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STABLE RELATIONS II: CORONA SEMIPROJECTIVITY AND DIMENSION-DROP C^*-ALGEBRAS

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We prove that the relations in any presentation of the dimension-drop interval are stable, meaning there is a perturbation of all approximate representations into exact representations. The dimension-drop interval is the algebra of all M_n-valued continuous function on the interval that are zero at one end-point and scalar at the other. This has applications to mod-p K-theory, lifting problems and classification problems in C^*-algebras. For many applications, the perturbation must respect precise functorial conditions. To make this possible, we develop a matricial version of Kasparov's technical theorem.

1. Introduction.

Suppose $\mathcal{R}$ is a finite set of relations on a finite set G of generators so that $C^*\langle G|\mathcal{R}\rangle$ is isomorphic to the dimension-drop interval

$$\tilde{\mathbb{I}}_n = \{f \in C[0,1] \mid f(0), f(1) \in \mathbb{C}I\}.$$

For simplicity, we assume the relations are of the form $p(g_1, \dots, g_n) = 0$ for some $*$-polynomial p. *Weak stability* means that an approximate representation $(x_1, \dots, x_n)$, meaning an n-tuple of elements in a C^*-algebra A such that each $p(x_1, \dots, x_n)$ is close zero, can be perturbed slightly within A to an actual representation $(\tilde{x}_1, \dots, \tilde{x}_n)$. That this (and a little more) can be done was shown in [**8**], but only for one specific set of relations. The relations $\mathcal{R}$ are *stable* if the pertubation can be done so that whenever there is a $*$-homomorphism $\varphi : A \to B$ which sends $(x_1, \dots, x_n)$ to an exact representation, then $\varphi(\tilde{x}_j) = \varphi(x_j)$.

There are several advantages to stability over weak stability. It is far more useful when dealing with extensions of C^*-algebras and it depends only on the universal C^*-algebra, not the choice of relations for that C^*-algebra. The reason for our focus on the dimension-drop interval is primarily that this is the most complicated building block used in the inductive limits, called AD algebras, that appeared in Elliott's first classification paper [**7**].

See [**5**] for an application of stable relations to the extension problem for AD algebras. See [**4**] for a discussion of the role of the dimension-drop interval in mod-p K-theory. Our results will be stated in the more general context of dimension-drop graphs, but certainly the dimension-drop interval is the most important case.

In §2 we give a characterization, in terms of lifting properties, of the universal C^*-algebras for stable relations. Since this property, called semiprojectivity, depends only on the C^*-algebra, this frees us from having to specify generators and relations in many cases. We have a third, equivalent property involving corona algebras. This characterization formalizes some of the ideas used by Olsen and Pedersen [**11**] to show that nilpotents always lift.

For any C^*-algebra A we let $M(A)$ denote the multiplier algebra of A and $C(A)$ denote the corona algebra $M(A)/A$.

By a dimension-drop graph, we mean a C^*-algebra of the form

$$\{f \in C(X, M_n) \mid f(v) \in \mathbb{C}I \text{ for all vertices } v\}$$

where X is the underlying topological space for a graph and n is a positive integer. We call this a dimension-drop interval in the special case where X is the unit interval with 0 and 1 as vertices.

To handle these algebras we need several generalizations of Kasparov's Technical Theorem. The purpose of these results is to show that, inside of a corona algebra, one can find good substitutes for elements that would exist if only the corona algebra were a von Neumann algebra. For example, there is an acceptable substitute for the logarithm of a unitary with full spectrum. Also, if $M_n(A)$ sits inside the corona algebra, there are elements that function just like matrix units in the way they multiply against $M_n(A)$, even if A is not unital but only σ-unital.

These technical lemmas are very similar to the second splitting lemma in BDF [**3**, Lemma 7.3]. The basic form of these results is to show that every $\varphi : A \to C(E)$ factors through some injection $A \to A_1$. In the BDF case, A and A_1 are commutative and $C(E)$ is the Calkin algebra.

Once we have shown that a dimension-drop graph is universal for a stable set of relations, a host of perturbation, lifting and homotopy results follow regarding homomorphisms (and asymptotic morphisms) out of dimension-drop C^*-algebras. For most of these we refer the reader to [**8**] but we will mention one of these, [**8**, Theorem 3.8]. If a separable C^*-algebra A has the property that any finite set of its elements can be approximated by elements of a C^*-subalgebra isomorphic to a quotient of a dimension-drop graph, then A is the inductive limit of dimension-drop graphs.

A C^*-algebra that will figure prominently in all this the cone $\mathrm{C}M_n = M_n(C_0(0,1])$. By [**8**, Theorem 4.9] we know that $\mathrm{C}M_n$ is projective. This is a very useful fact as there are many copies of C M_n inside of a dimension-drop graph.

