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TENSOR PRODUCTS WITH ANISOTROPIC PRINCIPAL SERIES REPRESENTATIONS OF FREE GROUPS

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Let τ be the tensor product of an anisotropic principal series representation of a free group Γ, not an endpoint representation, with an irreducible unitary finite dimensional Γ-representation. Usually τ is irreducible and has exactly two perfect boundary realizations. In a certain well specified anomalous case τ splits into two irreducible components and each component has exactly one boundary realization, which is not perfect.

1. Introduction.

Let Γ be a noncommutative free group on finitely many generators. Fix a basis A_+ for Γ and let $A = A_+ \cup A_+^{-1}$ consist of the basis elements and their inverses. Let π be a representation from one of the *anisotropic principal series* of [**Figà-Talamanca–Steger**]. Let ρ be an irreducible unitary finite dimensional representation of Γ. The aim of this paper is to apply the results of [**Steger**] to the tensor product $\pi \otimes \rho$. In particular, we show that the representation $\pi \otimes \rho$ is irreducible under most circumstances, and decomposes into two irreducible components when it does decompose.

Anisotropic principal series representations. The following is taken from [**Figà-Talamanca–Steger**]. (That paper deals with a free product of copies of $\mathbf{Z}/2$. The case of a free group is somewhat less complicated.) Let μ be a symmetric probability measure on A, suppose that $\mu(a) > 0$ for $a \in A$, and denote by $\mathrm{Spec}(\mu)$ the spectrum of μ acting on $\ell^2(\Gamma)$ by left convolution. Then $\mathrm{Spec}(\mu)$ consists of a closed interval symmetric about 0. Let σ belong to the interior of that interval. There is a unitary representation, π, described below, determined by the pair (μ, σ). Fixing μ and letting σ vary we obtain a series of representations whose direct integral is the regular representation of Γ. Call such a series an *anisotropic principal series* for (Γ, A), and call the representation π an *anisotropic principal series representation.*

Fix (μ, σ). Recall that each $x \in \Gamma$ has a unique shortest expression, $a_1 a_2 \cdots a_L$, as a product of elements in A. That shortest expression is the *reduced word* for x, characterized by the property that $a_l a_{l+1} \neq e$ for any l.

The *length*, $|x|$, of x as above is L. A nonzero function $f\colon \Gamma \to \mathbf{C}$ is called *multiplicative* if

$$f(xy) = f(x)f(y) \qquad \text{when } |xy| = |x| + |y|. \tag{1.1}$$

A multiplicative function is determined by its values on A. For $\gamma \notin \operatorname{Spec}(\mu)$, the inverse of left convolution by $\gamma\delta_e - \mu$ on $\ell^2(\Gamma)$ is left convolution by $g_\gamma = h_\gamma/(2w_\gamma)$, where h_γ is a multiplicative function and w_γ is a constant. Let

$$\begin{aligned} h_\pm &= h_{\sigma\pm i0} = \lim_{\delta\to 0+} h_{\sigma\pm i\delta} \\ w_\pm &= w_{\sigma\pm i0} = \lim_{\delta\to 0+} w_{\sigma\pm i\delta} \\ g_\pm &= g_{\sigma\pm i0} = h_\pm/(2w_\pm) \\ \phi &= \phi_\sigma = (g_+ - g_-)/(g_+(e) - g_-(e)) \,. \end{aligned} \tag{1.2}$$

Then ϕ is a positive definite function on Γ, and the representation $(\pi, \mathcal{H})$ is defined by applying the construction of Gel'fand and Raikov to ϕ.

For later use, observe that since μ is symmetric, $h_\gamma(a) = h_\gamma(a^{-1})$ for all $a \in A$, and consequently the same holds for $h_\pm$. Also note that since μ is real, $h_{\bar\gamma} = \bar h_\gamma$, $w_{\bar\gamma} = \bar w_\gamma$, and consequently $h_\pm$, $w_\pm$, and $g_\pm$ are complex conjugates of one another.

Results for $\tau = \pi \otimes \rho$. Let $\mathcal{H} \otimes \mathcal{V}$ be the representation space of τ, where $\mathcal{H}$ is the representation space of π and $\mathcal{V}$ is the representation space of ρ. Define sgn: $\Gamma \longrightarrow \mathbf{C}^\times$ by $\operatorname{sgn}(x) = (-1)^{|x|}$.

Theorem 1.1. *Exclude the anomalous case that $\sigma = 0$ and $\rho \cong \rho \otimes \operatorname{sgn}$. Then τ is irreducible.*

In the anomalous case, τ splits into two irreducible representations. The full story of the anomalous case is in Section 4.

Theorem 1.2. *Exclude the anomalous case that $\sigma = 0$ and $\rho \cong \rho \otimes \operatorname{sgn}$. Then there is a dense subspace $\mathcal{Z} \subseteq \mathcal{H} \otimes \mathcal{V}$ and a constant $C_0 > 0$ so that*

$$\lim_{\epsilon\to 0+} C_0\epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} \langle w_1, \tau(x)w_2\rangle \overline{\langle w_3, \tau(x)w_4\rangle} = \langle w_1, w_3\rangle \overline{\langle w_2, w_4\rangle}$$

for w_1, $w_3 \in \mathcal{H} \otimes \mathcal{V}$ and w_2, $w_4 \in \mathcal{Z}$.

The boundary, Ω, of Γ. The statements of the other main results, and the proofs of all the results, depend on the idea of the boundary of Γ. See [**Figà-Talamanca–Nebbia**] or the introduction to [**Kuhn–Steger**] for more detailed discussions of the boundary. Give Γ the structure of a tree by putting

an edge between x and xa for $x \in \Gamma$ and $a \in A$. Let $d(\cdot,\cdot)$ be the distance on Γ given by this tree structure. Note that $|x| = d(e,x)$. A *geodesic* is a sequence $(x_l)_l \subseteq \Gamma$ such that $d(x_l, x_m) = |l-m|$. There is a unique geodesic between any two elements of Γ. The *boundary*, Ω, is conveniently defined as $\{\text{semiinfinite geodesics } (x_l)_{l=0}^{\infty}\}/\sim$ where two such geodesics are considered equivalent when some shift of the indices makes them cofinally equal. This space is also called the space of *ends* of the tree. Given $x \in \Gamma$ and $\omega \in \Omega$, there is a unique geodesic $[x,\omega)$ starting at x and representing ω, and this we think of as the geodesic from x to ω. Let

$$[e,\omega) = (e, \omega_1, \omega_2, \dots) \,. \tag{1.3}$$

There will exist $(a_l)_{l=1}^{\infty} \subseteq A$ satisfying $a_l a_{l+1} \neq e$ such that $\omega_L = a_1 a_2 \cdots a_L$. We may think of ω as the infinite reduced word $a_1 a_2 a_3 \dots$. For $z \in \Gamma$, define

$$\Gamma(z) = \{x \in \Gamma;\ z \in [e,x]\} \qquad \text{and} \qquad \Omega(z) = \{\omega \in \Omega;\ z \in [e,\omega)\} \,,$$

those reduced words, respectively finite or infinite, which start out with the word for z. Topologize $\Gamma \cup \Omega$ by taking the singleton subsets of Γ and the sets $(\Gamma(z) \cup \Omega(z))_{z\in\Gamma}$ as a basis of open sets. This topology makes $\Gamma \cup \Omega$ and Ω compact, with Γ discrete and with every point of Ω in the closure of Γ. Indeed, $\omega = \lim_{l\to\infty} \omega_l$ where ω_l is as in (1.3).

The left action of Γ on itself preserves the tree structure, and therefore it induces an action of Γ on Ω. We write that action as multiplication, and indeed, it is the obvious action by left multiplication of finite reduced words on infinite reduced words. This left action of Γ on $\Gamma \cup \Omega$ leaves the topology fixed.

Realization of π on the boundary. Define a probability measure ν on Ω by

$$\nu(\Omega(xa)) = |h_+(xa)|^2/(1 + |h_+(a)|^2) \qquad \text{for } x \in \Gamma,\ a \in A,\ |xa| = |x| + 1 \,.$$

This works since $\sum_{a\in A} |h_+(a)|^2/(1+|h_+(a)|^2) = 1$. Define a unitary representation $\pi'_{+\Gamma}$ of Γ on $L^2(\Omega, d\nu)$ by

$$(\pi'_{+\Gamma}(a)F)(\omega) = \begin{cases} h_+(a)F(a^{-1}\omega), & \text{if } \omega \notin \Omega(a) \\ h_+(a)^{-1}F(a^{-1}\omega), & \text{if } \omega \in \Omega(a) \end{cases} \tag{1.4}$$

for $a \in A$. This extends to a representation of Γ because $\pi'_{+\Gamma}(a)\pi'_{+\Gamma}(a^{-1}) = 1$. The action of general $x \in \Gamma$ is given by

$$(\pi'_{+\Gamma}(x)F)(\omega) = P_+(x,\omega)F(x^{-1}\omega)$$

for

$$P_+(x,\omega) = \frac{h_+(x_2)}{h_+(x_1)} \tag{1.5}$$

where $x = x_1x_2$ and x_1 is the last element common to the geodesics $[e,x]$ and $[e,\omega)$, that is, the longest common initial segment of the reduced words for x and ω. Unitarity of $\pi'_{+\Gamma}$ depends on

$$|P_+(x,\omega)|^2 = \frac{d\nu(x^{-1}\omega)}{d\nu(\omega)}\ , \tag{1.6}$$

the right hand side being a Radon–Nikodym derivative. This is easy to check for $x \in A$ and then follows for all x. One defines $\pi'_{-\Gamma}$ in just the same way, using h_- instead of h_+.

