

*Pacific
Journal of
Mathematics*

NEVANLINNA'S COEFFICIENTS AND DOUGLAS ALGEBRAS

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Some relations between Douglas algebras and coefficients appearing in Nevanlinna's matrix parametrization of the solutions of the Nevanlinna Pick interpolation problem are studied.

1. Introduction.

Let U denote the analytic functions bounded by one in $\mathbb{D} = \{z : |z| < 1\}$. Given a sequence $\{z_n\} \subset \mathbb{D}$, we consider the classical Nevanlinna Pick interpolation problem

$$(NP) \quad f(z_n) = w_n, \quad n = 1, 2, \dots, \quad f \in U.$$

If this problem has more than one solution, R. Nevanlinna [4] found analytic functions P, Q, R and S such that the set of all solutions is given by

$$(1.1) \quad E = \left\{ \frac{P - Qw}{R - Sw}, \quad w \in U \right\}.$$

The functions P, Q, R and S are unique subject to the normalization $S(0) = 0$ and $PS - RQ = \pi$, where

$$\pi(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

is the Blaschke product corresponding to $\{z_n\}$.

While the functions P, Q, R and S arose from classical function theory, it turns out that they are also connected with more recent developments. It is part of Nevanlinna's theory that the functions $P/R, Q/R, S/R$ and $1/R$ belong to U and are linked with π in many ways. (See Lemma 1.)

Suppose (NP) has a solution f_0 satisfying $\sup\{|f_0(z)|, z \in D\} < 1$. Our main result is that then $P/R, Q/R, S/R$ and $1/R$ all belong to a certain subalgebra of H^∞ depending only on π which we shall denote by CDA_π . This algebra is part of the theory of Douglas algebras through the work of S.Y. Chang and D.E. Marshall ([1], [2?]). Our results in particular answer

a problem raised by V. Tolokonnikov in [11] where other relations between Douglas algebras and the Nevanlinna Pick problem are studied.

Our methods are based on Nevanlinna's ideas in [4] and last but not least on the more recent treatment of the Nevanlinna Pick problem given by J. Garnett in [2], where dual extremal methods are used. We also give a new proof of a recent result of Tolokonnikov concerning questions whether (NP) has a unique solution.

Next we introduce some notations and well known results.

Let m denote normalized Lebesgue measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. If $1 \leq p \leq \infty$, H^p denote the Hardy space consisting of all $f \in L^p(m)$ whose harmonic extension to D is analytic there. If $p = \infty$, the norm $\|f\|_p$ in $L^p(m)$ can also be given by

$$\|f\|_\infty = \sup\{|f(z)| : z \in D\} \quad f \in H^\infty.$$

For basic properties of H^p , we refer to Garnett's book [2].

We recall that $I \in H^\infty$ is called an inner function if $|I(e^{i\alpha})| = 1$ almost everywhere with respect to m . Any Blaschke product is inner, but there are many others ([2, p. 75]).

Considering H^∞ as a subalgebra of $L^\infty(m)$, let $D_\pi = [H^\infty, \bar{\pi}]$ be the Douglas algebra generated by H^∞ and the restriction $\bar{\pi}|_{\mathbb{T}}$ of $\bar{\pi}$ to \mathbb{T} . Then let $QD_\pi = D_\pi \cap \bar{D}_\pi$ be the maximal C^* -subalgebra of D_π . Define also $QDA_\pi = QD_\pi \cap H^\infty$ and let CDA_π denote the subalgebra of H^∞ generated by all inner functions I invertible in D_π . It is evident that $CDA_\pi \subset QDA_\pi$. For more about these algebras, see [1], and [2] for example. Let I be an inner function. The property of I being invertible in D_π has a very concrete formulation: If $\{\zeta_n\} \subset D$ and $|\pi(\zeta_n)| \rightarrow 1$, then $|I(\zeta_n)| \rightarrow 1$.