The author is grateful to Gert Pedersen for discussions which lead to much simplified proofs in Section four.

2. A characterization of stability.

We begin with a characterization of projectivity in terms of corona algebras that is suggested by [**11**]. This then generalizes to give a characterization of semiprojectivity and of stability for relations. One consequence is that two finite sets of relations that determine isomorphic universal C^*-algebras are either both stable, or both not.

All our definitions are with respect to the full category of not-necessarily-unital C^*-algebras and $*$-homomorphisms.

Definition 2.1. A C^*-algebra A is *projective* if, for every surjection $\pi : B \to C$ and every $*$-homomorphism $\varphi : A \to C$, there exists a $*$-homomorphism $\bar{\varphi} : A \to B$ such that $\pi \circ \bar{\varphi} = \varphi$. We call A *corona projective* if this holds only in the special case where $C = C(E)$ for some σ-unital C^*-algebra E.

Theorem 2.2. *Let A be a separable C^*-algebra. Then A is projective if and only if A is corona projective.*

Proof. The forward implication is trivial. Suppose that A is corona projective and that $\varphi : A \to C$ and a surjection $\pi : B \to C$ are given. Replacing B, if necessary, by the closed span of a lift of a dense sequence in $\varphi(A)$ reduces the problem to the case where B is separable.

Let $I = \ker(\pi)$ and let $I^\perp$ denote the annihilator of I in B. As $I \cap I^\perp = 0$ and $I + I^\perp$ is an essential ideal in B, we have the following commutative diagram with the left square a pull-back.

$$\begin{array}{ccccccc} & & B & \longrightarrow & B/I^\perp & \xrightarrow{\iota_1} & M(I+I^\perp)/I^\perp \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ A & \xrightarrow{\varphi} & B/I & \longrightarrow & B/(I+I^\perp) & \xrightarrow{\iota_2} & M(I+I^\perp)/(I+I^\perp) \end{array}$$

By the corona projectivity of A, we have

$$\psi : A \to M(I+I^\perp)/I^\perp$$

which is a lift of the composition of the bottom row:

We now claim that $\pi_2^{-1}(\mathrm{im}(\iota_2)) \subseteq \mathrm{im}(\iota_1)$. Suppose $b \in \pi_2^{-1}(\mathrm{im}(\iota_2))$. Thus $\pi_2(b) = \iota_2(c)$ for some c. But $c = \pi_1(a)$ for some a, so

$$\pi_2(\iota_1(a)) = \iota_2(\pi_1(a)) = \iota_2(c) = \pi_2(b).$$

This implies

$$\iota_1(a) - b \in \ker(\pi_2) = (I + I^\perp)/I^\perp \subseteq B/I^\perp = \mathrm{im}(\iota_1)$$

and hence $b \in \mathrm{im}(\iota_1)$.

By the claim, we may regard ψ as a map into $B/I^\perp$. The pull-back property now shows that φ and ψ together determine the desired lifting to B. □

Following Blackadar [**1**] we define semiprojectivity as a lifting property. This turns out to have better closure properties than the version of semiprojectivity due to Effros and Kaminker [**6**], which is better suited to some homotopy calculations.

Definition 2.3. A C^*-algebra A is called *semiprojective* if, for every $*$-homomorphism $\varphi : A \to B/\overline{\bigcup I_n}$, where the I_n are increasing ideals in B, and with $\pi_m : B/I_m \to B/\overline{\bigcup I_n}$ the natural quotient map, there exists, for some m, a $*$-homomorphism $\bar{\varphi} : A \to B/I_m$ such that $\pi_m \circ \bar{\varphi} = \varphi$. We call A *corona semiprojective* if this holds only in the special case where $B/\overline{\bigcup I_n} \cong C(E)$ for some σ-unital C^*-algebra E. □

Theorem 2.4. *Let A be a separable C^*-algebra. Then A is semiprojective if and only if A is corona semiprojective.*

Proof. The proof is similar to that of Theorem 2.2 except that one uses the following diagram, with $I = \overline{\bigcup I_n}$.

$$\begin{array}{ccccccc} & & B/I_n & \longrightarrow & B/(I_n + I^\perp) & \xrightarrow{\iota_1} & M(I + I^\perp)/I^\perp \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} \\ A & \xrightarrow{\varphi} & B/I & \longrightarrow & B/(I + I^\perp) & \xrightarrow{\iota_2} & M(I + I^\perp)/(I + I^\perp) \end{array}$$

Notice that $\overline{\bigcup I_n + I^\perp} = I + I^\perp$, so corona semiprojectivity applies, and the left square is still a pull-back since $I \cap (I_n + I^\perp) = I_n$. □

If A is unital, then it is easy to see that one need only check the corona semiprojectivity condition in the special case $\varphi(1) = 1$.