[**Figà-Talamanca–Steger**] exhibits unitary Γ-maps $\iota_\pm\colon \mathcal{H} \longrightarrow L^2(\Omega, d\nu)$, intertwining π to $\pi'_{\pm\Gamma}$ respectively. Also exhibited explicitly is the intertwining operator

$$J = \iota_-\iota_+^*\colon L^2(\Omega, d\nu) \longrightarrow L^2(\Omega, d\nu)$$

which satisfies $J\pi'_{+\Gamma}(x) = \pi'_{-\Gamma}(x)J$ for $x \in \Gamma$. Although complex conjugation intertwines $\pi'_{+\Gamma}$ and $\pi'_{-\Gamma}$ just as J does, J is a complex linear map, more complicated than simple conjugation.

Boundary representations. Define $*$-representations $\pi'_{\pm\Omega}$ of the commutative C^*-algebra $C(\Omega)$ on $L^2(\Omega, d\nu)$ by

$$(\pi'_{+\Omega}(G)F)(\omega) = (\pi'_{-\Omega}(G)F)(\omega) = G(\omega)F(\omega)\ . \tag{1.7}$$

The symbols $\pi'_{\pm\Gamma}$ and $\pi'_{\pm\Omega}$ are chosen so as to match the following definition from [**Steger**].

Definition 1.3. Let Γ be any free group and let Ω be its boundary. A $(\Gamma, C(\Omega))$-*representation* or *boundary representation,* π', of Γ on $\mathcal{H}'$ is a pair $(\pi'_\Gamma, \pi'_\Omega)$ such that

(1) π'_Γ is a unitary representation of Γ on $\mathcal{H}'$.

(2) π'_Ω is a $*$-representation of $C(\Omega)$ on $\mathcal{H}'$.

(3) For $x \in \Gamma$ and $G \in C(\Omega)$, $\pi'_\Gamma(x)\pi'_\Omega(G)\pi'_\Gamma(x^{-1}) = \pi'_\Omega(\lambda(x)G)$.

Here $(\lambda(x)G)(\omega) = G(x^{-1}\omega)$. The reader familiar with crossed product C^*-algebras will see that a boundary representation is just a representation of the crossed product algebra $\Gamma \ltimes C(\Omega)$. We will routinely drop the subscripts Γ and Ω on $\pi'_\pm$. One may easily verify that the two representations π'_+ defined by (1.4) and (1.7) fit together to give a boundary representation of Γ. One makes π'_- into a boundary representation in the same way. Note

that although π'_+ and π'_- are equivalent as Γ-representations and identical as $C(\Omega)$-representations, they are not equivalent as $(\Gamma, C(\Omega))$-representations.

Let ρ be an irreducible unitary representation of Γ on a finite dimensional Hilbert space $\mathcal{V}$. It is easily verified that the definitions

$$\tau'_{\pm\Gamma}(x) = \pi'_{\pm\Gamma}(x) \otimes \rho(x) \qquad \tau'_{\pm\Omega}(G) = \pi'_{\pm\Omega}(G) \otimes \mathrm{id}$$

give boundary representations $\tau'_\pm$ acting on $L^2(\Omega, d\nu) \otimes \mathcal{V}$. Indeed this simple construction applies generally, yielding a boundary representation as the tensor product of any boundary representation and any ordinary Γ-representation. Identifying $L^2(\Omega, d\nu) \otimes \mathcal{V}$ with $L^2(\Omega, d\nu; \mathcal{V})$, (see [**Reed–Simon**] Section 2.4, Theorem 10,) gives for any $F \in L^2(\Omega, d\nu; \mathcal{V})$

$$(\tau'_{\pm\Gamma}(x)F)(\omega) = P_\pm(x, \omega)\rho(x)F(x^{-1}\omega)$$

and

$$(\tau'_{\pm\Omega}(G)F)(\omega) = G(\omega)F(\omega)$$

where $x \in \Gamma$ and $G \in C(\Omega)$.

The following definitions are also from [**Steger**].

Definition 1.4. Let Γ be a free group, let Ω be its boundary, and let π be a fixed unitary representation of Γ on $\mathcal{H}$. A *boundary intertwiner* for π is a pair (ι, π') satisfying

(1) π' is a boundary representation of Γ on a Hilbert space $\mathcal{H}'$.

(2) ι is a bounded Γ-map from $\mathcal{H}$ to $\mathcal{H}'$.

(3) $\mathcal{H}'$ is generated as a $(\Gamma, C(\Omega))$-space by $\iota(\mathcal{H})$.

If ι is an isometric inclusion, the pair is called a *boundary realization* of π, and if ι is unitary, the pair is called a *perfect boundary realization* of π. Condition (3) is equivalent to requiring that $\pi'(C(\Omega))\iota(\mathcal{H})$ be dense in $\mathcal{H}'$.

Definition 1.5. Two boundary intertwiners (ι_1, π'_1) and (ι_2, π'_2) for the same given representation π of Γ are *equivalent* if there is a unitary $(\Gamma, C(\Omega))$-map $U\colon \mathcal{H}'_1 \longrightarrow \mathcal{H}'_2$ so that $U\iota_1 = \iota_2$.

Let (ι_1, π'_1) and (ι_2, π'_2) be two boundary intertwiners for a given representation π of Γ. The *direct sum* of these two intertwiners,

$$(\iota_s, \pi'_s) = (\iota_1, \pi'_1) + (\iota_2, \pi'_2)\ ,$$

is defined in two steps. Let $\tilde{\pi}'_s$ be the direct sum boundary representation on $\mathcal{H}'_1 \oplus \mathcal{H}'_2$. Let $\tilde{\iota}_s(v) = (\iota_1(v), \iota_2(v))$. Then $(\tilde{\iota}_s, \tilde{\pi}'_s)$ satisfies the definition of a boundary intertwiner except for condition (3). Let $\mathcal{H}'_s$ be the closure in $\mathcal{H}'_1 \oplus \mathcal{H}'_2$ of $\tilde{\pi}'_s(C(\Omega))\tilde{\iota}_s(\mathcal{H})$. Then $\mathcal{H}'_s$ is a subboundary representation of $\mathcal{H}'_1 \oplus \mathcal{H}'_2$. Denote the boundary representation on $\mathcal{H}'_s$ by π'_s. Finally, let $\iota_s\colon \mathcal{H} \longrightarrow \mathcal{H}'_s$ be $\tilde{\iota}_s$ with its range restricted to $\mathcal{H}'_s$.

For $G \in C(\Omega)$ one may verify that

$$\iota_s^* \pi_s'(G) \iota_s = \iota_1^* \pi_1'(G) \iota_1 + \iota_2^* \pi_2'(G) \iota_2 \ .$$

To be consistent with that definition define, for a boundary intertwiner (ι, π') and for real $t > 0$,

$$t(\iota, \pi') = (t^{1/2}\iota, \pi')$$

and define

$$0(\iota, \pi') = (0, \zeta')$$

where ζ' is the boundary representation on 0-dimensional Hilbert space. From these definitions it follows that if (ι_1, π_1') and (ι_2, π_2') are boundary realizations, that is, if ι_1 and ι_2 are isometric inclusions, and if $0 \leq t \leq 1$, then $t(\iota_1, \pi_1') + (1-t)(\iota_2, \pi_2')$ is also a boundary realization.

Results on boundary realizations of τ. Return to the specific case of the tensor product representation $(\tau, \mathcal{H} \otimes \mathcal{V})$ of an anisotropic principal series representation $(\pi, \mathcal{H})$ and an irreducible unitary finite dimensional representation $(\rho, \mathcal{V})$ of Γ. The maps

$$\iota_\pm \otimes \mathrm{id}\colon \mathcal{H} \otimes \mathcal{V} \longrightarrow L^2(\Omega, d\nu) \otimes \mathcal{V}$$

are unitary Γ-maps intertwining τ with $\tau'_{\pm\Gamma}$. Thus $(\iota_\pm \otimes \mathrm{id}, \tau'_\pm)$, where $\tau'_\pm = (\tau'_{\pm\Gamma}, \tau'_{\pm\Omega})$, are two perfect boundary realizations of τ.

Theorem 1.6. *Exclude the anomalous case that $\sigma = 0$ and $\rho \cong \rho \otimes \mathrm{sgn}$. Then, up to equivalence, the only perfect boundary realizations of τ are $(\iota_\pm \otimes \mathrm{id}, \tau'_\pm)$. Moreover, all boundary realizations, perfect or otherwise, are given, up to equivalence, by*

$$t(\iota_+ \otimes \mathrm{id}, \tau'_+) + (1-t)(\iota_- \otimes \mathrm{id}, \tau'_-) \qquad \textit{for } 0 \leq t \leq 1 \ .$$

Outline. Section 2 describes some general results on unitary representations of free groups and their boundary realizations. Section 3 establishes the applicability of those general results in the present context. Section 4 deals with the anomalous case, $\sigma = 0$ and $\rho \cong \rho \otimes \mathrm{sgn}$.