The special solutions I_α to (NP) given by

$$I_\alpha = \frac{P - Qe^{i\alpha}}{R - Se^{i\alpha}}$$

play an important role in this theory. Nevanlinna showed that each I_α is inner [4], and in fact almost all I_α are Blaschke products [9]. A Nevanlinna Pick problem is called scaled if it has a solution f_0 satisfying $\|f_0\|_\infty < 1$.

For general properties of Douglas algebras and more on the Nevanlinna Pick problem, Garnett's book [2] is a good reference.

The letter C_i will be used for different absolute constants, while $C(t)$ indicates a constant depending on the parameter t .

Acknowledgements. We thank the referee for several helpful remarks which have improved our work. Theorem 2, which is stronger than our previous result, is due to him. This work was done during a visit to University

of Bergen by the first author and to CRM in Barcelona by the second author. Both of us wish to express our appreciation of the hospitality and nice working conditions.

2. Main result.

If (NP) has more than one solution, R. Nevanlinna considered the “Wertevor-rat” $\Delta(z) = \{f(z) : f \text{ is a solution of (NP)}\}$, $z \in \mathbb{D}$. Using (1.1), one can easily check that $\Delta(z)$ is a disc of center $c(z) = (-Q(z)\overline{S(z)} + P(z)\overline{R(z)})/(|R(z)|^2 - |S(z)|^2)^{-1}$, and radius $\rho(z) = |\pi(z)|(|R(z)|^2 - |S(z)|^2)^{-1}$.

For later use, we collect some of the properties of Nevanlinna’s coefficients.

Lemma 1. *Assume (NP) has more than one solution and consider the Nevanlinna’s coefficients P, Q, R, S appearing in (1.1). Then*

- (i) P, Q, R, S have radial limit almost everywhere and $Q = -\pi\overline{R}$, $P = -\pi\overline{S}$, $|R|^2 - |S|^2 = 1$, $Q\overline{S} - P\overline{R} = 0$, almost everywhere on the unit circle.
- (ii) $|R(z)|^2 - |S(z)|^2 \geq 1$, $|R(z)|^2 - |P(z)|^2 \geq 1$, $z \in \mathbb{D}$.
- (iii) For any $e^{i\alpha} \in \partial\mathbb{D}$, $(R - Se^{i\alpha})^{-2}$ is an exposed point of H^1 .
- (iv) If $u \in U$ and $f = (P - Qu)(R - Su)^{-1}$, one has

$$\|f\|_\infty = \left\| \frac{\overline{S/R} - u}{1 - uS/R} \right\|_{L^\infty(\partial\mathbb{D})}.$$

- (v) If (NP) is scaled, one has $\rho(z) \rightarrow 1$ as $|\pi(z)| \rightarrow 1$.
- (vi) If (NP) is scaled and $\gamma = \inf\{\|f_0\|_\infty : f \text{ is a solution of (NP)}\}$, then $R \in H^p$ for all $p < \pi(\arcsin(\gamma))^{-1}$.

Proof. (i), (ii), (iii) are well known (see [8] and the references there given to [2]). Using the relations in (i)

$$\begin{aligned} \left| \frac{P - Qu}{R - Su}(e^{i\theta}) \right| &= \left| \frac{Q}{R}(e^{i\theta}) \right| \left| \frac{P/Q - u}{1 - uS/R}(e^{i\theta}) \right| \\ &= \left| \frac{\overline{S/R} - u}{1 - uS/R}(e^{i\theta}) \right|, \quad \text{a.e. } e^{i\theta} \in \partial\mathbb{D}, \end{aligned}$$

and this is (iv). A proof of (v) can be found in [10]. Now, let us prove (vi). Consider $I_\alpha = (P - Qe^{i\alpha})(R - Se^{i\alpha})^{-1}$, for fixed α , $0 \leq \alpha < 2\pi$. Using (i), one can easily check

$$I_\alpha \overline{\pi} = e^{i\alpha} \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|}, \quad \text{a.e. on } \partial\mathbb{D}.$$