We now recall the definition of stability from [**8**]. We shall assume that $G = \{g_1, \ldots, g_l\}$ is a finite set of generators and $\mathcal{R} = \{p_1, \ldots, p_k\}$ is a finite set of $*$-polynomials with zero constant terms. By $C^*\langle G|\mathcal{R}\rangle$, we denote the universal (not-necessarily-unital) C^*-algebra generated by $g_1, \ldots, g_l$ subject to

$$\|g_j\| \leq 1 \quad \text{and} \quad p_i(g_1, \ldots, g_l) = 0.$$

By $C^*_\epsilon\langle G|\mathcal{R}\rangle$, we denote the universal unital C^*-algebra generated by $g_1, \dots, g_l$ subject to

$$\|g_j\| \leq 1+\epsilon \quad \text{and} \quad \|p_i(g_1, \dots, g_l)\| \leq \epsilon.$$

Sometimes, to be more explicit, we will denote the generators of $C^*_\epsilon\langle G|\mathcal{R}\rangle$ by $g_1^\epsilon, \dots, g_l^\epsilon$. We let P_ϵ denote the surjection

$$P_\epsilon : C^*_\epsilon\langle G|\mathcal{R}\rangle \to C^*\langle G|R\rangle$$

which sends g_j^ϵ to g_j.

If, for every $\eta > 0$, there exists $\epsilon > 0$ and a $*$-homomorphism

$$\sigma_\epsilon : C^*\langle G|\mathcal{R}\rangle \to C^*_\epsilon\langle G|\mathcal{R}\rangle$$

such that

$$\|\sigma_\epsilon(g_j) - g_j^\epsilon\| \leq \eta, \quad j = 1, \dots, l$$

and $P_\epsilon \circ \sigma_\epsilon = \mathrm{id}$, then R is *stable*.

Theorem 2.5. *For a finitely presented C^*-algebra $C^*\langle G|\mathcal{R}\rangle$, the following conditions are equivalent:*

(1) *$\mathcal{R}$ is stable.*

(2) *$C^*\langle G|R\rangle$ is semiprojective.*

(3) *$C^*\langle G|R\rangle$ is corona semiprojective.*

Proof. The implication (1) $\Rightarrow$ (2) follows from [**8**, Theorem 3.2] while (2) $\Leftrightarrow$ (3) is a special case of Theorem 2.4. For (2) $\Rightarrow$ (1), applying semiprojectivity to the identity map immediately gives a map $\bar{\sigma}_{\bar{\epsilon}} : C^*\langle G|\mathcal{R}\rangle \to C^*_{\bar{\epsilon}}\langle G|\mathcal{R}\rangle$ with $P_{\bar{\epsilon}} \circ \bar{\sigma}_{\bar{\epsilon}} = \mathrm{id}$. Let σ_ϵ equal the composition of $\bar{\sigma}_{\bar{\epsilon}}$ with the natural surjection of $C^*_{\bar{\epsilon}}\langle G|\mathcal{R}\rangle$ onto $C^*_\epsilon\langle G|\mathcal{R}\rangle$ for ϵ sufficiently small, $0 < \epsilon < \bar{\epsilon}$. □

3. Generalizations of Kasparov's Technical Theorem.

Using the techniques of [**8**] and [**11**] we derive several generalizations of Kasparov's Technical Theorem (KTT). Our goal is to find the closest possible thing to matrix units inside a corona algebra for C^*-subalgebras of the form $A \otimes F$ where A is σ-unital and F is finite-dimensional.

All our theorems involve a subset D with which these ersatz matrix units are to commute. Easier proofs exist if one ignores D and sticks with the separable case. Indeed, one may use the projectivity of $\mathrm{C}M_n$, or $\bigoplus C_0(0,1]$, and [**12**, Proposition 3.12.1] along the lines of an observation of Cuntz described in [**2**, §12.4]. We will discuss this further in recent joint work with Gert Pedersen [**10**].

In this section, E will always denote a σ-unital C^*-algebra and $C(E)$ its corona algebra.

Theorem 3.1. *Suppose $A_1, \ldots, A_n$ are σ-unital C^*-subalgebras of $C(E)$. Let D be a separable, unital C^*-subalgebra of $C(E)$ such that*

$$A_j D A_k = 0, \quad j \neq k.$$

There exist $g_1, \ldots, g_n$ in $C(E) \cap D'$ such that

$$0 \leq g_j \leq 1, \quad j = 1, \ldots, n,$$

$$g_j g_k = 0, \quad j \neq k,$$

$$g_j a = a g_j = a, \quad \forall a \in A.$$

Proof. For $n = 2$ this is equivalent to KTT. Indeed, it is very close to the equivalent result [**11**, Theorem 3.7]. An induction argument gives the general case. □

Notice that $A_1 A_2 = 0$ implies that the C^*-algebra generated by $A_1 \cup A_2$ is isomorphic to $A_1 \oplus A_2$. Therefore, Kasparov's Technical Theorem implicitly involves a $*$-homomorphism $A_1 \oplus A_2 \to C(E)$. A natural setting for generalization is $M_n(A) \to C(E)$.

