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UNIT INDICES OF SOME IMAGINARY COMPOSITE  
QUADRATIC FIELDS. II

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## UNIT INDICES OF SOME IMAGINARY COMPOSITE QUADRATIC FIELDS II

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Let  $K$  be an imaginary abelian number field of type  $(2, 2, 2, 2)$  containing the 8-th cyclotomic field  $\mathbf{Q}(\sqrt{-1}, \sqrt{2})$ . Using the fundamental units of real quadratic subfields of  $K$ , we give a necessary and sufficient condition for the unit index  $Q_K$  of  $K$  to be equal to 2.

### 1. Introduction and Results.

Let  $K$  be an imaginary abelian number field and  $K_0$  the maximal real subfield of  $K$ . Let  $E$  and  $E_0$  be the groups of units of  $K$  and  $K_0$ , respectively, and let  $W$  be the group of roots of unity in  $K$ . Let  $Q_K$  be the unit index of  $K$ , i.e.,

$$Q_K = [E : WE_0].$$

In the previous paper [4] we gave a necessary and sufficient condition for  $Q_K$  to be equal to 2 when  $K$  is an imaginary abelian number field (whose Galois group is) of type  $(2, 2, 2, 2)$  not containing the 8-th cyclotomic field  $\mathbf{Q}(\sqrt{-1}, \sqrt{2})$ . In this paper we give such a condition when  $K$  contains  $\mathbf{Q}(\sqrt{-1}, \sqrt{2})$ .

In this paper we use the following notation, unless otherwise specified.

$\mathbf{N}, \mathbf{Z}, \mathbf{Q}$ : the sets of natural numbers, rational integers and rational numbers, respectively,

$\stackrel{=}{_2}$  (resp.  $\stackrel{=}{_2}$  in  $k$ ): the equality up to a rational quadratic factor (resp. the equality up to a square of a number of a field  $k$ ),

$d_1, d_2, \dots, d_7$ : square-free positive integers such that  $d_4 \stackrel{=}{_2} d_2 d_3$ ,  $d_5 \stackrel{=}{_2} d_3 d_1$ ,  $d_6 \stackrel{=}{_2} d_1 d_2$ ,  $d_7 \stackrel{=}{_2} d_1 d_2 d_3$  and that  $d_3 = 2$ .

$K = \mathbf{Q}(\sqrt{-1}, \sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}) = \mathbf{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{d_1}, \sqrt{d_2})$ : an imaginary abelian number field of type  $(2, 2, 2, 2)$ ,

$K_0 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$ ,

$E_0^+$ : the group of totally positive units of  $K_0$ ,

$$\begin{aligned} K_1 &= \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3}), & K_2 &= \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1}), & K_3 &= \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}), \\ K_4 &= \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2 d_3}), & K_5 &= \mathbf{Q}(\sqrt{d_2}, \sqrt{d_3 d_1}), & K_6 &= \mathbf{Q}(\sqrt{d_3}, \sqrt{d_1 d_2}), \\ K_7 &= \mathbf{Q}(\sqrt{d_2 d_3}, \sqrt{d_3 d_1}), \end{aligned}$$

$\sigma_i$  : a generator of  $\text{Gal}(K_0/K_i)$ , i.e.,  $\langle \sigma_i \rangle = \text{Gal}(K_0/K_i)$  ( $i = 1, 2, \dots, 7$ ),  
 $\varepsilon_i$  : the fundamental unit of  $k_i = \mathbf{Q}(\sqrt{d_i})$ ,  $\varepsilon_i > 1$  ( $i = 1, 2, \dots, 7$ ),  
 $N(x)$ ,  $Sp(x)$  : the absolute norm and the absolute trace of an algebraic number  $x$ , respectively.

For a totally positive unit  $\eta$  of  $K_0$ , let

$$\begin{aligned} (1) \quad & \xi = \xi(\eta) = \eta + \eta^{\sigma_1} + 2\sqrt{\eta\eta^{\sigma_1}}, \\ (2) \quad & \theta = \theta(\eta) = \xi + \xi^{\sigma_2} + 2\sqrt{\xi\xi^{\sigma_2}} \end{aligned}$$

under the condition that

$$(3) \quad \sqrt{\eta\eta^{\sigma_1}} \in K_1 \quad \text{and} \quad \sqrt{\xi\xi^{\sigma_2}} \in k_3.$$

Let  $\nu$  be the number of  $i$  for which  $N(\varepsilon_i) = -1$  ( $i = 1, 2, \dots, 7$ ), i.e.,

$$\nu = \#\{i \mid i = 1, 2, \dots, 7; N(\varepsilon_i) = -1\}.$$

**Remark 1.** Using Lemmas 3 and 6 we can show that the above condition (3) follows from the equations

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad \text{in } K_i \quad (i = 1, 2, 6).$$

Our result is

**Theorem.** (1) If  $\nu \geq 4$ , then  $Q_K = 1$ .

(2) Suppose that  $\nu = 3$  and that

$$N(\varepsilon_s) = N(\varepsilon_t) = N(\varepsilon_3) = -1$$

for  $s, t \in \{1, 2, \dots, 7\}$  ( $s \neq t$ ) different from 3. If  $d_s d_t \underset{2}{=} d_3$  does not hold, then  $Q_K = 1$ .