Since $\gamma = \text{dist}(I_\alpha \bar{\pi}, H^\infty) < 1$, there exists $g \in H^\infty$ satisfying

$$1 > \gamma = \left\| \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|} - g \right\|_\infty.$$

Since $I_\alpha(0) \in \partial\Delta(0)$, one has $\text{dist}(I_\alpha \bar{\pi}, H_0^\infty) = 1$, where $H_0^\infty = \{f \in H^\infty : f(0) = 0\}$. The proof of Lemma 4.3 in ([2, p. 386]) shows $|g(z)| \geq 1 - \gamma, z \in \mathbb{D}$. Let $\text{Arg}(z)$ be the principal branch of the argument. One has,

$$|\text{Arg}(g^{-1}(R - Se^{i\alpha})^{-2})| \leq \arcsin(\gamma), \quad \text{a.e. on } \partial\mathbb{D}.$$

So, the same is true on \mathbb{D} and using a result in ([2, p. 114]), one gets

$$(g^{-1}(R - Se^{i\alpha})^{-2})^{-1} \in H^p, \quad p < \frac{\pi}{2 \arcsin(\gamma)}.$$

Hence $(R - Se^{i\alpha})^2 \in H^p$, for $p < \pi(2 \arcsin(\gamma))^{-1}$ and it follows $R \in H^p$, for $p < \pi(\arcsin(\gamma))^{-1}$. This finishes the proof of Lemma 1. □

Let (NP) be an scaled Nevanlinna problem, V. Tolokonnikov proved that the extremal solutions I_α are invertible in D_π [11]. Our next result is an extension of this.

Proposition . *Let (NP) be a scaled Nevanlinna Pick problem and I_α one of its extremal solutions, $0 \leq \alpha < 2\pi$. Then $D_{I_\alpha} = D_\pi$.*

Proof. As mentioned before, it is known that I_α is invertible in D_π . We present another proof of it. From (v) of Lemma 1, $\rho(z) \rightarrow 1$ whenever $|\pi(z)| \rightarrow 1$. Since $I_\alpha(z) \in \partial\Delta(z)$, one gets $|I_\alpha(z)| \rightarrow 1$. Hence, I_α is invertible in D_π and $D_{I_\alpha} \subset D_\pi$.

For the converse assume

$$|I_\alpha(z_n)| \rightarrow 1.$$

Since the Nevanlinna Pick problem (NP) is scaled, the “Wertevorrat” $\Delta(z_n)$ must meet a fixed disc inside the unit disc. Actually, $f_0(z_n), I_\alpha(z_n) \in \Delta(z_n)$, where f_0 is a solution to (NP) with $\|f_0\|_\infty < 1$. Hence, for large n ,

$$|\pi(z_n)| \geq \rho(z_n) \geq \frac{1}{4}(1 - \|f_0\|_\infty) > 0$$

and one deduces that π is invertible in D_{I_α} .

The Proposition can also be immediately deduced from the proof of Theorem 2.1 in [1]. □

Remark. The hypothesis on the scaling of the Nevanlinna Pick problem is essential. In fact, there exist non scaled Nevanlinna Pick problems and points $\beta_n \in \mathbb{D}$ such that

$$\sup\{|w| : w \in \Delta(\beta_n)\} \xrightarrow{n \rightarrow \infty} 0, \quad |\pi(\beta_n)| \xrightarrow{n \rightarrow \infty} 1$$

see [5]. Then, $I_\alpha(\beta_n) \rightarrow 0$, $0 \leq \alpha < 2\pi$, and no I_α is invertible in D_π .

The following result is known although we have not found it in the literature. We thank the referee for pointing out it to us.