Theorem 3.2. *Suppose A is a σ-unital C^*-algebra, φ is a $*$-homomorphism*

$$\varphi : M_n(A) \to C(E)$$

and $\mathrm{im}(\varphi)$ *commutes with a separable subset D of $C(E)$. There exists a $*$-homomorphism*

$$\psi : \mathrm{C} M_n \to C(E) \cap D'$$

such that, setting $q_{ij} = \psi(t \otimes e_{ij})$,

$$q_{ij} \varphi(a \otimes e_{kl}) = \delta_{jk} \varphi(a \otimes e_{il}), \quad \forall a \in A.$$

Proof. Without loss of generality, D may be assumed to be a unital C^*-algebra. Applying Theorem 3.1 to

$$D, \varphi(A \otimes e_{11}), \ldots, \varphi(A \otimes e_{nn})$$

we obtain $g_1, \ldots, g_n$ in $C(E) \cap D'$ such that

$$0 \leq g_i \leq 1, \quad g_i g_j = 0 \ (i \neq j),$$

$$g_j\varphi(a \otimes e_{jj}) = \varphi(a \otimes e_{jj}).$$

Let h be a completely positive element of A. Since, for any a in A,

$$\begin{aligned} g_i\varphi(hah \otimes e_{jk}) &= g_i g_j \varphi(h \otimes e_{jj})\varphi(ah \otimes e_{jk}) \\ &= \delta_{ij}\varphi(hah \otimes e_{jk}) \end{aligned}$$

we conclude

$$g_i\varphi(a \otimes e_{jk}) = \delta_{ij}\varphi(a \otimes e_{jk}) \tag{1}$$

for all i, j, k and all $a \in A$.

Let $x = \varphi(h \otimes w)$ where

$$w = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix}.$$

Since x is normal and both x and $|x| = \varphi(h \otimes I)$ commute with D, we may apply [**11**, Theorem 4.4]. Thus, there exists u in $C(E) \cap D'$, with $\|u\| \leq 1$, such that $x = u|x|$ and $x^* = u^*|x|$.

Multiplying $x = u|x|$ by $\varphi(ah \otimes e_{ij})$ yields

$$u\varphi(hah \otimes e_{ij}) = \varphi(hah \otimes e_{i+1,j}).$$

(Addition taken mod n.) Therefore, by this and a similar calculation based on $x^* = u^*|x|$,

$$u\varphi(a \otimes e_{ij}) = \varphi(a \otimes e_{i+1,j}) \quad \text{and} \quad u^*\varphi(a \otimes e_{ij}) = \varphi(a \otimes e_{i-1,j}), \tag{2}$$

for all j, k and all $a \in A$.

We now make a first approximation on what shall be the images, under ψ, of the generators $t \otimes e_{j1}$ of $\mathrm{C}M_n$. Let

$$a_n = g_n u^{n-1} g_1,$$

and then for $j = n-1, \ldots, 2$,

$$a_{j-1} = g_{j-1} u^{j-2} |a_j|.$$

Clearly $a_j \in D'$ and

$$|a_2| \leq |a_3| \leq \cdots \leq |a_n| \leq 1. \tag{3}$$

By induction, $a_j \in \overline{g_j C(E) g_1}$. This forces some of the relations determining $\mathrm{C}\,M_n$ (as in [**8**, Proposition 2.7]) to hold, namely

$$a_j a_k = 0, \quad j,k = 2,\ldots,n,$$

$$a_j^* a_k = 0, \quad j \neq k. \tag{4}$$

We claim that, for all $b \in A$ and all i, j, k,

$$a_i \varphi(b \otimes e_{jk}) = \delta_{1j} \varphi(b \otimes e_{ik}) \text{ and } a_i^* \varphi(b \otimes e_{jk}) = \delta_{ij} \varphi(b \otimes e_{1k}). \tag{5}$$

For $i = n$ this follows directly from (1) and (2). But then

$$|a_n| \varphi(b \otimes e_{jk}) = \delta_{1j} \varphi(b \otimes e_{jk})$$

so one may handle the case $i = n - 1$, et cetera.