(3) Suppose that  $\nu \leq 2$  or that  $\nu = 3$  and  $d_s d_t \underset{2}{=} d_3$  holds for above  $s, t$ . Then  $Q_K = 2$  if and only if there exists a unit  $\eta$  in  $E_0^+$  such that

$$(4) \quad \eta = \prod_{i=1}^7 \varepsilon_i^{a_i} \cdot \sqrt{\prod_{N(\varepsilon_j)=+1} \varepsilon_j^{b_j}} \quad (a_i, b_j = 0 \text{ or } 1)$$

satisfying the following conditions (i), (ii) :

(i)

$$N_{K_0/K_\alpha}(\eta) \underset{2}{=} 1 \quad \text{in } K_\alpha \quad (\alpha = 1, 2, 6),$$

$$N_{K_0/K_\beta}(\eta) \underset{2}{=} 1 \quad \text{in } K_0, \text{ but not in } K_\beta \quad (\beta = 3, 4, 5, 7).$$

(ii)

$$\theta = \theta(\eta) = \frac{1}{2} \left( 2 + \sqrt{2} \right) d_1^{e_1} d_2^{e_2} \quad \text{in } k_3 = \mathbf{Q} \left( \sqrt{2} \right)$$

for some  $e_i \in \{0, 1\}$ .

Moreover, in the representation (4) of  $\eta$ , the number of  $j$ 's for which  $b_j = 1$  is greater than one.

**Remark 2.** When  $\nu = 3$  and  $d_s d_t = d_3$  holds for  $s, t$  in Theorem, we have examples of  $Q_K = 1$  and  $Q_K = 2$  :

If  $d_1 = 5, d_2 = 21$ , then  $Q_K = 1$ , which is checked by Proposition 1.

If  $d_1 = 7, d_2 = 41$ , then  $Q_K = 2$ . Because,

$$\eta = \sqrt{\varepsilon_1} \sqrt{\varepsilon_5} = \frac{1}{2} \left( 3\sqrt{2} + \sqrt{14} \right) \cdot \left( 2\sqrt{2} + \sqrt{7} \right)$$

satisfies the condition (3) of Theorem. In fact,

$$\theta = \theta(\eta) = \frac{1}{2} \left( 2 + \sqrt{2} \right) 7 \quad \text{in } k_3.$$

**Remark 3.** In the Theorem, when

$$\prod_{N(\varepsilon_j)=+1} \varepsilon_j^{b_j} = \varepsilon_{j_1} \varepsilon_{j_2},$$

it holds that  $d_{j_1} d_{j_2} = d_3 = 2$ , as seen in Lemma 5 (2).

The assertions (1) and (2) of the Theorem are easily obtained in §3 from

**Proposition 1.** Let  $L$  be the composite of a 2-power-th cyclotomic field  $\mathbf{Q}(\zeta)$  ( $\zeta = \exp(2\pi i/2^m), m \geq 2$ ) and  $n$  independent real quadratic fields  $\mathbf{Q}(\sqrt{D_i})$  where  $D_i$  are square-free positive integers ( $i = 1, 2, \dots, n$ ), that is,

$$L = \mathbf{Q} \left( \zeta, \sqrt{D_1}, \sqrt{D_2}, \dots, \sqrt{D_n} \right).$$

If  $D_1 \equiv D_2 \equiv \dots \equiv D_n \equiv 1 \pmod{4}$ , then  $Q_L = 1$ .

## 2. Characterization of $\eta \in \overline{E}_0$ .

Our argument depends on

**Lemma 1** (cf. [3, Satz 15]).  $Q_K = 2$  if and only if there exists a unit  $\eta \in E_0^+$  such that  $K_0(\sqrt{\eta}) = K_0(\sqrt{2 + \sqrt{2}})$ .