Lemma 2. *Given $u, |u| = 1$ and $z, |z| \leq 1$, one has that*

$$z = \int_0^{2\pi} \frac{z - ue^{i\alpha}}{1 - \bar{z}ue^{i\alpha}} \frac{d\alpha}{2\pi}$$

can be uniformly approximated by its Riemann sums.

Proof. Multiplying by \bar{u} if necessary, one may assume $u = 1$. For $w = e^{2\pi in^{-1}}$, one has

$$z - \frac{1}{n} \sum_{k=1}^n \frac{z - w^k}{1 - w^k \bar{z}} = \bar{z}^{n-1} \frac{1 - |z|^2}{1 - \bar{z}^n}, \quad |z| < 1.$$

This can be shown expanding in a series and using

$$\sum_{k=1}^n w^{pk} = 0$$

unless $p \equiv 0 \pmod n$. By continuity the same holds if $\bar{z}^n \neq 1$. Now, the inequalities

$$\begin{aligned} \left| z - \frac{1}{n} \sum_{k=1}^n \frac{z - w^k}{1 - w^k \bar{z}} \right| &\leq \frac{|z|^{n-1}(1 + |z|)(1 - |z|)}{1 - |z|^n} \\ &= \frac{1 + |z|}{1 + |z|^{-1} + \dots + |z|^{-(n-1)}} \leq \frac{2}{n} \end{aligned}$$

finish the proof. □

Assume (NP) is scaled. In [11] it is proved that the functions $P/R, \pi R^{-2}(S/R)^k, k \geq 0$, belong to CDA_π and it is asked if $R^{-1} \in CDA_\pi$. Next, we complete these results.

Theorem 1. *Let (NP) be a scaled Nevanlinna Pick problem, E the set of its solutions and*

$$E = \left\{ \frac{P - Qw}{R - Sw} : w \in U \right\}$$

its Nevanlinna's parametrization. Let D_π be the Douglas algebra generated by H^∞ and $\bar{\pi}|_{\mathbb{T}}$. Then, the functions $P/R, Q/R, S/R, 1/R$ belong to the algebra CDA_π .

Proof. Since $|S/R(e^{i\theta})| \leq 1$, Lemma 2 shows

$$\frac{1}{2\pi} \int_0^{2\pi} I_\alpha(e^{i\theta}) d\alpha = P/R(e^{i\theta}), \quad \text{a.e. } e^{i\theta} \in \mathbb{T},$$

and the integral can be uniformly approximated by its Riemann sums. Since I_α are inner functions invertible in D_π , one gets $P/R \in CDA_\pi$.

Since Q/R is an inner function, one only has to show that Q/R is invertible in D_π . If $|\pi(z)| \rightarrow 1$, by (v) of Lemma 1, the disc $\Delta(z)$ tends to the unit disc, that is to say,

$$\begin{aligned} \rho(z) &= \frac{|Q/R(z) - P/R(z)S/R(z)|}{1 - |S/R(z)|^2} \rightarrow 1 \\ c(z) &= \frac{P/R(z) - Q/R(z)\overline{S/R(z)}}{1 - |S/R(z)|^2} \rightarrow 0. \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\leftarrow \frac{P/R(z)S/R(z) - Q/R(z)}{1 - |S/R(z)|^2} + Q/R(z) \\ &= \frac{P/R(z)S/R(z) - Q/R(z)|S/R(z)|^2}{1 - |S/R(z)|^2} \end{aligned}$$

and one gets $|Q/R(z)| \rightarrow 1$. Therefore $Q/R \in CDA_\pi$.

Since by (i) of Lemma 1 $Q\bar{S} = P\bar{R}$ a.e. on the unit circle, one has $S/R = \overline{(P/R)}(Q/R) \in CD_\pi$ and since it is analytic, $S/R \in CDA_\pi$.