As done in the proof of [**8**, Lemma 4.8], for $j = 2, \ldots, n$ we define

$$\tilde{a}_j = \lim_{m \to \infty} a_j ((1/m) + a_j^* a_j)^{-1/2} (a_2^* a_2)^{1/2}.$$

By the calculations done in the proof of [**8**, Lemma 4.8] we conclude that setting $\psi(t \otimes e_{i1}) = \tilde{a}_i$ defines a homomorphism

$$\psi : \mathrm{C}\,M_n \to C(E) \cap D'.$$

For every $b \in A$, (5) implies

$$\tilde{a}_i \varphi(b \otimes e_{jk}) = \delta_{1j} \varphi(b \otimes e_{ik}) \text{ and } \tilde{a}_i^* \varphi(b \otimes e_{jk}) = \delta_{ij} \varphi(b \otimes e_{1k}) \tag{6}$$

whence

$$\psi(t \otimes e_{ij}) \varphi(b \otimes e_{kl}) = \delta_{jk} \varphi(b \otimes e_{il}).$$

□

4. Interval stretching in corona algebras.

We continue in this section to assume $C(E)$ is the corona algebra of some σ-unital C^*-algebra.

Let us consider a simple case of Kasparov's Technical Theorem. Given h_1, h_2 in $C(E)$ such that

$$0 \leq h_i \leq 1 \ (i = 1,2) \quad \text{and} \quad h_1 h_2 = 0, \tag{7}$$

the conclusion is there exists an additional element so that now

$$0 \le z \le 1,\ 0 \le h_i \le 1\ (i = 1, 2),$$

$$h_1 z = 0,\ h_2 z = h_2 \text{ and } \quad h_1 h_2 = 0. \tag{8}$$

The universal C^*-algebra for these relations are as follows:

$$C^*\langle h_1, h_2 \mid (7) \text{ holds } \rangle \cong C_0([-1,0) \cup (0,1])$$

and

$$C^*\langle h_1, h_2, z \mid (8) \text{ holds } \rangle \cong C_0([-1,0) \cup (0,2]).$$

For this reason, we think of Kasparov's Technical Theorem as a device for stretching an interval algebra at a point.

We introduce some notation to be used for the rest of this section.

Let $X \subseteq \mathbb{C}$ denote the union of the unit circle and the interval $[-2,-1]$. Let

$$A_n = \{f \in C(X, M_n) \mid f(-2) \text{ is scalar}\}$$

and let $\alpha : M_n(C_0(0,1))^\sim \to A_n$ denote the inclusion of the subalgebra of functions in $C(X, M_n)$ that are constant and scalar on $[-2,-1]$.

Lemma 4.1. *Let B denote any separable, unital C^*-algebra. Given a $*$-homomorphism*

$$\varphi : M_n(C_0(0,1))^\sim \otimes B \to C(E)$$

whose image commutes with a separable subset $D \subseteq C(E)$, there exists $*$-homomorphism

$$\tilde{\varphi} : A_n \otimes B \to C(E)$$

such that $\tilde{\varphi} \circ (\alpha \otimes \mathrm{id}_B) = \varphi$ and whose image commutes with D.

Proof. Since A_n and $M_n(C_0(0,1))^\sim$ are nuclear there is no ambiguity in the tensor product. As the tensor products involve unital C^*-algebras they are characterized as the universal C^*-algebras containing commuting copies of the two factors. By altering the subset D one easily shows that it suffices to prove this result only when $B = \mathbb{C}$.

Proposition 2.8 of [**8**] shows that $M_n(C_0(0,1))^\sim$ is the universal unital C^*-algebra generated by $x, a_2, a_3, \ldots, a_n$ subject to the relations

$$\|a_j\| \le 1, \quad j = 2, \ldots, n,$$

$$a_j a_k = 0, \quad 2 \le j, k \le n,$$

$$a_j^* a_k = 0, \quad j \ne k,$$

$$a_j^* a_j = x^* x,$$
$$x^* x = x x^* = -x - x^*.$$

Similarly, one may show that A_n is the universal unital C^*-algebra generated by $x, b_2, b_3, \dots, b_n$ subject to the relations

$$\|b_j\| \leq 1, \quad j = 2, \dots, n,$$
$$b_j b_k = 0, \quad 2 \leq j, k \leq n,$$
$$b_j^* b_k = 0, \quad j \neq k,$$
$$b_j^* b_j = b_k^* b_k, \quad 2 \leq j, k \leq n,$$
$$(b_j^* b_j - 1)(x x^* + x^* x) = 0,$$
$$x x^* = x^* x = -x - x^*,$$

and the inclusion α corresponds to the $*$-homomorphism determined by the assignment $x \mapsto x, a_j \mapsto b_j|x|$. Working with the same relations, but in nonunital category, one sees that this is a special case of Theorem 3.2.

□

Lemma 4.2. *Suppose J is an ideal in A and A is a sub-C^*-algebra of B. Let J_B denote the ideal of B generated by J. There is an isomorphism*

$$\Phi : B/J_B \to B *_A (A/J)$$

defined by $\Phi(b + J_B) = b$.