Therefore, in order to determine the alternative  $Q_K = 1$  or 2, we investigate such  $\eta \in E_0^+$ . We replace the definition of  $\overline{E}_0$  in [4] by

$$\overline{E}_0 = \left\{ \eta \in E_0^+ \mid K_0(\sqrt{\eta}) = K_0(\sqrt{2 + \sqrt{2}}) \right\}.$$

Here we note that if  $\eta \in \overline{E}_0$ ,  $\eta$  is totally positive.

**Lemma 2** (cf. [4, Lemma 1]). *For  $\eta \in \overline{E}_0$ , we have*

$$\eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7}$$

for some  $x_i \in \mathbf{Z}$ .

*Proof.* For  $\eta \in \overline{E}_0$ , we can put

$$\eta^4 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

In fact, for a  $(2, 2)$ -extension  $K/k$  with Galois group  $\text{Gal}(K/k) = \langle \sigma, \tau \rangle$  we have

$$\alpha^2 = \frac{\alpha^{1+\sigma} \alpha^{1+\tau}}{(\alpha^\sigma)^{1+\sigma\tau}}$$

for any  $\alpha \in K, \alpha \neq 0$ . By this formula we see that  $E_0^4 \subseteq E_0^*$ , where  $E_0^*$  is the subgroup of  $E_0$  generated by  $\pm \varepsilon_i$  ( $i = 1, 2, \dots, 7$ ).

We show that every  $x_i$  is even.

Since  $K_0(\sqrt{\eta}) = K_0\left(\sqrt{2 + \sqrt{2}}\right)$ , we have  $\eta = (2 + \sqrt{2})\alpha_0^2$  for some  $\alpha_0 \in K_0$ . Then

$$(5) \quad (2 + \sqrt{2})^4 \alpha_0^8 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7}.$$

Taking the norms  $N_{K_0/k_3}$  and  $N_{K_0/k_i}$  ( $i \neq 3$ ) of this equation (5) and then the positive fourth root, we have

$$(2 + \sqrt{2})^4 N_{K_0/k_3}(\alpha_0)^2 = \varepsilon_3^{x_3} \text{ and } 2^2 N_{K_0/k_i}(\alpha_0)^2 = \varepsilon_i^{x_i},$$

respectively. Here we recall that  $\varepsilon_3$  and  $\varepsilon_i$  are positive. These equations show that  $\varepsilon_i^{x_i}$  is square in  $k_i$  and hence  $x_i \equiv 0 \pmod{2}$  for every  $i$ .  $\square$

**Lemma 3** ([2, Satz 1]). *Let  $K_1$  be a field with  $\text{char}(K_1) \neq 2$  and  $K_0$  a quadratic extension over  $K_1$ . Let  $\eta$  be an element of  $K_0$  which is not a square in  $K_0$ .*

- (1)  $K_0(\sqrt{\eta})/K_1$  is Galois  $\iff N_{K_0/K_1}(\eta) \equiv 1 \pmod{2}$  in  $K_0$ .
- (2)  $K_0(\sqrt{\eta})/K_1$  is an extension of type  $(2, 2)$   $\iff N_{K_0/K_1}(\eta) \equiv 1 \pmod{2}$  in  $K_1$ .
- (3)  $K_0(\sqrt{\eta})/K_1$  is cyclic  $\iff N_{K_0/K_1}(\eta) \equiv 1 \pmod{2}$  in  $K_0$ , but not in  $K_1$ .

**Lemma 4** (cf. [4, Lemma 3]). *Let  $\eta \in \overline{E}_0$  and put*

$$\eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

- (1) If there exists an even  $x_i$ , then  $N(\varepsilon_j) = +1$  for each odd  $x_j$ .  
 (2) If  $x_1 \equiv x_2 \equiv \cdots \equiv x_7 \equiv 1 \pmod{2}$ , then  $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7)$ .

We can prove this Lemma 4 as in the same way in [4, Lemma 3].

**Lemma 5.** Let  $\eta \in \overline{E}_0$  and put

$$(6) \quad \eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

- (1) There exist at least two odd integers among the  $x_i$ 's.  
 (2) If  $x_i, x_j$  ( $i \neq j$ ) are odd and the others  $x_k$  are even, then  $d_i \neq 2, d_j \neq 2$  and  $d_i d_j = 2$ .