2. General Results.

If $\mathcal{H}$ and $\mathcal{H}_1$ are Hilbert spaces, we denote by $\mathcal{B}(\mathcal{H}, \mathcal{H}_1)$ the vector space of all bounded linear maps from $\mathcal{H}$ to $\mathcal{H}_1$. If $\mathcal{H} = \mathcal{H}_1$ then $\mathcal{B}(\mathcal{H}) \equiv \mathcal{B}(\mathcal{H}, \mathcal{H})$.

Definition 2.1. Let $T \in \mathcal{B}(\mathcal{H})$. We say that T is positive if

$$\langle Tv, v \rangle_{\mathcal{H}} \geq 0 \quad \text{for } v \in \mathcal{H} \ .$$

We denote by $\mathcal{B}^+(\mathcal{H})$ the set of all positive operators in $\mathcal{B}(\mathcal{H})$.

Definition 2.2. Let $T \in \mathcal{B}^+(\mathcal{H})$. We define the trace of T as

$$\operatorname{tr}(T) = \sum_{n=1}^{+\infty} \langle Te_n, e_n \rangle_{\mathcal{H}}$$

where $(e_n)_{n=1}^{+\infty}$ is a fixed orthonormal basis for $\mathcal{H}$.

Properties 2.3. Let S and T be in $\mathcal{B}^+(\mathcal{H})$.

(1) $\operatorname{tr}(T) \in [0, +\infty]$.

(2) $\operatorname{tr}(\alpha\, S + \beta\, T) = \alpha \operatorname{tr}(S) + \beta \operatorname{tr}(T)$ for $\alpha, \beta \in \mathbf{R}^+$.

(3) $\operatorname{tr}(SS^*) = \|S\|_{HS}^2$ where $\| \cdot \|_{HS}$ is the Hilbert–Schmidt norm.

(4) If $U \in \mathcal{B}(\mathcal{H})$ is unitary, then $\operatorname{tr}(UTU^{-1}) = \operatorname{tr}(T)$.

(5) $\operatorname{tr}(T)$ is independent of the choice of basis.

(6) Let $\mathcal{V}$ be a finite dimensional Hilbert space, then $\operatorname{tr}(T \otimes \mathrm{id}_{\mathcal{V}}) = \dim \mathcal{V} \operatorname{tr}(T)$.

Proof. For (1)–(5), see [**Dixmier**] Section 1.6.6, Theorem 5. Let $(v_m)_{m=1}^N$ be an orthonormal basis of $\mathcal{V}$. Then using $(e_n \otimes v_m)_{n,m}$ as a basis of $\mathcal{H} \otimes \mathcal{V}$ gives (6). □

Definition 2.4. Let $S, T \in \mathcal{B}^+(\mathcal{H})$. We define the following inner product

$$(S, T)_{\mathcal{B}^+(\mathcal{H})} = \operatorname{tr}(\sqrt{S}T\sqrt{S}) \,.$$

Properties 2.5. Let S and T be in $\mathcal{B}^+(\mathcal{H})$.

(1) $(S, T)_{\mathcal{B}^+(\mathcal{H})} \in [0, +\infty]$.

(2) $(S, T)_{\mathcal{B}^+(\mathcal{H})} = \|\sqrt{S}\sqrt{T}\|_{HS}^2$.

(3) $(S, T)_{\mathcal{B}^+(\mathcal{H})} = (T, S)_{\mathcal{B}^+(\mathcal{H})}$.

(4) $(S, T)_{\mathcal{B}^+(\mathcal{H})}$ is bilinear in S and T.

(5) Let $\mathcal{V}$ be a finite dimensional Hilbert space, then $(S \otimes \mathrm{id}, T \otimes \mathrm{id})_{\mathcal{B}^+(\mathcal{H} \otimes \mathcal{V})} = \dim \mathcal{V}\ (S, T)_{\mathcal{B}^+(\mathcal{H})}$.

Proof. (1) follows directly from the definition. For (2),

$$\begin{aligned}(S, T)_{\mathcal{B}^+(\mathcal{H})} &= \operatorname{tr}(\sqrt{S}T\sqrt{S}) = \operatorname{tr}(\sqrt{S}\sqrt{T}\sqrt{T}\sqrt{S}) \\ &= \operatorname{tr}((\sqrt{S}\sqrt{T})(\sqrt{S}\sqrt{T})^*) = \|\sqrt{S}\sqrt{T}\|_{HS}^2 \,.\end{aligned}$$

For (3),

$$(S,T)_{\mathcal{B}^+(\mathcal{H})} = \|\sqrt{S}\sqrt{T}\|_{HS}^2 = \|(\sqrt{S}\sqrt{T})^*\|_{HS}^2 = \|\sqrt{T}\sqrt{S}\|_{HS}^2 = (T,S)_{\mathcal{B}^+(\mathcal{H})} .$$

Finally (4) follows from the definition and (3), and (5) follows from (6) of Properties 2.3. □

One may easily verify the following lemma.

Lemma 2.6. *Let Γ be a free group. Let π be a unitary representation of Γ and let (ι, π') be a boundary intertwiner for π. For any $G \in C(\Omega)$ with $G \geq 0$ the operator*

$$\iota^* \pi'(G) \iota \colon \mathcal{H} \longrightarrow \mathcal{H}$$

is linear, bounded and positive.

Now we can give the definition of the Finite Trace Condition (FTC) for boundary intertwiners.

Definition 2.7. Let A be a set of generators and their inverses for a free group Γ. Let (ι_1, π_1') and (ι_2, π_2') be two boundary intertwiners for a representation $(\pi, \mathcal{H})$ of Γ. We say that (ι_1, π_1') and (ι_2, π_2') satisfy the (FTC) if

$$(\iota_1^* \pi_1'(\mathbf{1}_{\Omega(a)})\iota_1, \iota_2^* \pi_2'(\mathbf{1} - \mathbf{1}_{\Omega(a)})\iota_2)_{\mathcal{B}^+(\mathcal{H})} < +\infty$$

for all $a \in A$.

Lemma 2.8. *Let Γ be a free group with A as a set of generators and inverses. Let π be a unitary representation of Γ and let (ι_1, π_1') and (ι_2, π_2') be boundary intertwiners for Γ. Let ρ be a unitary representation of Γ on a finite dimensional Hilbert space $\mathcal{V}$. Let $\tau = \pi \otimes \rho$, $\tau_{1,\Gamma}' = \pi_{1,\Gamma}' \otimes \rho$, $\tau_{2,\Gamma}' = \pi_{2,\Gamma}' \otimes \rho$, $\tau_{1,\Omega}' = \pi_{1,\Omega}' \otimes \mathrm{id}$, and $\tau_{2,\Omega}' = \pi_{2,\Omega}' \otimes \mathrm{id}$. Then $(\iota_1 \otimes \mathrm{id}, \tau_1')$ and $(\iota_2 \otimes \mathrm{id}, \tau_2')$ satisfy the (FTC) as realizations of τ if and only if (ι_1, π_1') and (ι_2, π_2') satisfy the (FTC) as realizations of π.*

Proof. Let $a \in A$. Then

$$\begin{aligned} &\left((\iota_1 \otimes \mathrm{id})^* \tau_1' \left(\mathbf{1}_{\Omega(a)}\right)(\iota_1 \otimes \mathrm{id}), (\iota_2 \otimes \mathrm{id})^* \tau_2' \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right)(\iota_2 \otimes \mathrm{id})\right)_{\mathcal{B}^+(\mathcal{H}\otimes\mathcal{V})} \\ &= \left(\iota_1^* \pi_1' \left(\mathbf{1}_{\Omega(a)}\right)\iota_1 \otimes \mathrm{id}, \iota_2^* \pi_2' \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right)\iota_2 \otimes \mathrm{id}\right)_{\mathcal{B}^+(\mathcal{H}\otimes\mathcal{V})} \\ &= \dim \mathcal{V} \left(\iota_1^* \pi_1' \left(\mathbf{1}_{\Omega(a)}\right)\iota_1, \iota_2^* \pi_2' \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right)\iota_2\right)_{\mathcal{B}^+(\mathcal{H})} \end{aligned}$$

according to (5) of Properties 2.5. □

The following result, which is our main tool, is from [**Steger**].

Theorem 2.9. *Let Γ be a free group and let A be a set of generators and their inverses. Let $(\pi, \mathcal{H})$ be a unitary representation of Γ on a Hilbert space $\mathcal{H}$. If (ι_1, π_1') and (ι_2, π_2') are two perfect boundary realizations of π satisfying the (FTC), and if π_1' and π_2' are irreducible and inequivalent as boundary representations, then*

(1) *π is irreducible.*

(2) *Up to equivalence (ι_1, π_1') and (ι_2, π_2') are the only perfect boundary realizations of π.*

(3) *Up to equivalence all boundary realizations of π are obtained as*

$$t(\iota_1, \pi_1') + (1-t)(\iota_2, \pi_2') \quad \textit{with} \quad t \in [0,1].$$

(4) *There exists a dense subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ and a constant $C_0 > 0$ so that*

$$\lim_{\epsilon \to 0+} C_0 \epsilon \sum_{x \in \Gamma} e^{-\epsilon |x|_\Gamma} \langle v_1, \pi(x) v_2 \rangle \overline{\langle v_3, \pi(x) v_4 \rangle} = \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}$$

for v_1, $v_3 \in \mathcal{H}$ and v_2, $v_4 \in \mathcal{H}_0$.