We will need to prove technical results regarding maps from general dimension-drop graphs into corona algebras. For clarity we will concentrate on the most important case, that of the dimension-drop intervals, $\tilde{\mathbb{I}}_n$. Recall

$$\tilde{\mathbb{I}}_n = \{f \in C[0,1] \mid f(0), f(1) \in \mathbb{C}I\},$$

this being the unital version of the dimension-drop interval.

Although isomorphic to $\tilde{\mathbb{I}}_n$ we also consider

$$\mathbb{J}_n = \{f \in C[-1,2] \mid f(-1) \text{ and } f(2) \text{ are scalar}\}.$$

Let $\iota : \tilde{\mathbb{I}}_n \to \mathbb{J}_n$ denote the inclusion that extends a function to be constant on $[-1,0]$ and on $[1,2]$.

Theorem 4.3. *Suppose $\varphi : \tilde{\mathbb{I}}_n \to C(E)$ is a $*$-homomorphism whose image commutes with a separable subset D. Then there exists a $*$-homomorphism $\bar{\varphi} : \mathbb{J}_n \to C(E) \cap D'$ such that $\bar{\varphi} \circ \iota = \varphi$.*

Proof. Consider $M_n(C_0(0,1))^\sim \otimes C[0,1]$ which we identify with

$$C_n = \{f \in C([0,1]^2, M_n) \mid f(0,t) = f(1,t) \in \mathbb{C}I, \ \forall t\}.$$

Restriction to the diagonal gives us a surjection

$$\rho : M_n(C_0(0,1))^\sim \otimes C[0,1] \to \tilde{\mathbb{I}}_n.$$

One can check that by the last lemma we have the commutative diagram

$$\begin{array}{ccc} (A_n \otimes C[0,1]) *_{C_n} \tilde{\mathbb{I}}_n & \xrightarrow{\cong} & \mathbb{J}_n \\ \uparrow{\scriptstyle (\alpha\otimes\mathrm{id})*\mathrm{id}} & & \uparrow{\scriptstyle \iota} \\ C_n *_{C_n} \tilde{\mathbb{I}}_n & \xrightarrow{\cong} & \tilde{\mathbb{I}}_n \end{array}$$

and so this result thus follows from Lemma 4.1. □

Remark. The generalization of Theorem 4.3 to the case of extending maps of dimension-drop graphs into corona algebras follows by the same methods, but the notation is significantly worse.

5. Stability for dimension-drop graphs.

Suppose X is a graph. We denote the associated dimension-drop C^*-algebra by

$$C_{\text{vert}}(X, M_n) = \{f \in C(X, M_n) \mid f(v) \in \mathbb{C}I \text{ for all vertices } v\}.$$

Theorem 5.1. *For every graph X, and every positive integer n, the C^*-algebra $C_{\text{vert}}(X, M_n)$ is universal for a stable set of relations.*

Proof. We may reduce to the case of X connected using Proposition 3.10 and [**8**, Theorem 5.1]. For connected graphs, the proof is by induction on the number of vertices. If there is but one vertex then

$$C_{\text{vert}}(X, M_n) \cong \left(\bigoplus_{j=1}^{J} M_n(C_0(0,1)) \right)^\sim$$

where J is the number of edges. This has stable relations by [**8**, Theorem 5.1].

Now suppose X has at least two vertices, v_0 and v_1. We will need an auxiliary space, $\tilde{X}$, which is obtained from X by stretching all edges attached to v_0 or v_1. Topologically, $\tilde{X}$ will be a copy of X. We shall use v_0 and v_1 to denote the appropriate vertices in $\tilde{X}$.

Choose a function

$$h_0 : \tilde{X} \to [-1, 2]$$

such that $h_0^{-1}([-1, 0])$ consists of the union of half-closed subintervals, containing v_0, of each edge adjacent to v_0. We may assume a similar statement holds for $h_0^{-1}([1, 2])$ and v_1.

We will identify X with the quotient of $\tilde{X}$ obtained by collapsing $h_0^{-1}([-1, 0])$ to a point and $h_0^{-1}([1, 2])$ to a different point. We will also consider two copies of the graph obtained from X by collapsing the two designated vertices together. We let $\tilde{Y}$ denote the quotient of $\tilde{X}$ obtained by identifying v_0 with v_1 and Y denote the quotient of $\tilde{X}$ obtained by collapsing $h_0^{-1}([-1, 0]) \cup h_0^{-1}([1, 2])$ to a point.