Using $R = \overline{Q}\pi$ a.e. on the unit circle, one gets $\overline{(1/R)}Q/R = \pi/R \in H^\infty$. Then, for $0 < \delta < 1$,

$$\delta \frac{1}{R} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\frac{Q}{R}e^{i\alpha} + \frac{1}{R}\delta}{1 + e^{i\alpha}(\delta/R)Q/R} d\alpha$$

uniformly on the unit circle. Since Q/R is an inner function invertible in D_π , so is

$$\frac{Q/R e^{i\alpha} + \delta/R}{1 + e^{i\alpha}(\delta/R)Q/R}, \quad e^{i\alpha} \in \partial\mathbb{D},$$

and one gets $R^{-1} \in CDA_\pi$. □

3. An example.

The results of last section may suggest that if one takes $w \in CDA_\pi$, $w \in U$ in Nevanlinna's formula, the resulting function $f = (P - Qw)(R - Sw)^{-1}$ may also belong to CDA_π . This is of course the case if $\|w\|_\infty < 1$, because of the relation

$$f = (P/R - wQ/R) \sum_{n=0}^{\infty} (wS/R)^n.$$

It has been surprising to us that for general $w \in U \cap CDA_\pi$, the function f may not belong to CDA_π . In fact, f may not belong to the bigger algebra QA_π , which consists of the holomorphic functions in the unit disc which belong to $D_\pi \cap \overline{D_\pi}$. To show this, we need to construct a scaled Nevanlinna Pick problem such that the corresponding function R is not bounded. We will do the construction in the upper half plane.

Consider $z_n = iy_n$, where $y_{n+1} < cy_n$, for some fixed $0 < c < 1$ and $z_n^* = x_n + iy_n$, where $x_n > 0$ is a decreasing sequence, $\sup x_n y_n^{-1}$ is a small number to be chosen later, $x_n y_n^{-1} \rightarrow 0$ as $n \rightarrow \infty$, but

$$(3.1) \quad \sum_n (x_n y_n^{-1})^2 = +\infty.$$

Let B and B^* be the Blaschke products in the upper half plane with zeros $\{z_n\}$ and $\{z_n^*\}$ and B_1, B_1^* the Blaschke products with zeros $\{\varphi(z_n)\}, \{\varphi(z_n^*)\}$, where φ is a conformal map from the upper half plane to the unit disc.

Lemma 3. *With the notations above, the Nevanlinna Pick problem*

$$(*) \quad f(\varphi(z_n)) = B_1^*(\varphi(z_n)), \quad n = 1, 2, \dots, \quad f \in U$$

is scaled. Moreover, if

$$\{f \in H^\infty : f \text{ solves } (*)\} = \left\{ \frac{P - Qw}{R - Sw} : w \in U \right\}$$

is Nevanlinna's parametrization of the set of its solutions, one has

$$\lim_{\theta \rightarrow 0} |R(e^{i\theta})| = +\infty.$$

Proof. We will prove the Lemma in the upper half plane. Let $x \in \mathbb{R}$, as in ([2, p. 432]), one can compute

$$(3.2) \quad \text{Arg} \frac{B^*(x)}{B(x)} = \sum_n \text{Arg} \left(\frac{x - z_n}{x - \bar{z}_n} \right) - \text{Arg} \left(\frac{x - z_n^*}{x - \bar{z}_n^*} \right) = 2 \int_0^{x_n} \frac{y_n}{(x - t)^2 + y_n^2} dt.$$

Now, if $F \in H^1$, one has

$$\begin{aligned} \left| \int_{\mathbb{R}} F(x) \operatorname{Arg} \frac{B^*(x)}{B(x)} dx \right| &= 2 \left| \sum_n \int_0^{x_n} \int_{\mathbb{R}} \frac{y_n}{(x-t)^2 + y_n^2} F(x) dx dt \right| \\ &\leq 2 \sum_n \int_0^{x_n} |F(t + iy_n)| dt \leq 2K \sup(x_n y_n^{-1}) \|F\|_1 \end{aligned}$$

