^4. \end{aligned}$$

Thus we have

$$\begin{aligned} \alpha_{7,2} &= -A(6,7) + A(3,7)A(4,5) \\ \alpha_{7,3} &= 3A(5,7) - A(3,7)A(3,5) \\ \alpha_{7,4} &= -3A(4,7). \end{aligned}$$

In the same way we calculate $\alpha_{p,r}$ for $p = 8, 9, 10, 11$ and $r = 2, 3, 4$. We get the following 5×3-matrix:

$A(3,7)A(4,5)$ $-A(6,7)$	$-A(3,7)A(3,5)$ $+3A(5,7)$	$-3A(4,7)$
$1/2A(3,8)A(5,6)$ $-2A(4,8)A(4,5)$ $-A(7,8)$	$2A(4,8)A(3,5)$ $+2A(6,8)$	$-1/2A(3,8)A(3,6)$
$-5/2A(4,9)A(5,6)$ $-3A(8,9)$	$3A(3,9)A(5,7)$ $+5A(7,9)$	$-3A(3,9)A(4,7)$ $+5/2A(4,9)A(3,6)$
$-1/2A(3,10)A(4,8)A(4,5)$ $-3/2A(5,10)A(5,6)$ $-A(9,10)$	$1/2A(3,10)A(4,8)A(3,5)$ $+1/2A(3,10)A(6,8)$	$3/2A(5,10)A(3,6)$ $+3A(7,10)$
$7/2A(4,11)A(4,8)A(4,5)$ $+A(3,11)A(8,9)$ $-5A(10,11)$	$-A(3,11)A(3,9)A(5,7)$ $-7/2A(4,11)A(4,8)A(3,5)$ $-7/2A(4,11)A(6,8)$ $+7A(9,11)$	$A(3,11)A(3,9)A(4,7)$

We have to compute the rank of the matrix (δ_2). It is clear that the rank can not be bigger than 2. Direct computation shows that $\mathrm{rk}(\delta_2) = 1$ if and only if $\lambda = -1$, $\mu = -1, 1, 2, 3$; $\lambda = \mu = 0$; $\lambda = \mu = 1$. From this, using formula $(*)$, it follows

Theorem 2.

$$\dim H_1^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 2 & \textit{if} \quad \lambda = m + \mu = -1 \\ 1 & \textit{if} \quad \lambda = -1, m + \mu = 1, 2, 3 \\ & \quad \textit{or } \lambda = 0 \textit{ and } m + \mu = 0 \\ & \quad \textit{or } \lambda = 1 \textit{ and } m + \mu = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

3. The spaces $H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ for $q \geq 2$.

The next differential δ_3 is a 5×7-matrix. Its rank can not be bigger than 3 for any λ and μ. On the other hand, computation shows that $\mathrm{rk}(\delta_3) = 3$ for every λ, μ; namely, the first three rows of the matrix are linearly independent for every λ, μ. From this it follows that the dimension of the space $H_2^{(m)}(L_2; F_{\lambda,\mu})$ drops only if the rank of the previous matrix (δ_2) does. This proves

Theorem 3.

$$\dim H_2^{(m)}(L_2;\mathcal{F}_{\lambda,\mu}) = \begin{cases} 1 & \textit{if} \quad \lambda=-1, m+\mu=-1,1,2,3 \\ & \quad \textit{or } \lambda = 0 \textit{ and } m+\mu=0 \\ & \quad \textit{or } \lambda = 1 \textit{ and } m+\mu=1 \\ 0 & \textit{otherwise.} \end{cases}$$

By this theorem, for generic λ, μ, $\dim H_2^{(m)}(L_2;\mathcal{F}_{\lambda,\mu}) = 0$.