Accordingly, we will be making identifications of the various dimension-drop algebras with subalgebras of $C(\tilde{X}, M_n)$. Of course, $C_{\text{vert}}(\tilde{X}, M_n)$ is defined as such a subalgebra. The remaining identifications are:

$$\begin{aligned} C_{\text{vert}}(X, M_n) &= \{f \mid f(x) = f(v_0) \text{ if } h_0(x) \le 0 \\ &\qquad\qquad \text{and } f(x) = f(v_1) \text{ if } h_0(x) \ge 1\}, \\ C_{\text{vert}}(Y, M_n) &= \{f \mid f(x) = f(v_0) \text{ if } h_0(x) \le 0 \text{ or } h_0(x) \ge 1\} \\ C_{\text{vert}}(\tilde{Y}, M_n) &= \{f \mid f(v_0) = f(v_1)\}. \end{aligned}$$

Our strategy is based on the observation that $C_{\text{vert}}(X, M_n)$ is generated by the subalgebra $C_{\text{vert}}(Y, M_n)$ and the element

$$h = h_1 \otimes I \quad \text{where} \quad h_1(x) = \max(\min(h_0(x), 1), 0).$$

A way to express the relation between h and $C_{\text{vert}}(Y, M_n)$ is that

$$e^{2\pi i h} = e^{2\pi i h_1} \otimes I.$$

By Theorem 2.6, our task is reduced to proving corona semiprojectivity for $C_{\text{vert}}(X, M_n)$ while assuming it for $C_{\text{vert}}(\tilde{Y}, M_n)$. So suppose that we are given a unital $*$-homomorphism

$$\varphi : C_{\text{vert}}(X, M_n) \to C(E) \cong B/\overline{\bigcup I_m}.$$

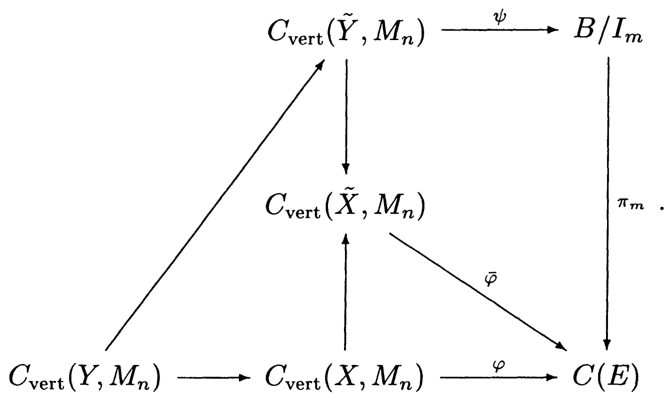
By Theorem 4.3 and the remark following, there is an extension of φ to

$$\bar{\varphi} : C_{\text{vert}}(\tilde{X}, M_n) \to C(E).$$

By the induction hypothesis, the restriction of $\bar{\varphi}$ to $C_{\text{vert}}(\tilde{Y}, M_n)$ can be lifted to

$$\psi : C_{\text{vert}}(\tilde{Y}, M_n) \to B/I_m$$

for some m. This leads to the following commutative diagram:



Let H be any lift of $\varphi(h)$ to B/I_m such that $0 \le H \le 1$. Now define

$$\tilde{H} = \psi(l(h_0) \otimes I) + \psi(m(h_0)^{1/2} \otimes I) H \psi(m(h_0)^{1/2} \otimes I)$$

where l and m are the functions

$$l(t) = \begin{cases} 0, & t \le 0, \\ t, & 0 \le t \le 1, \\ 2 - t, & 1 \le t \le 2, \end{cases} \quad m(t) = \begin{cases} -t, & t \le 0, \\ 0, & 0 \le t \le 1, \\ t - 1, & 1 \le t \le 2. \end{cases}$$

These are defined so that $l + mh_2 = h_2$ where h_2 is the function

$$h_2(t) = \begin{cases} 0, & t \le 0, \\ t, & 0 \le t \le 1, \\ 1, & 1 \le t \le 2. \end{cases}$$

Notice also that $h_2(h_0) = h_1$.

Clearly $\tilde{H}$ is selfadjoint. In fact, it is also a lift of $\varphi(h)$ since

$$\begin{aligned} \pi_m(\tilde{H}) &= \bar{\varphi}(l(h_0) \otimes I) + \bar{\varphi}(m(h_0) \otimes I)\bar{\varphi}(h_2(h_0) \otimes I) \\ &= \bar{\varphi}((l + mh_2)(h_0) \otimes I) = \varphi(h). \end{aligned}$$

For any $f \otimes T \in C_{\text{vert}}(Y, M_n)$

$$(f \otimes T)(m(h_0)^{1/2} \otimes I) = 0 \quad \Rightarrow \quad \psi(f \otimes T)\tilde{H} = \tilde{H}\psi(f \otimes T).$$

By replacing $\tilde{H}$ by $h_2(\tilde{H})$, we have found a lift of $\varphi(h)$, with $0 \le \tilde{H} \le 1$, and a lift of $\varphi|_{C_{\text{vert}}(Y, M_n)}$ that commute.