because the linear measure σ on $\cup_n [iy_n, x_n + iy_n]$ is a Carleson measure, with $\sigma(Q) \leq \sup_n(x_n y_n^{-1})l(Q)$ where Q is a square lying on the real line and $l(Q)$ is the length of its side. So, given $\varepsilon > 0$, if $\sup_n x_n y_n^{-1}$ is sufficiently small, one gets $\|\operatorname{Arg}(B^*/B)\|_{BMO} < \varepsilon$, and hence

$$(3.3) \quad \operatorname{Arg}(B^*/B) = u + \tilde{v}, \quad \|u\|_{\infty} \leq C\varepsilon, \quad \|v\|_{\infty} \leq C\varepsilon,$$

where \tilde{v} is the conjugate function of v and C is an absolute constant ([2, p. 248]).

Now,

$$\|B^*/B - e^{v+i\tilde{v}}\|_{\infty} \leq 2C\varepsilon,$$

hence

$$(3.4) \quad \operatorname{dist}(B^*/B, H^{\infty}) \leq 2C\varepsilon < 1$$

and (*) is scaled.

On other hand,

$$\|B/B^* - e^{-v-i\tilde{v}}\|_{\infty} \leq 2C\varepsilon,$$

so

$$(3.5) \quad \operatorname{dist}(B/B^*, H^{\infty}) \leq 2C\varepsilon < 1.$$

Now, (3.4) and (3.5) give that B^* is an extremal solution of (*), that is to say, there exists $0 \leq \alpha < 2\pi$,

$$B^* = \frac{P - Qe^{i\alpha}}{R - Se^{i\alpha}}.$$

Thus, applying (3.3) and (i) of Lemma 1,

$$\exp(i(u + \tilde{v})) = B^*/B = \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|}.$$

Consider $H = \exp(iu - \tilde{u} + v + i\tilde{v}) \in H^1$ and hence

$$\frac{H}{|H|} = \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|}.$$

By (iii) of Lemma 1 $(R - Se^{i\alpha})^{-2}$ is an exposed point of H^1 , so

$$H = M(R - Se^{i\alpha})^{-2}, \quad M \in \mathbb{C},$$

and $|M(R - Se^{i\alpha})^{-2}(x)| = \exp(v(x) - \tilde{u}(x))$. Now, by (3.3),

$$\begin{aligned} v(x) - \tilde{u}(x) &= -\widetilde{\text{Arg}}(B^*/B)(x) = \frac{-2}{\pi} \sum_n \int_0^{x_n} \frac{x-t}{(x-t)^2 + y_n^2} dt \\ &= \frac{1}{\pi} \sum_n \ln \left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2} \right). \end{aligned}$$

Now, let $x > 0$. Using the inequality $\ln(t^{-1}) \leq c(\delta)(1-t)$ for $\delta \leq t \leq 1$, one gets

$$\begin{aligned} \left| \sum_{x_n: |x_n-x| < x} \ln \left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2} \right) \right| &= \sum_{x_n: |x_n-x| < x} \ln \left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2} \right)^{-1} \\ &\leq C \sum_{x_n: |x_n-x| < x} \frac{2x_n x - x_n^2}{x^2 + y_n^2} \\ &\leq \frac{2C}{x} \sum_{x_n: |x_n-x| < x} x_n \leq C_1. \end{aligned}$$

On the other hand, considering k with $x_k > 2x > x_{k+1}$ one has

$$\begin{aligned} \sum_{x_n: |x_n-x| \geq x} \ln \left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2} \right) &= \sum_{n=1}^k \ln \left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2} \right) \\ &\geq C \sum_{n=1}^k \frac{x_n(x_n - 2x)}{x^2 + y_n^2} \geq C_2 \sum_{n=1}^{k-1} x_n^2 y_n^{-2}. \end{aligned}$$