3. Technical Results.

In this section, let Γ be a free group with A as a fixed set of generators and their inverses. Let $(\pi, \mathcal{H})$ be an anisotropic principal series representation of Γ and let $(\iota_\pm, \pi_\pm')$ be the two perfect boundary realizations of π described in the introduction. Let ρ be an irreducible unitary representation of Γ on a Hilbert space $\mathcal{V}$ of finite dimension N. Let $\tau = \pi \otimes \rho$ be the tensor product representation of Γ on the Hilbert space $\mathcal{H} \otimes \mathcal{V}$. We will establish that the hypotheses of Theorem 2.9 apply to τ and its two perfect boundary realizations $(\iota_\pm \otimes \mathrm{id}, \tau_\pm')$ except in the anomalous case $\sigma = 0$ and $\rho \cong \rho \otimes \mathrm{sgn}$. The various conclusions of Theorem 2.9 give Theorems 1.1, 1.2, and 1.6.

Lemma 3.1. *Let $T \in \mathcal{B}(L^2(\Omega, d\nu; \mathcal{V}))$ be a linear bounded $C(\Omega)$-map. Then there exists $t \in L^\infty(\Omega, d\nu; GL(\mathcal{V}))$ such that*

$$(TF)(\omega) = t(\omega) F(\omega) \quad \textit{for } F \in L^2(\Omega, d\nu; \mathcal{V}) \ .$$

Proof. We identify $\mathcal{V}$ with $\mathbf{C}^N$ and use subscripts on elements of $\mathcal{V}$ to pick out their coordinates. For every $n = 1, \ldots, N$ define $E^n \in L^2(\Omega, d\nu; \mathcal{V})$ as follows

$$(E^n)_m(\omega) = \delta(m,n) \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} .$$

Hence for any $F \in C(\Omega; \mathcal{V})$ we have $F = \sum_{n=1}^{N} \pi'_+(F_n)E^n$. The operator T acts as follows

$$TF = T\left(\sum_{n=1}^{N} \pi'_+(F_n)E^n\right) = \sum_{n=1}^{N} \pi'_+(F_n)TE^n \ .$$

We define

$$t_{mn}(\omega) = (TE^n)_m(\omega) \quad \text{where } t_{mn} \in L^2(\Omega, d\nu) \ .$$

Then

$$(TF)_m(\omega) = \sum_{n=1}^{N} t_{mn}(\omega)F_n(\omega) = (t(\omega)F(\omega))_m$$

where $t(\omega) \equiv (t_{mn}(\omega))_{N\times N}$ belongs to $L^2(\Omega, d\nu; GL(\mathcal{V}))$. Since T is bounded, it follows from Lusin's Theorem that the entries of t are almost everywhere bounded. Then $t \in L^\infty(\Omega, d\nu; GL(\mathcal{V}))$. Consequently extending by density, we get

$$(TF)(\omega) = t(\omega)F(\omega) \quad \text{for } F \in L^2(\Omega, d\nu; \mathcal{V}) \ .$$

□

Endow $GL(\mathcal{V})$ with the norm $\| \ \|_\infty$ obtained by identifying any linear operator $t\colon \mathcal{V} \longrightarrow \mathcal{V}$ with its matrix $(t_{jk})_{j,k}$ and setting $\|t\|_\infty = \sup_{j,k} |t_{jk}|$.

Lemma 3.2. *Let f be a function in $L^1(\Omega, d\nu; GL(\mathcal{V}))$. Then*

$$\lim_{n\to+\infty} \frac{1}{\nu(\Omega(\omega_n))} \int_{\Omega(\omega_n)} \|f(\omega') - f(\omega)\|_\infty d\nu(\omega') = 0$$

for almost all $\omega = (\omega_0, \omega_1, \dots, \omega_n, \dots) \in \Omega$.

Proof. This is a standard consequence of the boundedness of the maximal function for L^1-martingales. □

Lemma 3.3. *If $\gamma \in \Gamma$, $a \in A$, and $|\gamma a| = |\gamma| + 1$, then $\gamma^{-1}\Omega(\gamma a) = \Omega(a)$.*

Proof. Thinking of boundary points of Ω as infinite reduced words, one sees easily that

$$\gamma^{-1}\Omega(\gamma a) \subseteq \Omega(a) \quad \text{and} \quad \gamma\Omega(a) \subseteq \Omega(\gamma a) \ .$$

□

Lemma 3.4. *If $\gamma \in \Gamma$, $a \in A$, and $|\gamma a| = |\gamma| + 1$, then $P_+(\gamma, \omega)$ is constant for $\omega \in \Omega(\gamma a)$.*

Proof. By (1.5)

$$P_+(\gamma, \omega) = \frac{1}{h_+(\gamma)} \quad \text{for } \omega \in \Omega(\gamma a) \ .$$ □

Lemma 3.5. *If $\gamma \in \Gamma$, $a \in A$, and $|\gamma a| = |\gamma| + 1$, then*

$$\frac{d\nu(\gamma^{-1}\omega)}{d\nu(\omega)} = \frac{\nu(\Omega(a))}{\nu(\Omega(\gamma a))} \quad \text{for } \omega \in \Omega(\gamma a) .$$

Proof. From (1.6) we have that

$$\frac{d\nu(\gamma^{-1}\omega)}{d\nu(\omega)} = |P_+(\gamma, \omega)|^2 .$$

By Lemma 3.4 above, $|P_+(\gamma, \omega)|^2$ is constant for $\omega \in \Omega(\gamma a)$. Clearly this constant also gives

$$\frac{\nu(\gamma^{-1}\Omega(\gamma a))}{\nu(\Omega(\gamma a))} = \frac{\nu(\Omega(a))}{\nu(\Omega(\gamma a))} .$$

□

Lemma 3.6. *Let f be a function in $L^\infty(\Omega, d\nu; GL(\mathcal{V}))$. Let ρ be a unitary representation of Γ on a finite dimensional Hilbert space $\mathcal{V}$ and let $Q\colon \Gamma \times \Omega \longrightarrow \mathbf{C}$ be a function, ν-measurable in ω, such that*

(1) $|Q(x, \omega)| = 1$ *for all $x \in \Gamma$ and $\omega \in \Omega$.*

(2) $f(x^{-1}\omega) = Q(x, \omega)\rho(x^{-1})f(\omega)\rho(x)$ *for all $x \in \Gamma$ and $\omega \in \Omega$.*

(3) *If γ, $y \in \Gamma$ and $\gamma \notin \Gamma(y)$ then $Q(\gamma, \omega)$ is constant for $\omega \in \Omega(y)$.*

Then there exists $a \in A$ and $t_0 \in GL(\mathcal{V})$ such that

$$f(\omega) = t_0 \quad \text{for almost all } \omega \in \Omega(a) .$$

Proof. By Lemma 3.2 there exists $\omega = (\omega_0, \omega_1, \dots, \omega_n, \dots) \in \Omega$ such that

$$\lim_{n \to +\infty} \frac{1}{\nu(\Omega(\omega_n))} \int_{\Omega(\omega_n)} \|f(\omega') - s\|_\infty d\nu(\omega') = 0$$

where $s = f(\omega)$. For every $n \geq 1$ let $a_n = \gamma_n^{-1}\omega_n$ where $\gamma_n = \omega_{n-1}$. According to (3) $Q(\gamma_n, \omega')s$ has a constant value for $\omega' \in \Omega(\omega_n)$. Let $s_n = Q(\gamma_n, \omega')s$ be that constant value. Then

$$\begin{aligned}
&\frac{1}{\nu(\Omega(a_n))} \int_{\Omega(a_n)} \|\rho(\gamma_n)f(\omega')\rho(\gamma_n^{-1}) - s_n\|_\infty d\nu(\omega') \\
&\quad = \frac{\nu(\Omega(\omega_n))}{\nu(\Omega(a_n))} \frac{1}{\nu(\Omega(\omega_n))} \int_{\gamma_n \Omega(a_n)} \|\rho(\gamma_n)f(\gamma_n^{-1}\omega')\rho(\gamma_n^{-1}) - s_n\|_\infty d\nu(\gamma_n^{-1}\omega') \\
&\quad = \frac{1}{\nu(\Omega(\omega_n))} \int_{\Omega(\omega_n)} \|f(\omega') - s\|_\infty d\nu(\omega') \quad \text{(using Lemmas 3.3 and 3.5).}
\end{aligned}$$

Hence

$$\lim_{n\to+\infty} \frac{1}{\nu(\Omega(a_n))} \int_{\Omega(a_n)} \|\rho(\gamma_n) f(\omega') \rho(\gamma_n^{-1}) - s_n\|_\infty d\nu(\omega') = 0 .$$

Since $\mathcal{U}(\mathcal{V})$ is compact, since $s_n \in \{\alpha s \ ; \ |\alpha| = 1\}$, and since A is finite, we may select a sequence (n_j) so that $\rho(\gamma_{n_j}) \to \rho_0$ for some ρ_0 in $\mathcal{U}(\mathcal{V})$, so that $s_{n_j} \to t$ for some $t \in GL(\mathcal{V})$, and so that $a_{n_j} = a$ for all j and for some $a \in A$. Then

$$\int_{\Omega(a)} \|\rho_0 f(\omega') \rho_0^{-1} - t\|_\infty d\nu(\omega') = 0 .$$

That is

$$f(\omega) = t_0 \quad \text{for almost all } \omega \in \Omega(a)$$

where $t_0 \equiv \rho_0^{-1} t \rho_0$. □

In analogy with $\Gamma(y)$, the set of all words which start with the letters of y, define $\tilde{\Gamma}(y) = \Gamma(y^{-1})^{-1}$, the set of all words that end with the letters of y.