*Proof of Lemma 5.* (1) First we suppose that all  $x_i$  are even. Then  $\eta$  is a product of some of  $\varepsilon_i$ 's. Noting that  $\eta$  is contained in  $(E_0^*)^+ = E_0^* \cap E_0^+$ , we see by [4, Proposition 1] that  $\eta$  is, up to a square, a product of some of following totally positive units :

$$\begin{aligned} \varepsilon_i & \quad (\text{when } N(\varepsilon_i) = +1), \\ \eta_{ij} &:= \varepsilon_i \varepsilon_j \varepsilon_k \quad (\text{when } d_i d_j = d_k \text{ and } N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = -1), \\ \eta_{ijk} &:= \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l \quad (\text{when } d_i d_j d_k = d_l \text{ and } N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = N(\varepsilon_l) \\ & \quad = -1). \end{aligned}$$

For a unit  $\varepsilon_i$  with  $N(\varepsilon_i) = +1$  we have

$$\eta Sp(\xi) = \xi^2$$

where  $\eta = \varepsilon_i$  and  $\xi = \varepsilon_i + 1$ . For  $\eta = \eta_{ij}$  or  $\eta_{ijk}$  we also have by [5, Proof of Zusatz 1] or by [4, Lemma 6] that

$$\eta Sp(\xi) = \xi^2$$

where

$$\xi = \varepsilon_i \varepsilon_j \varepsilon_k - \varepsilon_i - \varepsilon_j - \varepsilon_k$$

or

$$\xi = \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l + 1 - (\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_k + \varepsilon_k \varepsilon_i + \varepsilon_i \varepsilon_l + \varepsilon_j \varepsilon_l + \varepsilon_k \varepsilon_l),$$

respectively. Therefore,  $K_0(\sqrt{\varepsilon_i})$ ,  $K_0(\sqrt{\eta_{ij}})$  and  $K_0(\sqrt{\eta_{ijk}})$  are 2-elementary extensions over  $\mathbf{Q}$  and so is  $K_0(\sqrt{\eta})$ , which contradicts  $\eta \in \overline{E}_0$ .

Next we suppose that  $x_i$  is odd and the other  $x_k$  are even. Choose  $K_j$  for which  $\sqrt{d_i} \notin K_j$ . Taking the norm  $N_{K_0/K_j}$  of the equation (6), we have

$$N_{K_0/K_j}(\eta)^2 = N(\varepsilon_i)^{x_i} \varepsilon_u^{2x_u} \varepsilon_v^{2x_v} \varepsilon_w^{2x_w}$$

where  $K_j = \mathbf{Q}(\sqrt{d_u}, \sqrt{d_v})$  and  $d_w = d_u d_v$ . Hence,  $N(\varepsilon_i) = +1$  and so  $i \neq 3$ . (Then, as for above  $j$ , we can take  $j = 3, 4, 5$  or  $7$ .) Moreover, since  $x_u, x_v$  and  $x_w$  are even, we have

$$N_{K_0/K_j}(\eta) = \varepsilon_u^{x_u} \varepsilon_v^{x_v} \varepsilon_w^{x_w} = 1 \quad \text{in } K_j.$$

Therefore it follows from Lemma 3 that  $K_0(\sqrt{\eta})/K_j$  is of type  $(2, 2)$ . However, the extension  $K_0(\sqrt{\eta})/K_j = K_0\left(\sqrt{2 + \sqrt{2}}\right)/K_j$  is itself a cyclic extension of degree 4. Thus we get a contradiction.

(2) Choose  $k \in \{1, 2, \dots, 7\}$  for which  $\sqrt{d_i} \in K_k$  and  $\sqrt{d_j} \notin K_k$ . Taking the norm  $N_{K_0/K_k}$  of the equation (6), we have

$$N_{K_0/K_k}(\eta)^2 = \varepsilon_i^{2x_i} N(\varepsilon_j)^{x_j} \eta_k^2$$

where  $\eta_k$  is a unit of  $K_k$ . Hence  $N(\varepsilon_j) = +1$  and so  $d_j \neq d_3 = 2$ .

By exchanging  $i$  and  $j$ , we also have  $N(\varepsilon_i) = +1$  and  $d_i \neq d_3$ .

Finally we show that  $d_i d_j = 2$ . Assume that this is false. Then,  $K_l := \mathbf{Q}(\sqrt{d_i d_3}, \sqrt{d_j d_3})$  contains neither  $\sqrt{d_i}$  nor  $\sqrt{d_j}$ . Taking the norm  $N_{K_0/K_l}$  of (6) and then the positive square root, we obtain

$$N_{K_0/K_l}(\eta) = \varepsilon_\alpha^{x_\alpha} \varepsilon_\beta^{x_\beta} \varepsilon_\gamma^{x_\gamma} = 1 \quad \text{in } K_l$$

where  $d_\alpha = d_i d_3, d_\beta = d_j d_3$  and  $d_\gamma = d_\alpha d_\beta$ , because,  $x_\alpha, x_\beta$  and  $x_\gamma$  are even. Therefore, it follows from Lemma 3 (2) that  $K_0(\sqrt{\eta})/K_l$  is an extension of type  $(2, 2)$ . However, by the definition of  $K_l$ ,  $K_l$  does not contain  $\sqrt{d_3}$  and so  $K_l \neq K_1, K_2$  or  $K_6$ . Hence  $K_0(\sqrt{\eta})/K_l$  is a cyclic extension of degree 4, which is a contradiction.  $\square$