Also, if $x < 0$, $v(x) - \tilde{u}(x) \geq -C_3 + v(-x) - \tilde{u}(-x)$. So, (3.1) gives

$$\lim_{x \rightarrow 0} |(R - Se^{i\alpha})^{-2}(x)| = +\infty,$$

and thus $\lim_{x \rightarrow 0} |S/R(x)| = 1$. So, by (i) of Lemma 1, $\lim_{x \rightarrow 0} |R(x)| = +\infty$ and this finishes the proof of Lemma 3. □

Now, consider the Nevanlinna Pick problem (*) given by Lemma 3 and

$$\gamma = \inf\{\|f\|_\infty : f \text{ is solution of } (*)\}.$$

For $1 > t > \gamma$, Proposition of last section gives that there exists an inner function J , $tJ = (P - Qw_0)(R - Sw_0)^{-1} \in CDA_\pi$. Using Theorem 1 one

can see that $w_0 \in CDA_\pi$. Now consider an interpolating sequence $\{\alpha_n\}$ approaching to 1, with $|\pi(\alpha_n)| \rightarrow 1$ as $n \rightarrow \infty$, where $\pi = B_1$, and let I be the Blaschke product with zeros $\{\alpha_n\}$. Then, by Lemma 3, $R^{-2}I$ is continuous up to the circle. Also (iv) of Lemma 1 gives

$$(3.6) \quad \left\| \frac{\overline{S/R} - w_0}{1 - w_0 S/R} \right\|_{L^\infty(\partial\mathbb{D})} = t$$

and then $|w_0(e^{i\theta})| \leq |S/R(e^{i\theta})| + c(1 - |S/R(e^{i\theta})|)$, $0 \leq \theta < 2\pi$, for some fixed $c = c(t) < 1$. Therefore $w_1 = w_0 + (1 - c)R^{-2}I \in U \cap CDA_\pi$.

Now, assume $f = (P - Qw_1)(R - Sw_1)^{-1} \in QA_\pi$. Thus,

$$f - tJ = \pi(w_1 - w_0)(R - Sw_0)^{-1}(R - Sw_1)^{-1} \in QA_\pi.$$

Let σ denote the pseudohyperbolic metric, $\sigma(z, w) = |z - w| |1 - \overline{w}z|^{-1}$. Since $|\pi(\alpha_n)| \rightarrow 1$ as $n \rightarrow \infty$, writing $g = (w_1 - w_0)(R - Sw_0)^{-1}(R - Sw_1)^{-1}$, from the fact that $\pi g \in QA_\pi$ one can deduce

$$\max_{\sigma(z, \alpha_n) \leq r} |g(z)| - \min_{\sigma(z, \alpha_n) \leq r} |g(z)| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for any $r < 1$, because otherwise, taking a subsequence of $\{\alpha_n\}$, for some fixed $r < 1$, there would exist $\delta > 0$ and $z_n, \sigma(\alpha_n, z_n) \leq r$, such that

$$(1 - |z_n|) |g'(z_n)| \geq \delta.$$

Then, by subharmonicity, for $m < 1$, it would follow

$$\int_{D_n} |g'(w)|^2 dm(w) \geq C_1(m)\delta$$

where D_n is the disc of center z_n and radius $m(1 - |z_n|)$. So,

$$\int_{D_n} |g'(w)|^2 (1 - |w|) dm(w) \geq C_2(m)\delta(1 - |z_n|)$$

and using a result in [2, p. 381], this would contradict the fact $\pi g \in QA_\pi$.

Since $g(\alpha_n) = 0$, one gets

$$(3.7) \quad \max_{\sigma(z, \alpha_n) \leq r} |g(z)| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

But, (3.6) and (v) of Lemma 1 give

$$|1 - S/R(z)w_i(z)| \leq C_1(t) (1 - |S/R(z)|^2) \leq C_1(t) |R(z)|^{-2}, \quad i = 0, 1, z \in \mathbb{D},$$

so,

$$\max_{\sigma(z, \alpha_n) \leq r} |g(z)| \geq \frac{1 - c}{1 - C_1} \max_{\sigma(z, \alpha_n) \leq r} |I(z)|.$$

Since $\{\alpha_n\}$ is an interpolating sequence, this contradicts (3.7). Therefore $f \notin QA_\pi$.