Lemma 3.7. *Let (Γ, A) be a free group. Let ρ be a unitary representation of Γ on a finite dimensional Hilbert space $\mathcal{V}$ and let $q\colon \Gamma \longrightarrow \mathbf{C}$ be a multiplicative function (not a character, see (1.1)). If there exists $a \in A$ and $t_0 \in GL(\mathcal{V})$ such that*

$$t_0 \rho(x) = q(x) \rho(x) t_0 \quad \textit{for } x \in \Gamma(a) \backslash \tilde{\Gamma}(a^{-1})$$

then

$$t_0 \rho(x) = q(x) \rho(x) t_0 \quad \textit{for } x \in \Gamma .$$

Proof. Fix $x \in \Gamma(a_1) \cap \tilde{\Gamma}(a_2)$ where $a_1, a_2 \in A$ and $a_2 \neq a^{-1}$. Choose $w_1 \in \Gamma(a)$ such that $w_1 \notin \tilde{\Gamma}(a^{-1})$ and $w_1 \notin \tilde{\Gamma}(a_1^{-1})$. Choose $w_2 \in \Gamma(a)$ such that $w_2 \notin \tilde{\Gamma}(a^{-1})$. Then $w_1, w_2, w_1 x w_2 \in \Gamma(a) \backslash \tilde{\Gamma}(a^{-1})$, and $|w_1 x w_2| = |w_1| + |x| + |w_2|$. Hence

$$\begin{aligned} & t_0 \rho(w_1 x w_2) = q(w_1 x w_2) \rho(w_1 x w_2) t_0 \\ & \quad \Rightarrow q(w_1) \rho(w_1) t_0 \rho(x) \rho(w_2) = q(w_1) \rho(w_1) q(x w_2) \rho(x w_2) t_0 \\ & \quad \Rightarrow t_0 \rho(x) \rho(w_2) = q(x) \rho(x) q(w_2) \rho(w_2) t_0 = q(x) \rho(x) t_0 \rho(w_2) \\ & \quad \Rightarrow t_0 \rho(x) = q(x) \rho(x) t_0 . \end{aligned}$$

For $x \in \tilde{\Gamma}(a^{-1})$ choose $b \in A$ such that $b \neq a$ and $b \neq a^{-1}$. Then b and xb are words which don't end with a^{-1} and $|xb| = |x| + 1$. In this case we already know that

$$t_0 \rho(xb) = q(xb) \rho(xb) t_0 \quad \text{and} \quad t_0 \rho(b) = q(b) \rho(b) t_0 .$$

Hence

$$t_0\rho(x)\rho(b) = q(x)\rho(x)q(b)\rho(b)t_0 = q(x)\rho(x)t_0\rho(b) .$$

□

Recall that $\sigma \in \mathrm{Spec}(\mu)$ is not allowed to be an endpoint. We define

$$q(x) = \frac{h_+(x)}{h_-(x)} \quad \text{for } x \in \Gamma .$$

This makes q a multiplicative function of modulus one on Γ.

Lemma 3.8. *If $q^2(a) = 1$ for all $a \in A$, then $\sigma = 0$ and $q(x) = \mathrm{sgn}(x) \equiv (-1)^{|x|}$ for all $x \in \Gamma$.*

Proof. From [**Figà-Talamanca–Steger**], Proposition 1.3.4, we will use
(1) $2w_+ = \mu(c)(h_+^{-1}(c) - h_+(c))$ for any $c \in A$ and
(2) $2w_+ = \sigma - \sum_{b\in A} \mu(b)h_+(b)$.
We proceed, dividing the proof into two cases.

(A) There is a $c \in A$ such that $h_+(c) \in \mathbf{R}$. By (1) we note first that w_+ is real and then that $h_+(b)$ is real for all $b \in A$. By [**Figà-Talamanca–Steger**], Lemma 2.3.1, and the discussion thereafter, we know that $h_+ = h_-$ and $w_+ = w_-$ is impossible for σ in the interior of $\mathrm{Spec}(\mu)$. This eliminates case (A).

(B) For all $a \in A$ we have $h_+(a) \in i\mathbf{R}$. Since $h_+(a) \in i\mathbf{R}$ for all $a \in A$, $w_+ \in i\mathbf{R}$. By (2) $\sigma = 0$. We also have that $q(a) = -1$ for all $a \in A$, so $q(x) = (-1)^{|x|}$ for all $x \in \Gamma$. □

Proposition 3.9. *The two boundary representations $\tau'_\pm$ are irreducible.*

Proof. We only need to work with τ'_+. Let $\mathcal{H}_1$ be a closed $(\Gamma, C(\Omega))$-stable subspace in $L^2(\Omega, d\nu; \mathcal{V})$. Then $\mathcal{H}_2 = \mathcal{H}_1^\perp$ has the same property, so the orthogonal projection $T\colon L^2(\Omega, d\nu; \mathcal{V}) \longrightarrow L^2(\Omega, d\nu; \mathcal{V})$ onto $\mathcal{H}_1$ commutes with $\tau'_+(x)$ and $\tau'_+(G)$ for all $x \in \Gamma$ and $G \in C(\Omega)$. So, as usual, we only need to prove that any bounded $(\Gamma, C(\Omega))$-map T on $L^2(\Omega, d\nu; \mathcal{V})$ is scalar. Applying Lemma 3.1 there exists $t \in L^\infty(\Omega, d\nu; GL(\mathcal{V}))$ such that

$$(TF)(\omega) = t(\omega)F(\omega) \quad \text{for } F \in L^2(\Omega, d\nu; \mathcal{V}) .$$

On the other hand, for all $x \in \Gamma$ and $F \in L^2(\Omega, d\nu; \mathcal{V})$

$$(\tau'_+(x)TF)(\omega) = P_+(x, \omega)\rho(x)t(x^{-1}\omega)F(x^{-1}\omega)$$

and

$$(T\tau'_+(x)F)(\omega) = t(\omega)P_+(x, \omega)\rho(x)F(x^{-1}\omega) .$$

Therefore

$$t(x^{-1}\omega) = \rho(x^{-1})t(\omega)\rho(x) \quad \text{for any } x \in \Gamma .$$

From Lemma 3.6, with $Q(x,\omega) = 1$ there exists $a \in \Gamma$ and $t_0 \in GL(\mathcal{V})$ such that $t(\omega) = t_0$ for almost all $\omega \in \Omega(a)$. Fix $x \in \Gamma(a)\backslash\tilde{\Gamma}(a^{-1})$. Then $x\omega \in \Omega(a)$ and the product $x\omega$ does not simplify when $\omega \in \Omega(a)$. Therefore

$$t_0 = t(x\omega) = \rho(x)t(\omega)\rho(x^{-1}) = \rho(x)t_0\rho(x^{-1})$$

for almost all $\omega \in \Omega(a)$. By Lemma 3.7, with $q(x) = 1$, it follows that

$$t_0\rho(x) = \rho(x)t_0 \quad \text{for } x \in \Gamma .$$

Then t_0 commutes with $\rho(x)$ for all $x \in \Gamma$. As a consequence of Schur's Lemma, t_0 is scalar. That is, there exists $\alpha \in \mathbf{C}$ such that

$$t(\omega) = \alpha \, \mathrm{id}_{\mathcal{V}} \quad \text{for almost all } \omega \in \Omega(a) .$$

We want now to extend this result to Ω. Fix $b \in A$. Find $z \in \Gamma(a) \cap \tilde{\Gamma}(b)$. Then for $\omega \notin \Omega(b^{-1})$, $z\omega \in \Omega(a)$. Thus $t(\omega) = \alpha \, \mathrm{id}_{\mathcal{V}}$ almost everywhere on $\Omega\backslash\Omega(b^{-1})$. Vary b to see that this holds almost everywhere on Ω. Thus T is multiplication by α. □

Proposition 3.10. *Excluding the anomalous case that $\sigma = 0$ and $\rho \cong \rho \otimes \mathrm{sgn}$, the two boundary representations $\tau'_{\pm}$ are inequivalent.*

Proof. We proceed by contradiction. Suppose that

$$T\colon L^2(\Omega, d\nu; \mathcal{V}) \longrightarrow L^2(\Omega, d\nu; \mathcal{V})$$

is a unitary $(\Gamma, C(\Omega))$-map that intertwines $\tau'_+(x)$ with $\tau'_-(x)$ and $\tau'_+(G)$ with $\tau'_-(G)$ for any x in Γ and G in $C(\Omega)$. By Lemma 3.1 there exists $t \in L^\infty(\Omega, d\nu; GL(\mathcal{V}))$ such that

$$(TF)(\omega) = t(\omega)F(\omega) \quad \text{for } F \in L^2(\Omega, d\nu; \mathcal{V}) .$$