Expressing this conclusion differently, we have shown that given a unital map

$$C_{\text{vert}}(X, M_n) \to C(E)$$

we can find an m and a map making the diagram commute where D is the universal unital C^*-algebra generated by a copy of $C_{\text{vert}}(Y, M_n)$ and a central element h such that $0 \leq h \leq 1$. I.e.,

$$D \cong C_{\text{vert}}(Y, M_n) \otimes C[0,1].$$

We have no further need for $\tilde{X}$ so v_0 and v_1 again denote the specified vertices in X. We regard Y as the quotient of X, with quotient map $\eta : X \to Y$ which collapses v_0 and v_1 to a single vertex we call w_0.

Let us identify D with

$$\{g \in C(Y \times [0,1], M_n) \mid g(v,t) \in \mathbb{C}I \text{ for all vertices}\}.$$

The copy of $C_{\text{vert}}(Y, M_n)$ and the extra element h appear as functions in D constant in one variable or the other. There is a sort of diagonal map

$$\Delta : X \to Y \times [0,1], \quad \Delta(x) = (\eta(x), h_1(x))$$

which induces a surjection $\beta : D \to C_{\text{vert}}(X, M_n)$.

We need also a quotient of D where the relation (9) holds approximately. Consider

$$Z_\delta = \{(\eta(x), t) \in Y \times [0,1] \mid |e^{2\pi i h_1(x)} - e^{2\pi i t}| \leq \delta\},$$

where δ is a small number to be named later, and let

$$D_\delta = \{g \in C(Z, M_n) \mid g(v,t) \in \mathbb{C}I \text{ for all vertices}\}.$$

Since Δ maps into Z it induces

$$\beta_0 : D_\delta \to C_{\text{vert}}(X, M_n).$$

By increasing m we may assume that the map $D \to B/I_m$ factors through D_δ. Therefore, we are done if we exhibit a right-inverse to β_0. This exists because there is a retraction of Z_δ onto $\text{im}(\Delta)$ which sends (v,t) to (v,t') for every vertex v. To be able to describe this retraction we break up Z_δ as $Z_\delta = Z_1 \cup Z_2 \cup Z_3$ where

$$\begin{aligned} Z_1 &= \{(\eta(x), t) \mid |h_1(x) - t| \leq 1/4, 0 < t < 1\}, \\ Z_2 &= \{(\eta(x), t) \mid |h_1(x) + 1 - t| \leq 1/4\}, \\ Z_3 &= \{(\eta(x), t) \mid |h_1(x) - 1 - t| \leq 1/4\}. \end{aligned}$$

The retraction sends Z_2 to $(w_0, 1)$ and Z_3 to $(w_0, 0)$. Each point $(\eta(x), t)$ in Z_1 is sent to $(\eta(x), s)$ where s is the unique number in $(0,1)$ such that

$e^{2\pi is} = e^{2\pi ih_1(x)}$. By choosing δ sufficiently small, we ensure that $(v,t) \notin Z_2 \cup Z_3$ for any vertex v except for $v = w_0$. Therefore this is the desired retraction. □

References

[1] B. Blackadar, *Shape theory for C^*-algebras*, Math. Scand., **56** (1985), 249-275.

[2] ———, *K-theory for Operator Algebras*, M.S.R.I. Monographs No. 5, Springer Verlag, Berlin and New York, 1986.

[3] L. Brown, *Unitary equivalence modulo the compact operators and extensions of C^*-algebras*, Proceedings of a Conference on Operator Theory, Lecture Notes in Mathematics, vol. 345, Springer-Verlag, Berlin, 1973, 58-140.

[4] M. Dadarlat and T.A. Loring, *K-homology, asymptotic morphisms and unsuspended E-theory*, J. Funct. Anal., **126** (1994), 367-383.

[5] ———, *Extensions of certain real rank zero C^*-algebras*, Ann. Inst. Fourier (Grenoble), **44** (1994), 906-925.

[6] E. Effros and J. Kaminker, *Homotopy continuity and shape theory for C^*-algebras*, Geometric Methods in Operator Algebras, U.S. - Japan Joint Seminar at Kyoto, 1983, Pitman.

[7] G.A. Elliott, *On the classification of C^*-algebras of the real rank zero*, J. reine angew Math., **443** (1993), 179-219.

[8] T.A. Loring, *C^*-algebras generated by stable relations*, J. Funct. Anal., **112** (1993), 159-201.

[9] ———, *Projective C^*-algebras*, Math. Scan., **173** (1993), 274-280.

[10] T.A. Loring and G.K. Pedersen, *Projectivity, transitivity and AF telescopes*, preprint.

[11] C.L. Olsen and G.K. Pedersen, *Corona C^*-algebras and their applications to lifting problems*, Math. Scand., **164** (1989), 63-86.

[12] G.K. Pedersen, *C^*-algebras and Their Automorphism Groups*, Academic Press, New York, 1979.

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