4. A question about uniqueness.

The question whether (NP) has a unique solution is in general delicate. A necessary condition for uniqueness is of course that $\|f\|_\infty = 1$ for any solution f to (NP). If there is $f_0 \in H^\infty$ with $\|f_0\|_\infty < 1$ solving the reduced problem $f(z_n) = w_n$, $n \geq N$ for some $N \geq 2$, we shall call (NP) semiscaled. In [11], Tolokonnikov obtained the following nice result

Theorem 2. (Tolokonnikov). *If a Nevanlinna Pick problem is semiscaled, but not scaled, then any solution is inner and hence must be unique.*

It should be observed that previous results due to T. Nakazi [3] and K.O. Oyma [7] easily follow from Theorem 2.

Proof. Let us use the notation from the introduction and assume that the Nevanlinna Pick problem (NP) is scaled. One can assume $N = 1$. If $\{z_0, w_0\}$ is an extra pair of points consider the extended problem

$$(*) \quad f(z_n) = w_n, \quad n = 0, 1, 2, \dots, \quad f \in U.$$

One can assume $z_0 = 0$. The sets $F = \{f \in H^\infty : \|f\|_\infty \leq 1, f(z_n) = w_n, n \geq 1\}$ and $B = \{f(0) : f \in F, \|f\|_\infty < 1\}$ are convex. Suppose B is non-empty and that the only functions in F with $f(0) = w_0$ have norm 1. We will show that such f are inner. Since the average of two inner functions is not inner, this will also prove uniqueness.

If $\|f\|_\infty \leq 1$, $\|g\|_\infty < 1$ and $0 < \epsilon < 1$, then $\|\epsilon g + (1 - \epsilon)f\|_\infty < 1$, and hence $\overline{B} = \{f(0) : f \in F\}$. The assumptions mean that $w_0 \in \overline{B} \setminus B$. The proof in [2, p. 152] works verbatim, and shows that any $f \in F$ with $f(z_0) \in \partial B$ must be inner. \square

References

- [1] S.Y. Chang and D. Marshall, *Some algebras of bounded analytic functions containing the disc algebra*, Lecture Notes in Math. vol. 604, p. 12-20, Springer-Verlag.
- [2] J. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.
- [3] T. Nakazi, *Notes on interpolation by bounded analytic functions*, Canad. Math. Bull., **31** (4) 1988.
- [4] R. Nevanlinna, *Über veschränkte analytische Funktionen*, Ann. Acad. Sci. Fenn, **32** n.7, (1929).
- [5] A. Nicolau, *Interpolating Blaschke products solving Pick Nevanlinna problems*, Journal d'Analyse Mat., **62** (1994), 199-224.
- [6] N.K. Nikolskii, *Treasure on the shift operator*, Springer-Verlag 1986.
- [7] K.O. Oyma, *Extremal interpolating functions in H^∞* , Proc. Amer. Math. Soc., **64** (1977), 272-276.

- [8] A. Stray, *Minimal interpolation by Blaschke products, I*, J. London Math. Soc. (2), **32** (1985), 488-496.
- [9] ———, *Minimal interpolation by Blaschke products, II*, Bull. London Math. Soc., **20** (1988), 329-332.
- [10] ———, *Interpolating sequences and the Nevanlinna Pick problem*, Publicacions Matemàtiques vol. 35, n.2, 1991, 507-516.
- [11] V. Tolokonnikov, *Extremal functions of the Nevanlinna Pick problem and Douglas algebras*, preprint.

Received May 24, 1993 and revised March 7, 1994. The first author was supported in part by DGICYT grant PB89-0311, Spain. The second author was supported in part by NAVF.

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