For all $x \in \Gamma$ and $F \in L^2(\Omega, d\nu; \mathcal{V})$ the following holds.

$$(\tau'_-(x)TF)(\omega) = P_-(x,\omega)\rho(x)t(x^{-1}\omega)F(x^{-1}\omega)$$

and

$$(T\tau'_+(x)F)(\omega) = t(\omega)P_+(x,\omega)\rho(x)F(x^{-1}\omega) .$$

Hence

$$t(x^{-1}\omega) = \frac{P_+(x,\omega)}{P_-(x,\omega)}\rho(x^{-1})t(\omega)\rho(x) .$$

By Lemma 3.6 with

$$Q(x,\omega) = \frac{P_+(x,\omega)}{P_-(x,\omega)} = \frac{P_+(x,\omega)}{P_+(x,\omega)^-} ,$$

there exists $a \in \Gamma$ and $t_0 \in GL(\mathcal{V})$ such that

$$t(\omega) = t_0 \quad \text{for almost all } \omega \in \Omega(a) .$$

Fix $x \in \Gamma(a)\backslash\tilde{\Gamma}(a^{-1})$. When $\omega \in \Omega(a)$, the product $x\omega$ does not simplify and $x\omega \in \Omega(a)$. By the absence of cancellation

$$Q(x^{-1},\omega) = \frac{h_+(x^{-1})}{h_-(x^{-1})} = \frac{h_+(x)}{h_-(x)} = q(x)$$

for $\omega \in \Omega(a)$. For almost every $\omega \in \Omega(a)$

$$t_0 = t(x\omega) = Q(x^{-1},\omega)\rho(x)t(\omega)\rho(x^{-1}) = q(x)\rho(x)t_0\rho(x^{-1}) .$$

Thus, by Lemma 3.7, the relation $t_0\rho(x) = q(x)\rho(x)t_0$ holds for all $x \in \Gamma$. For x, $y \in \Gamma$

$$\begin{aligned} q(xy)\rho(xy)t_0 &= t_0\rho(xy) = t_0\rho(x)\rho(y) \\ &= q(x)\rho(x)t_0\rho(y) = q(x)\rho(x)q(y)\rho(y)t_0 = q(x)q(y)\rho(xy)t_0 . \end{aligned}$$

Hence

$$q(x)q(y) = q(xy) \quad \text{for } x, y \in \Gamma .$$

This means that q is a character of Γ. Then for any $a \in A$ we have

$$1 = q(e) = q(a^{-1}a) = q(a^{-1})q(a) .$$

That is

$$q^2(a) = 1 \quad \text{for } a \in A .$$

By Lemma 3.8 it follows that $\sigma = 0$ and $q(x) = \operatorname{sgn}(x)$ for all $x \in \Gamma$. Since $t_0\rho(x) = \operatorname{sgn}(x)\rho(x)t_0$, t_0 is an intertwiner between ρ and $\rho \otimes \operatorname{sgn}$. Since the anomalous case is excluded, and since $\sigma = 0$, ρ and $\rho \otimes \operatorname{sgn}$ cannot be equivalent. Therefore t_0 must be 0. □

Proposition 3.11. *The two boundary realizations* $(\iota_\pm \otimes \mathrm{id}, \tau'_\pm)$ *satisfy the* (FTC).

Proof. Given Lemma 2.8, we need only to work with $(\iota_\pm, \pi'_\pm)$. Remember that $\iota_\pm$ are unitary, and that the Hilbert–Schmidt norm is invariant under unitary operator action on either side. Observe also that for any

$a \in A$ the functions $\mathbf{1}_{\Omega(a)}$ and $\mathbf{1} - \mathbf{1}_{\Omega(a)}$ are idempotent, so $\pi'_+(\mathbf{1}_{\Omega(a)})$ and $\pi'_-(\mathbf{1} - \mathbf{1}_{\Omega(a)})$ are projections and consequently the operators $\iota_+^*\pi'_+(\mathbf{1}_{\Omega(a)})\iota_+$ and $\iota_-^*\pi'_-(\mathbf{1} - \mathbf{1}_{\Omega(a)})\iota_-$ are too. Recall that $J = \iota_-\iota_+^*$ intertwines

$$(\pi'_+(x), L^2(\Omega, d\nu)) \quad \text{with} \quad (\pi'_-(x), L^2(\Omega, d\nu)).$$

Therefore, our problem reduces as follows

$$\begin{aligned}
&\left(\iota_+^*\pi'_+\left(\mathbf{1}_{\Omega(a)}\right)\iota_+ \,,\, \iota_-^*\pi'_-\left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right)\iota_-\right)_{B^+(\mathcal{H})} \\
&\quad = \left\|\iota_+^*\pi'_+\left(\mathbf{1}_{\Omega(a)}\right)\iota_+\iota_-^*\pi'_-\left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right)\iota_-\right\|_{HS}^2 \\
&\quad = \left\|\pi'_+\left(\mathbf{1}_{\Omega(a)}\right)J^*\pi'_-\left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right)\right\|_{HS}^2 .
\end{aligned}$$

Section 3 of [**Figà-Talamanca–Steger**] gives the following description of J. Define a set $(F_x)_{x\in\Gamma}$ of functions in $\mathcal{K}(\Omega)$, the space of locally constant functions in $L^2(\Omega, d\nu)$, by

$$\begin{aligned}
F_e &= \mathbf{1}_\Omega \\
F_{xa} &= \frac{1}{\nu(\Omega(a))}\mathbf{1}_{\Omega(ax)} - \frac{1}{\nu(\Omega(x))}\mathbf{1}_{\Omega(x)} \quad \text{for } |a| = 1 \text{ and } |xa| = |x| + 1 .
\end{aligned}$$

We know that $(F_x)_{x\in\Gamma}$ spans $\mathcal{K}(\Omega)$ and

$$\begin{aligned}
JF_e &= F_e \\
JF_{xa} &= \frac{w_-}{w_+}\frac{h_+(x)}{h_-(x)}F_{xa} \quad \text{for } |a| = 1 \text{ and } |xa| = |x| + 1 .
\end{aligned}$$

For $x \in \Gamma$ let $\mathcal{K}_x$ be the linear span of $\{F_{xa} \,;\, |xa| = |x| + 1 \text{ and } a \in A\}$. This means that

$$\mathcal{K}_x = \{F \in \mathcal{K}(\Omega) \,;\, \operatorname{supp}(F) \subseteq \Omega(x), \quad \int_\Omega F(\omega)d\nu(\omega) = 0,$$
$$\text{and } F \text{ depends only on the first } |x| + 1 \text{ letters of } \omega\} .$$

Define $\mathcal{K}_0 = \mathbf{C} \cdot \mathbf{1}_\Omega$. Then the spaces $\mathcal{K}_x$ together with $\mathcal{K}_0$ span $\mathcal{K}(\Omega)$. Each element of $\mathcal{K}_x$ is an eigenvector of J. In addition, for $x \neq y$, $\mathcal{K}_x$ and $\mathcal{K}_y$ are orthogonal to each other and to $\mathcal{K}_0$, with respect to integration by ν. For these reasons, we can write

$$\begin{aligned}
L^2(\Omega, d\nu) &\cong \mathcal{K}_0 \oplus \bigoplus_{x\in\Gamma} \mathcal{K}_x \\
J|_{\mathcal{K}_0} &= \mathrm{id}_{\mathcal{K}_0} \\
J|_{\mathcal{K}_x} &= \frac{w_-}{w_+}\frac{h_+(x)}{h_-(x)}\mathrm{id}_{\mathcal{K}_x} .
\end{aligned}$$

Let $x \in \Gamma$ and $|x| \geq 1$. If $x \in \Omega(a)$ then $\pi'_-(\mathbf{1} - \mathbf{1}_{\Omega(a)})(\mathcal{K}_x) = 0$. If $x \notin \Omega(a)$ we have

$$\pi'_+(\mathbf{1}_{\Omega(a)})J^*\pi'_-(\mathbf{1} - \mathbf{1}_{\Omega(a)})(\mathcal{K}_x) \subseteq \pi'_+(\mathbf{1}_{\Omega(a)})(\mathcal{K}_x) = 0 \,.$$

Hence $\pi'_+(\mathbf{1}_{\Omega(a)})J^*\pi'_-(\mathbf{1} - \mathbf{1}_{\Omega(a)})$ is identically zero on $\bigoplus_{|x|\geq 1} \mathcal{K}_x$. Therefore to compute the Hilbert–Schmidt norm, we need only to work with the restriction to $\mathcal{K}_0 \oplus \mathcal{K}_e$. Since $\mathcal{K}_0 \oplus \mathcal{K}_e$ is finite dimensional, the Hilbert–Schmidt norm is finite. □

4. Anomalous case.

Let Γ be a free group and let A be a set of generators and inverses for Γ. As in the introduction, fix a symmetric probability measure μ on A. The anisotropic principal series representation π is constructed from the pair (μ, σ) where σ is in the interior of $\mathrm{Spec}(\mu)$. The burden of this section is to explain the special case $\sigma = 0$.

Since the function sgn is a character

$$((f_1)\mathrm{sgn}) * ((f_2)\mathrm{sgn}) = (f_1 * f_2)\mathrm{sgn} \tag{4.1}$$

where f_1 and f_2 are functions on Γ and "$*$" denotes convolution. The symbols in the following lemma are from (1.2).