### 3. Proofs of Proposition 1 and Theorem.

*Proof of Proposition 1.* Let  $f(\chi)$  be the conductor of a Dirichlet character  $\chi$ . For any even character  $\chi_0$  of  $L$ , we have  $2 \nmid f(\chi_0)$  or  $2^3 \mid f(\chi_0)$  and  $2^{m+1} \nmid f(\chi_0)$ . Then, from [2, Satz 22] it follows that  $Q_L = 1$ .  $\square$

**Remark 4.** Proposition 1 is also proved in [1 (14.7) Corollary and the comment on p. 87 - 88].

*Proof of (1), (2) of Theorem.* By the assumption we have

$$K = \mathbf{Q}\left(\sqrt{-1}, \sqrt{2}, \sqrt{d_s}, \sqrt{d_t}\right), \quad N(\varepsilon_s) = N(\varepsilon_t) = N(\varepsilon_3) = -1$$

for suitable  $d_s, d_t \neq d_3$ . Then for every odd prime  $p$  dividing  $d_s d_t$ , we have  $p \equiv 1 \pmod{4}$ . In fact, for example, by  $N(\varepsilon_s) = -1$  we have  $x^2 - d_s y^2 = -4$

for some  $x, y \in \mathbf{Z}$ . Then, for an odd prime  $p$  dividing  $d_s$ ,  $x^2 \equiv -4 \pmod{p}$  and hence  $(-1/p) = (-1)^{\frac{p-1}{2}} = 1$ , where  $(\ / )$  is the Legendre symbol. Thus we get  $p \equiv 1 \pmod{4}$ .

Therefore

$$K = \mathbf{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{D_s}, \sqrt{D_t})$$

for some  $D_s, D_t \in \mathbf{N}$ ,  $D_s \equiv D_t \equiv 1 \pmod{4}$ . Thus Proposition 1 implies that  $Q_K = 1$ .  $\square$

In the following we prove the assertion (3) of Theorem, for which we need

**Proposition 2.** *Let  $K$  and  $K_0$  be as in the notation in §1. Let  $\eta$  be an element of  $K_0$  which is not square in  $K_0$ .*

(1)  $K_0(\sqrt{\eta})/\mathbf{Q}$  is a Galois extension if and only if

$$(7) \quad N_{K_0/K_i}(\eta) \equiv 1 \pmod{2} \text{ in } K_0 \quad (i = 1, 2, \dots, 7).$$

(2)  $K_0(\sqrt{\eta})/\mathbf{Q}$  is an abelian extension of type  $(2, 2, 2, 2)$  if and only if

$$(8) \quad N_{K_0/K_i}(\eta) \equiv 1 \pmod{2} \text{ in } K_i \quad (i = 1, 2, \dots, 7).$$

(3)  $K_0(\sqrt{\eta})/\mathbf{Q}$  is an abelian extension of type  $(2, 2, 4)$  and  $K_0(\sqrt{\eta})/k_3$  of type  $(2, 2, 2)$  if and only if

$$(9) \quad \begin{cases} N_{K_0/K_\alpha}(\eta) \equiv 1 \pmod{2} \text{ in } K_\alpha & (\alpha = 1, 2, 6), \\ N_{K_0/K_\beta}(\eta) \equiv 1 \pmod{2} \text{ in } K_0, \text{ but not in } K_\beta & (\beta = 3, 4, 5, 7). \end{cases}$$

**Remark 5.** This Proposition 2 remains valid if  $K_0 = \mathbf{Q}(\sqrt{2}, \sqrt{d_1}, \sqrt{d_2})$  is replaced by  $K_0 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$  with arbitrary  $d_3 \in \mathbf{N}$  ( $d_3$  : square-free,  $d_3 \geq 2$ ). Therefore, the condition (8) leads to the condition (5) of [4].

For the proof of Proposition 2, we need the following two lemmas.