It seems very likely that the next differential matrices (δ_k), $k \geq 4$, have the same rank for every λ and μ ($\mathrm{rk}(\delta_k) = q$) which would imply our

Conjecture 1. *$H_q(L_2;\mathcal{F}_{\lambda,\mu}) = 0$ for every λ, μ for $q > 2$.*

III. Computations of $H_q^{(m)}(L_2;\mathcal{F}_{\lambda,\mu})$.

Recall that the L_0-modules $F_{\lambda,\mu}$ differ from the W_1-modules $\mathcal{F}_{\lambda,\mu}$ only in requiring the non-negativity of j for the generators f_j. Consequently the spectral sequence is basically the same, only it is truncated as follows:

$$E_r^p(\lambda,\mu,m) = 0 \quad \text{if} \quad m - p < 0.$$

The space $C_q^{(m)}(L_2;F_{\lambda,\mu})$ is generated by the chains

$$e_{i_1} \wedge \ldots \wedge e_{i_q} \otimes f_j$$

with $2 \leq i_1 \leq \ldots \leq i_q$, $j \geq 0$ and $i_1 + \ldots + i_q = m$. This way, for computing homology, we have to compute the rank of truncated matrices, consisting of some of the upper rows of the previous matrices.

Let us compute the space $H_0(L_2;F_{\lambda,\mu})$. Obviously,

$$H_0^{(0)}(L_2;F_{\lambda,\mu}) = H_0^{(1)}(L_2;F_{\lambda,\mu}) = \mathbb{C}.$$

For $m = 2$ the differential is the following:

$$e_2 \otimes f_0 \to (\mu - 3\lambda) f_2$$

which shows that if $\mu = 3\lambda$, then $\dim H_0^{(2)} = 1$, otherwise $H_0^{(2)}(L_2;F_{\lambda,\mu}) = 0$.

For $m > 2$

$$\dim H_0^{(m)}(L_2;F_{\lambda,\mu}) = \begin{cases} 1 & \text{if } \lambda = -1 \text{ and } m+\mu = -1 \\ 0 & \text{otherwise.} \end{cases}$$

So we get

Theorem 4.

$$H_0^{(m)}(L_2; F_{\lambda,\mu}) = \begin{cases} \mathbb{C} & \text{if } m = 0, 1 \\ & \quad \text{or } m = 2 \text{ and } \mu = 3\lambda \\ & \quad \text{or } \lambda = -1 \text{ and } m + \mu = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary. *For generic* λ, μ $\quad H_0(L_2; F_{\lambda,\mu}) = 2$.

Direct computation proves the result for the space $H_1^{(m)}(L_2; F_{\lambda,\mu})$.

Theorem 5.

$$\dim H_1^{(2)}(L_2; F_{\lambda,\mu}) = \begin{cases} 1 & \text{if } \mu = 3\lambda \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(3)}(L_2; F_{\lambda,\mu}) = \begin{cases} 2 & \text{for } \lambda = -1, \mu = -4 \\ 1 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(4)}(L_2; F_{\lambda,\mu}) = \dim H_1^{(5)}(L_2; F_{\lambda,\mu}) = \dim H_1^{(6)}(L_2; F_{\lambda,\mu})$$
$$= \begin{cases} 3 & \text{for } \mu = -4, \lambda = -1 \\ 2 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(7)}(L_2; F_{\lambda,\mu}) = \begin{cases} 2 & \text{if } \mu = -8, \lambda = -1 \text{ or } \mu = 0, \lambda = 0 \\ 1 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(8)}(L_2; F_{\lambda,\mu}) = \begin{cases} 2 & \text{if } \mu = -9, \lambda = -1 \\ 1 & \text{for } \lambda \text{ and } \mu \text{ lying on the curve} \\ & -36\lambda + 147\lambda^2 - 27\lambda^3 + 8\mu - 72\lambda\mu + 27\lambda^2\mu \\ & +9\mu^2 - 9\lambda\mu^2 + \mu^3 = 0 \\ 0 & \text{otherwise;} \end{cases}$$

for $m > 8$, $\quad \dim H_1^{(m)}(L_2; F_{\lambda,\mu}) = \dim H_1^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ (*see Theorem* 2).

Corollary. *For generic* λ, μ, $\dim H_1(L_2; F_{\lambda,\mu}) = 8$.

Conjecture 2. *For generic* λ, μ,

$$\dim H_q(L_2; F_{\lambda,\mu}) = 2(q+1)^2,$$

or, in more details,

$$H_q^{(m)}(L_2; F_{\lambda,\mu}) \simeq H_q^{(m)}(L_3; \mathbb{C}) \otimes H_q^{(m-1)}(L_3; \mathbb{C}).$$

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University of California
Davis, CA 95616-8633
E-mail address: fialowsk@math.ucdavis.edu.

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