Lemma 4.1. *When $\sigma = 0$*

(1) $g_+ = -(g_-)\mathrm{sgn}$

(2) $w_+ = -w_-$

(3) $h_+ = (h_-)\mathrm{sgn}$

(4) $w_\pm, h_\pm(a) \in i\mathbf{R}$ *for* $a \in A$

(5) $\phi(x) = 0$ *if* $|x|$ *is odd.*

Proof. For $\gamma \notin \mathrm{Spec}(\mu)$ recall that g_γ is defined as the inverse of $\gamma\delta_e - \mu$, both functions acting on $\ell^2(\Gamma)$ by left convolution. From (4.1)

$$-g_{-\gamma} = (\gamma\delta_e + \mu)^{-1} = ((\gamma\delta_e - \mu)\mathrm{sgn})^{-1} = (\gamma\delta_e - \mu)^{-1}\mathrm{sgn} = (g_\gamma)\mathrm{sgn}$$

where multiplication by sgn is pointwise multiplication but all inverses are inverses for the convolution algebra. (1) then follows from the definition of $g_\pm$ as the limit $g_{0\pm i0}$. (2) and (3) follow from the equality $g_\pm = h_\pm/(2w_\pm)$ given that $h_\pm$ is a multiplicative function and $w_\pm$ is a constant. Since $w_+ = \bar{w}_-$ and $h_+ = \bar{h}_-$, (4) is a consequence of (2) and (3). (5) is immediate from (1) and the definition $\phi = (g_+ - g_-)/(g_+(e) - g_-(e))$. □

Since $(\pi, \mathcal{H})$ is the result of the Gel'fand–Raikov construction applied to ϕ, there is a cyclic vector $k_0 \in \mathcal{H}$ such that

$$\langle k_0, \pi(x)k_0 \rangle = \phi(x) \quad \text{for } x \in \Gamma .$$

Let

$$\begin{aligned} \mathcal{H}_e &= \text{closed-span}\{\pi(x)k_0;\ x \in \Gamma,\ |x| \text{ is even}\} \\ \mathcal{H}_o &= \text{closed-span}\{\pi(y)k_0;\ y \in \Gamma,\ |y| \text{ is odd}\} . \end{aligned} \tag{4.2}$$

By (5) of Lemma 4.1, these two spaces are orthogonal to one another, so $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o$. Define $S\colon \mathcal{H} \to \mathcal{H}$ by

$$S|_{\mathcal{H}_e} = \mathrm{id}_{\mathcal{H}_e} \qquad S|_{\mathcal{H}_o} = -\mathrm{id}_{\mathcal{H}_o} .$$

Lemma 4.2.

(1) $S\pi(x) = \mathrm{sgn}(x)\pi(x)S$ *for* $x \in \Gamma$.

(2) $\pi'_+(x) = \mathrm{sgn}(x)\pi'_-(x)$ *for* $x \in \Gamma$.

(3) $\iota_+ S = \iota_-$.

Proof. (1) follows from the definitions. To establish (2), it is enough to do so for $x \in A$, and that is immediate by the definition (1.4) of $\pi'_\pm$ and by (3) of Lemma 4.1. Since $\iota_+ S k_0 = \mathbf{1} = \iota_- k_0$,

$$\begin{aligned} \iota_+ S\pi(x)k_0 &= \mathrm{sgn}(x)\iota_+\pi(x)Sk_0 = \mathrm{sgn}(x)\pi'_+(x)\iota_+ Sk_0 \\ &= \pi'_-(x)\iota_- k_0 = \iota_-\pi(x)k_0 \end{aligned}$$

for $x \in \Gamma$. Since the Γ-translates of k_0 are dense in $\mathcal{H}$, this proves (3). □

Fix ρ an irreducible unitary representation of Γ on a finite dimensional Hilbert space $\mathcal{V}$, such that $\rho \cong \rho \otimes \mathrm{sgn}$. Then there exists a unitary map $s\colon \mathcal{V} \longrightarrow \mathcal{V}$ such that

$$s\rho(x) = \mathrm{sgn}(x)\rho(x)s . \tag{4.3}$$

Observe that since $s^2\rho(x) = \rho(x)s^2$, s^2 is a scalar operator. Multiplying s by an opportune constant, we may assume s^2 is the identity operator.

We consider, the operator $S \otimes s\colon \mathcal{H} \otimes \mathcal{V} \longrightarrow \mathcal{H} \otimes \mathcal{V}$.

Lemma 4.3.

(1) $(S \otimes s)\tau(x) = \tau(x)(S \otimes s)$ *for all* $x \in \Gamma$.

(2) $(S \otimes s)^2 = \mathrm{id}_{\mathcal{H}\otimes\mathcal{V}}$.

(3) $S \otimes s$ *is unitary and self adjoint.*

(4) $\tau'_+(x) = \operatorname{sgn}(x)\tau'_-(x)$ *for all* $x \in \Gamma$.

(5) $(\iota_+ \otimes \mathrm{id})(S \otimes s) = (\iota_- \otimes s)$.

Proof. (1) follows from (4.3) and (1) of Lemma 4.2. In fact

$$\begin{aligned}(S \otimes s)\tau(x) &= (S \otimes s)(\pi(x) \otimes \rho(x)) = (S\pi(x) \otimes s\rho(x)) \\ &= (\operatorname{sgn}(x)\pi(x)S \otimes \operatorname{sgn}(x)\rho(x)s) = \tau(x)(S \otimes s)\ .\end{aligned}$$

For (2) $(S \otimes s)^2 = (S^2 \otimes s^2) = \mathrm{id}_{\mathcal{H}} \otimes \mathrm{id}_{\mathcal{V}} = \mathrm{id}_{\mathcal{H}\otimes\mathcal{V}}$. Remember that $S \otimes s$ is the tensor product of unitary self adjoint operators. This gives (3). By (2) of Lemma 4.2 $\pi'_+(x) = \operatorname{sgn}(x)\pi'_-(x)$ for any $x \in \Gamma$, and so $\tau'_+(x) = \pi'_+(x) \otimes \rho(x)$ $= \operatorname{sgn}(x)\pi'_-(x) \otimes \rho(x) = \operatorname{sgn}(x)\tau'_-(x)$, proving (4). Finally, (3) of Lemma 4.2 gives (5). □

Let $\mathcal{V}_e$ and $\mathcal{V}_o$ be the 1 and -1 eigenspaces of s respectively. This means that

$$s|_{\mathcal{V}_e} = \mathrm{id}|_{\mathcal{V}_e} \qquad s|_{\mathcal{V}_o} = -\mathrm{id}|_{\mathcal{V}_o} \qquad \mathcal{V} = \mathcal{V}_e \oplus \mathcal{V}_o\ .$$

Then the 1 and -1 eigenspaces of $S \otimes s$ are

$$\mathcal{W}_e = (\mathcal{H}_e \otimes \mathcal{V}_e) \oplus (\mathcal{H}_o \otimes \mathcal{V}_o), \qquad \mathcal{W}_o = (\mathcal{H}_e \otimes \mathcal{V}_o) \oplus (\mathcal{H}_o \otimes \mathcal{V}_e)\ .$$

Let τ_e and τ_o be the representations of Γ on $\mathcal{W}_e$ and $\mathcal{W}_o$ respectively. From the perfect boundary realization $(\iota_+ \otimes \mathrm{id}, \tau'_+)$ of τ construct boundary realizations, no longer perfect, (ι_e, τ'_+) and (ι_o, τ'_+) of τ_e and τ_o respectively, by restricting $\iota_+ \otimes \mathrm{id}$ to $\mathcal{W}_e$ and $\mathcal{W}_o$. Condition (3) in Definition 1.4 holds on account of Proposition 3.9, which says that τ'_+ is irreducible as a $(\Gamma, C(\Omega))$-representation.

The following result from [**Steger**] is parallel to Theorem 2.9.

Theorem 4.4. *Let* Γ *be a free group, let* A *be some set of generators and their inverses for* Γ, *and define length on* Γ *in terms of* A. *Let* $(\pi_1, \mathcal{H}_1)$ *be an arbitrary unitary representation of* Γ *and let* (ι_1, π') *be a boundary realization of* π_1. *Suppose that* (ι_1, π') *is not perfect and that it satisfies the* (FTC) *with itself. Suppose further that* π' *is irreducible as a* $(\Gamma, C(\Omega))$*-representation. Then*

(1) π_1 *is irreducible.*

(2) *Up to equivalence,* (ι_1, π') *is the only boundary realization of* π_1.

(3) *There is a dense subspace* $\mathcal{H}_{01} \subseteq \mathcal{H}_1$ *and a constant* $C_0 > 0$ *so that*

$$\lim_{\epsilon \to 0+} C_0 \epsilon \sum_{x \in \Gamma} e^{-\epsilon|x|_\Gamma} \langle v_1, \pi(x)v_2\rangle \overline{\langle v_3, \pi(x)v_4\rangle} = \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}$$

for v_1, $v_3 \in \mathcal{H}_1$ *and* v_2, $v_4 \in \mathcal{H}_{01}$.

The rest of this section is devoted to showing that the abstract Theorem 4.4 applies to our particular situation.