**Lemma 6.** *Let  $k$  be an algebraic number field. Let  $K_0/k$  be an abelian extension of type  $(2, 2)$ . Let  $K_1, K_2$  and  $K_3$  be the intermediate fields of  $K_0/k$ . Let  $\eta$  be an element of  $K_0$ .*

(1)  $K_0(\sqrt{\eta})/k$  is a Galois extension if and only if



$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad \text{in } K_0 \quad (i = 1, 2, 3).$$

(2) Suppose that  $K_0(\sqrt{\eta})/k$  is a Galois extension. Let

$$\mu = \#\{i \mid i = 1, 2, 3; N_{K_0/K_i}(\eta) \underset{2}{=} 1 \text{ in } K_i\}.$$

Then,  $K_0(\sqrt{\eta})/k$  is quaternion, abelian of type (2, 4), dihedral or abelian of type (2, 2, 2) if and only if  $\mu = 0, 1, 2$  or 3, respectively.

**Lemma 7.** Let  $G$  be a group of order 16. Assume that there exists a normal subgroup  $N$  of  $G$  of order 2 with quotient group  $G/N$  of type (2, 2, 2). Then  $G$  is isomorphic to one of the followings :

- (a) a 2-elementary group
- (b) an abelian group of type (2, 2, 4)
- (c) a central product of an abelian subgroup  $A$  and a dihedral or quaternion subgroup  $B$  of order 8 such that  $AB = G, A \cap B = N$ . ( $A$  is the center of  $G$ .)

Lemma 6 is an immediate consequence of Lemma 3. Lemma 7 is a special case of [6, (4.16) and Theorem 4.18].

*Proof of Proposition 2.* (1) Suppose that  $K_0(\sqrt{\eta})/\mathbf{Q}$  is a Galois extension. Then, for any quadratic subfield  $k$  of  $K_0$ ,  $K_0(\sqrt{\eta})/k$  is also a Galois extension. Hence, by Lemma 6 (1) we have

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad \text{in } K_0$$

for every intermediate field  $K_i$  of  $K_0/k$ .

Conversely, suppose that the condition (7) is satisfied. For an automorphism  $\sigma$  of the algebraic closure  $\overline{\mathbf{Q}}$  of  $\mathbf{Q}$ , the restriction  $\sigma|_{K_0}$  of  $\sigma$  to  $K_0$  belongs to the Galois group  $\text{Gal}(K_0/\mathbf{Q}) = \{\sigma_0 = 1, \sigma_1, \dots, \sigma_7\}$ . Then

$$\sigma|_{K_0} = \sigma_i$$

for some  $i$ . By the assumption, we have

$$\eta\eta^{\sigma_i} = \eta_i^2$$

for some  $\eta_i \in K_0$ . Therefore,

$$\sqrt{\eta}^\sigma = \pm\sqrt{\eta}^\sigma = \pm\frac{\eta_i}{\sqrt{\eta}}$$

is contained in  $K_0(\sqrt{\eta})$  and whence  $K_0(\sqrt{\eta})/\mathbf{Q}$  is a Galois extension.

(2), (3) At first, we suppose that  $K_0(\sqrt{\eta})/\mathbf{Q}$  is a Galois extension with Galois group  $G$ . Let  $N$  be the subgroup of  $G$  corresponding to  $K_0$ .

Here we assume that  $G$  is not abelian. Then, it follows from Lemma 7 that  $G$  is a central product of an abelian subgroup  $A$  and a non-abelian subgroup  $B$  of degree 8. Let  $k$  be the subfield of  $K_0(\sqrt{\eta})$  corresponding to  $B$ . Since  $A \cap B = N$  and since  $B$  is of order 8,  $k$  is a quadratic subfield of  $K_0$ , i.e.,  $k = k_a$  for some  $a \in \{1, 2, \dots, 7\}$ . Then,  $K_0(\sqrt{\eta})/k_a$  is a quaternion or dihedral extension. Let  $K'_i$  ( $i = 1, 2, 3$ ) be the intermediate fields of  $K_0/k_a$  and let

$$\mu = \#\{i \mid N_{K_0/K'_i}(\eta) = 1 \text{ in } K'_i\}.$$

Then, by Lemma 6 (2) we have  $\mu = 0$  or 2.

Now, suppose that the condition (9) is satisfied. Then,  $K_0(\sqrt{\eta})/\mathbf{Q}$  is a Galois extension with Galois group  $G$ . If  $G$  is not abelian, then, for above  $\mu$  and  $a$ , we have by the condition (9) that  $\mu = 3$  or 1 according as  $a = 3$  or not, which is a contradiction. Therefore  $G$  must be abelian.