Theorem 4.5. *Let $\sigma = 0$ and $\rho \cong \rho \otimes \operatorname{sgn}$. Let τ_e and τ_o be the two direct summands of τ defined previously. The conclusions of Theorem 4.4 apply to τ_e and its realization (ι_e, τ'_+) and also to τ_o and its realization (ι_o, τ'_+).*

Accordingly we have $\tau'_+ \cong \tau_e \oplus \tau_o$ as representations of Γ, where τ'_+ is realized perfectly on the boundary, but neither of its two irreducible summands can be realized except in conjunction with the other.

Neither (ι_e, τ'_+) nor (ι_o, τ'_+) is a perfect realization, verifying one of the hypotheses of Theorem 4.4. Proposition 3.9 asserts that τ'_+ is an irreducible $(\Gamma, C(\Omega))$-representation, verifying another. It remains only to show that (ι_e, τ'_+) satisfies the (FTC) with itself as a realization of τ_e and similarly for (ι_o, τ'_+). This can be reduced to Proposition 3.11, which says that $(\iota_+ \otimes \mathrm{id}, \tau'_+)$ and $(\iota_- \otimes \mathrm{id}, \tau'_-)$ satisfy the (FTC) as realizations of τ.

Lemma 4.6. *Let $j\colon \mathcal{H}_1 \longrightarrow \mathcal{H}$ be an isometric inclusion. If $S, T \in \mathcal{B}^+(\mathcal{H})$, then*

(1) $j(j^*Sj)^{1/2}j^* = (jj^*Sjj^*)^{1/2}$.

(2) $(j^*Sj, j^*Tj)_{\mathcal{B}^+(\mathcal{H}_1)} = (jj^*Sjj^*, jj^*Tjj^*)_{\mathcal{B}^+(\mathcal{H})}$.

Proof. For (1), check that both sides are positive square roots of jj^*Sjj^*. For (2), let $(e_m)_m$ be an orthonormal basis for $\mathcal{H}_1$ and suppose that $(je_m)_m$ and $(f_n)_n$ together make up an orthonormal basis for $\mathcal{H}$. Then

$$\begin{aligned}
(j^*Sj, j^*Tj)_{\mathcal{B}^+(\mathcal{H}_1)} &= \sum_m \left\langle j^*Tj(j^*Sj)^{1/2}e_m, (j^*Sj)^{1/2}e_m \right\rangle_{\mathcal{H}_1} \\
&= \sum_m \left\langle jj^*Tjj^*j(j^*Sj)^{1/2}j^*je_m, j(j^*Sj)^{1/2}j^*je_m \right\rangle_{\mathcal{H}} \\
&= \sum_m \left\langle jj^*Tjj^*(jj^*Sjj^*)^{1/2}je_m, (jj^*Sjj^*)^{1/2}je_m \right\rangle_{\mathcal{H}} \\
&= (jj^*Sjj^*, jj^*Tjj^*)_{\mathcal{B}^+(\mathcal{H})}
\end{aligned}$$

since

$$\left\langle jj^*Tjj^*(jj^*Sjj^*)^{1/2}f_n, (jj^*Sjj^*)^{1/2}f_n \right\rangle_{\mathcal{H}} = 0$$

for all n. □

Lemma 4.7. *Let $\mathcal{H}$ be a Hilbert space. If $X, Y \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}^+(\mathcal{H})$, then*

$$(X+Y)^* T (X+Y) \leq 2(X^*TX + Y^*TY).$$

Proof.

$$\begin{aligned}\langle (X+Y)^* T (X+Y) v, v\rangle &= \|T^{1/2}(Xv+Yv)\|^2 \\ &\leq \left(\|T^{1/2}Xv\| + \|T^{1/2}Yv\|\right)^2 \\ &\leq 2\left(\|T^{1/2}Xv\|^2 + \|T^{1/2}Yv\|^2\right) \\ &= 2\left(\langle X^*TXv, v\rangle + \langle Y^*TYv, v\rangle\right) .\end{aligned}$$

□

For fixed $a \in A$ we must show that

$$\left(\iota_e^* \tau'_+ \left(\mathbf{1}_{\Omega(a)}\right) \iota_e, \iota_e^* \tau'_+ \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right) \iota_e\right)_{\mathcal{B}^+(\mathcal{W}_e)} \tag{4.4}$$

is finite. Let $j_e \colon \mathcal{W}_e \longrightarrow \mathcal{H} \otimes \mathcal{V}$ be inclusion. Then

$$(\iota_+ \otimes \mathrm{id})\, j_e = \iota_e \qquad j_e^* j_e = \mathrm{id}_{\mathcal{W}_e} \qquad 2 j_e j_e^* = \mathrm{id}_{\mathcal{H}\otimes\mathcal{V}} + (S \otimes s) .$$

Together with these identities Lemma 4.6 says that up to a factor (4.4) equals

$$\begin{aligned}&\left((\mathrm{id} + (S\otimes s))^* (\iota_+ \otimes \mathrm{id})^* \tau'_+ \left(\mathbf{1}_{\Omega(a)}\right) (\iota_+ \otimes \mathrm{id}) (\mathrm{id} + (S\otimes s)) ,\right. \\ &\left. (\mathrm{id} + (S\otimes s))^* (\iota_+ \otimes \mathrm{id})^* \tau'_+ \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right) (\iota_+ \otimes \mathrm{id}) (\mathrm{id} + (S\otimes s))\right)_{\mathcal{B}^+(\mathcal{H}\otimes\mathcal{V})} .\end{aligned} \tag{4.5}$$

By Lemma 4.7 the first of the two operators in this inner product is bounded by

$$\begin{aligned}&2\left((\iota_+ \otimes \mathrm{id})^* \tau'_+ \left(\mathbf{1}_{\Omega(a)}\right) (\iota_+ \otimes \mathrm{id})\right. \\ &\qquad \left. + (S\otimes s)^* (\iota_+ \otimes \mathrm{id})^* \tau'_+ \left(\mathbf{1}_{\Omega(a)}\right) (\iota_+ \otimes \mathrm{id}) (S\otimes s)\right) .\end{aligned} \tag{4.6}$$

Keeping in mind (5) of Lemma 4.3, the identity of τ'_+ and τ'_- as representations of $C(\Omega)$ and the fact that $s^*s = \mathrm{id}$, one sees that (4.6) is equal to

$$\begin{aligned}&2\left((\iota_+ \otimes \mathrm{id})^* \tau'_+ \left(\mathbf{1}_{\Omega(a)}\right) (\iota_+ \otimes \mathrm{id}) + (\iota_- \otimes s)^* \tau'_- \left(\mathbf{1}_{\Omega(a)}\right) (\iota_- \otimes s)\right) \\ &\qquad = 2\left(\iota_+^* \pi'_+ \left(\mathbf{1}_{\Omega(a)}\right) \iota_+ \otimes \mathrm{id} + \iota_-^* \pi'_- \left(\mathbf{1}_{\Omega(a)}\right) \iota_- \otimes \mathrm{id}\right) .\end{aligned}$$

This bound, and the similar bound on the second operator in (4.5), together with the bilinearity and positivity of the $\mathcal{B}^+(\mathcal{H}\otimes\mathcal{V})$-inner product, show that up to a factor (4.5) is bounded by

$$\begin{aligned}&\left(\iota_+^* \pi'_+ \left(\mathbf{1}_{\Omega(a)}\right) \iota_+ \otimes \mathrm{id} + \iota_-^* \pi'_- \left(\mathbf{1}_{\Omega(a)}\right) \iota_- \otimes \mathrm{id},\right. \\ &\qquad \left. \iota_+^* \pi'_+ \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right) \iota_+ \otimes \mathrm{id} + \iota_-^* \pi'_- \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right) \iota_- \otimes \mathrm{id}\right)_{\mathcal{B}^+(\mathcal{H}\otimes\mathcal{V})} .\end{aligned}$$

This expands into four terms. The term

$$\left(\iota_+^* \pi'_+ \left(\mathbf{1}_{\Omega(a)}\right) \iota_+ \otimes \mathrm{id}, \iota_+^* \pi'_+ \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right) \iota_+ \otimes \mathrm{id}\right)_{\mathcal{B}^+(\mathcal{H}\otimes\mathcal{V})}$$

is zero since the two operators involved are complementary orthogonal projections.

$$\left(\iota_+^* \pi'_+ \left(\mathbf{1}_{\Omega(a)}\right) \iota_+ \otimes \mathrm{id}, \iota_-^* \pi'_- \left(\mathbf{1} - \mathbf{1}_{\Omega(a)}\right) \iota_- \otimes \mathrm{id}\right)_{\mathcal{B}^+(\mathcal{H}\otimes\mathcal{V})}$$

is finite by Proposition 3.11. The other two terms are zero and finite respectively by symmetry. This concludes the proof that (4.4) is finite, and so concludes the proof of Theorem 4.4 for τ_e. To prove the (FTC) for (ι_o, τ'_+) with itself use the same method, letting j_o: $\mathcal{W}_o \longrightarrow \mathcal{H} \otimes \mathcal{V}$ be inclusion and making use of

$$(\iota_+ \otimes \mathrm{id})\, j_o = \iota_o \qquad j_o^* j_o = \mathrm{id}_{\mathcal{W}_o} \qquad 2 j_o j_o^* = \mathrm{id}_{\mathcal{H}\otimes\mathcal{V}} - (S \otimes s) \ .$$

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