Moreover, the equations

$$N_{K_0/K_\beta}(\eta) = 1 \text{ not in } K_\beta \quad (\beta = 3, 4, 5, 7)$$

imply that  $K_0(\sqrt{\eta})/K_\beta$  is cyclic. Hence it follows from Lemma 7 that  $K_0(\sqrt{\eta})/\mathbf{Q}$  is an abelian extension of type  $(2, 2, 4)$ . And the equations

$$N_{K_0/K_\alpha}(\eta) = 1 \text{ in } K_\alpha \quad (\alpha = 1, 2, 6)$$

imply that  $K_0(\sqrt{\eta})/k_3$  is an abelian extension of type  $(2, 2, 2)$ .

Next, suppose that the condition (8) is satisfied. In a similar way we see that  $K_0(\sqrt{\eta})/\mathbf{Q}$  is an abelian extension.

We show that  $K_0(\sqrt{\eta})/\mathbf{Q}$  is of type  $(2, 2, 2, 2)$ . Assume that this is false, i.e., assume that  $K_0(\sqrt{\eta})/\mathbf{Q}$  is of type  $(2, 2, 4)$ . Let, as above,

$$G = \text{Gal}(K_0(\sqrt{\eta})/\mathbf{Q}), \quad N = \text{Gal}(K_0(\sqrt{\eta})/K_0).$$

Then,

$$G/N \cong \text{Gal}(K_0/\mathbf{Q})$$

is of type  $(2, 2, 2)$ . By the assumption there exists an element  $\sigma$  of  $G$  of order 4. Since the order of the coset  $\sigma N$  of  $G/N$  is at most 2,  $\sigma^2$  is contained in  $N$ . Hence  $N = \langle \sigma^2 \rangle$ , because  $N$  has order 2. Let  $K_i$  be the subfield of  $K_0$  corresponding to  $\langle \sigma^i \rangle$ . Then  $K_0(\sqrt{\eta})/K_i$  is cyclic. Hence, by Lemma 3 (3), we have

$$N_{K_0/K_i}(\eta) = 1 \text{ not in } K_i,$$

which is a contradiction to the condition (8).

Thus we have proved the sufficiencies of (2) and (3) of Proposition 2.

Conversely, their necessities are immediately deduced from Lemma 3 .

□

For the proof of (3) of Theorem, we also need

**Lemma 8** ([4, Lemma 5]). *Let  $K_1$  be an algebraic number field and  $K_0$  a quadratic extension of  $K_1$ . Let  $K_0(\sqrt{\eta_0})$  ( $\eta_0 \in K_0$ ,  $\eta_0 \notin K_1$ ) be a bi-quadratic bicyclic extension of  $K_1$  with  $\text{Gal}(K_0(\sqrt{\eta_0})/K_1) = \langle \sigma, \tau \rangle$  and  $\text{Gal}(K_0(\sqrt{\eta_0})/K_0) = \langle \tau \rangle$ . Let  $F$  be the intermediate field of  $K_0(\sqrt{\eta_0})/K_1$  fixed by  $\sigma$ . Then we have*

$$F = K_1(\sqrt{\eta_0} + \sqrt{\eta_0}^\sigma).$$

*Proof of (3) of Theorem.* Suppose that  $Q_K = 2$ . Then, by Lemma 1 there exists a unit  $\eta$  in  $E_0^+$  such that

$$K_0(\sqrt{\eta}) = K_0\left(\sqrt{2 + \sqrt{2}}\right).$$

By Lemma 2 we have

$$\eta^2 = \varepsilon_1^{x_1} \varepsilon_2^{x_2} \cdots \varepsilon_7^{x_7}$$

for some  $x_i \in \mathbf{Z}$  ( $i = 1, 2, \dots, 7$ ). And we see by Lemma 5 (1) that there are at least two odd integers among  $x_i$ 's.

If all  $x_i$  are odd, then it follows from Lemma 4 (2) that

$$N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = \cdots = N(\varepsilon_7) = -1,$$

and so  $\nu = 7$ , which contradicts our assumption  $\nu \leq 3$ . Then there exists at least one even integer among  $x_i$ 's. Hence Lemma 4 (1) implies that  $N(\varepsilon_i) = +1$  for odd  $x_i$ . Therefore we may represent the  $\eta$  in question as

$$\eta = \prod_{i=1}^7 \varepsilon_i^{a_i} \cdot \sqrt[N(\varepsilon_j)=+1]{\prod \varepsilon_j^{b_j}} \quad (a_i, b_j = 0 \text{ or } 1),$$

and Lemma 5 (1) shows that there are at least two  $b_j = 1$ .

Since  $K_0(\sqrt{\eta}) = K_0\left(\sqrt{2 + \sqrt{2}}\right)$  is an extension of type  $(2, 2, 4)$  over  $\mathbf{Q}$  and of type  $(2, 2, 2)$  over  $k_3 = \mathbf{Q}(\sqrt{2})$ , Proposition 2 (3) implies the condition (3) (i) of Theorem.

Moreover, it follows from Lemma 8 that  $K_1(\sqrt{\xi}) = K_1(\sqrt{\eta_0} \pm \sqrt{\eta_0}^\sigma)$  is the intermediate field of  $K_0(\sqrt{\eta})/K_1$  fixed by  $\sigma$  or  $\tau\sigma$ , where  $\sigma$  is an automorphism of  $\overline{\mathbf{Q}}$  over  $\mathbf{Q}$  such that  $\sigma|_{K_0} = \sigma_1$ ,  $\langle \sigma_1 \rangle = \text{Gal}(K_0/K_1)$  and  $\tau$  is a generator of  $\text{Gal}(K_0(\sqrt{\eta})/K_0)$ . Consequently we have  $K_1(\sqrt{\xi}) \neq K_0$ . Similarly we can show that  $k_3(\sqrt{\theta})$  is an intermediate field of  $K_1(\sqrt{\xi})/k_3$  and that  $k_3(\sqrt{\theta}) \neq K_1$ . Therefore

$$k_3(\sqrt{\theta}) = k_3\left(\sqrt{(2 + \sqrt{2})d_1^{e_1}d_2^{e_2}}\right)$$

for some  $e_i \in \{0, 1\}$ . Thus we obtain the condition (3) (ii) of Theorem.

Conversely, suppose that there exists a unit  $\eta \in E_0^+$  satisfying the conditions (3) (i), (ii) of Theorem. Then, it follows from Proposition 2 (3) that  $K_0(\sqrt{\eta})$  is of type  $(2, 2, 4)$  over  $\mathbf{Q}$  and of type  $(2, 2, 2)$  over  $k_3 = \mathbf{Q}(\sqrt{2})$ . By Lemma 8, we see that  $K_1(\sqrt{\xi})$  is an intermediate field of  $K_0(\sqrt{\eta})/K_1$  and  $K_1(\sqrt{\xi}) \neq K_0$ . Then we have

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\xi}).$$

In the same way we get

$$K_1(\sqrt{\xi}) = K_1(\sqrt{\theta}).$$

Therefore,

$$K_0(\sqrt{\eta}) = K_0(\sqrt{\xi}) = K_0(\sqrt{\theta}).$$

By the condition (3) (ii) of Theorem we have

$$K_0(\sqrt{\theta}) = K_0\left(\sqrt{2 + \sqrt{2}}\right).$$

Thus we obtain

$$K_0(\sqrt{\eta}) = K_0\left(\sqrt{2 + \sqrt{2}}\right),$$

from which Lemma 1 implies  $Q_K = 2$ , as desired.  $\square$

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# PACIFIC JOURNAL OF MATHEMATICS

Volume 173      No. 1      March 1996

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Isometric immersions of $H_1^n$ into $H_1^{n+1}$	1
KINETSU ABE	
Rotationally symmetric hypersurfaces with prescribed mean curvature	29
MARIE-FRANÇOISE BIDAUT-VÉRON	
The covers of a Noetherian module	69
JIAN-JUN CHUAI	
On the odd primary cohomology of higher projective planes	77
MARK FOSKEY and MICHAEL DAVID SLACK	
Unit indices of some imaginary composite quadratic fields. II	93
MIKIHITO HIRABAYASHI	
Mixed automorphic vector bundles on Shimura varieties	105
MIN HO LEE	
Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights	127
PENG LIN and RICHARD ROCHBERG	
On quadratic reciprocity over function fields	147
KATHY DONOVAN MERRILL and LYNNE WALLING	
$(A_2)$ -conditions and Carleson inequalities in Bergman spaces	151
TAKAHIKO NAKAZI and MASAHIRO YAMADA	
A note on a paper of E. Boasso and A. Larotonda: "A spectral theory for solvable Lie algebras of operators"	173
C. OTT	
Tensor products with anisotropic principal series representations of free groups	181
CARLO PENSAVALLE and TIM STEGER	
On Ricci deformation of a Riemannian metric on manifold with boundary	203
YING SHEN	
The Weyl quantization of Poisson $SU(2)$	223
ALBERT JEU-LIANG SHEU	
Weyl's law for $SL(3, \mathbb{Z}) \backslash SL(3, \mathbb{R}) / SO(3, \mathbb{R})$	241
ERIC GEORGE STADE and DOROTHY IRENE WALLACE (ANDREOLI)	
Minimal hyperspheres in two-point homogeneous spaces	263
PER TOMTER	
Subalgebras of little Lipschitz algebras	283
NIKOLAI ISAAC WEAVER	