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COHOMOLOGY COMPLEX PROJECTIVE SPACE WITH DEGREE ONE CODIMENSION-TWO FIXED SUBMANIFOLDS

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If M^{2n} is a cohomology $\mathbb{C}P^n$ and p is a prime, let $D_p(M^{2n})$ be the set of positive integers d such that $d \in D_p(M^{2n})$ if there exists a diffeomorphism of M^{2n} of order p fixing an orientable, codimension-2 submanifold of degree d. If $p = 2$ or n is odd, then $1 \in D_p(M^{2n})$ implies that $D_p(M^{2n}) = \{1\}$. The case p odd and n even is also investigated. If M^{4m} is a homotopy $\mathbb{C}P^{2m}$ and $m \not\equiv 0, 4$, or $7 \pmod 8$, then $1 \in D_3(M^{4m})$ implies that $D_3(M^{4m}) = \{1\}$.

1. Introduction.

A cohomology complex projective n-space is a smooth, closed, orientable $2n$-manifold M^{2n} such that there is a class $x \in H^2(M;\mathbb{Z})$ with the property that $H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$. If $i : K^{2n-2} \subset M^{2n}$ is the inclusion map of a closed, connected, orientable submanifold and d is an integer, we will say that the degree of K^{2n-2} is d if $i_*[K]$ is the Poincaré dual of dx. We will always assume that the orientation of K^{2n-2} is chosen in such a way that d is nonnegative. Let p be a prime number and let G_p denote the cyclic group of order p. Let $D_p(M^{2n})$ be the set of positive integers d defined by the condition that $d \in D_p(M^{2n})$ if M^{2n} admits a smooth G_p action such that the fixed point set of the action contains a codimension-2 submanifold of degree d. If $d \in D_p(M^{2n})$, then $d \not\equiv 0 \pmod p$, (see [**2**, pp. 378-383]). The following conjecture is motivated by the work of several authors ([**3**], [**4**], [**6**], [**8**]).

Conjecture 1.0. *If $D_p(M^{2n})$ is nonempty, then $D_p(M^{2n}) = \{1\}$.*

This conjecture has been verified for small values of n ([**3**], Theorem A ($n = 3, p \geq 3, n = 4, p > 3$), [**4**], Corollary 4.5 ($n = 4, p = 2$), [**7**], Theorem 1.7 ($n = 4, p = 3$)). We begin this paper with the observation that a weaker version of the conjecture is true if $p = 2$ or n is odd.

Theorem 1.1. *Let M^{2n} be a cohomology complex projective n-space.*

(1) *If $p = 2$ and $1 \in D_2(M^{2n})$, then $D_2(M^{2n}) = \{1\}$.*

(2) *If n is odd and p and q are primes, then $1 \in D_p(M^{2n})$ implies that either $D_q(M^{2n})$ is empty or $D_q(M^{2n}) = \{1\}$.*

(3) *If n is odd or $p = 2$, then $D_p(M^{2n})$ is a finite subset of the odd natural numbers.*

It is easy to see that if $\mathbb{C}P^n$ is complex projective n-space, then $1 \in D_p(\mathbb{C}P^n)$, and so, if $p = 2$ or n is odd, it follows from Theorem 1.1 that $D_p(\mathbb{C}P^n) = \{1\}$. This result appears in the literature ([**4**], Theorem A), but Theorem 1.1 does not. Less is known about $D_p(M^{2n})$ if p is odd and n is even. We will state a theorem similar to Theorem 1.1 about the case p odd and n even after some preparation.

Suppose that n and p are arbitrary and that M^{2n} admits a smooth G_p action fixing a closed, connected submanifold F^{2n-2}. If $n > 2$ and $p \geq 2$ or $n = 2$ and $p > 2$, then the fixed point set of the action consists of F^{2n-2} and an isolated point and F^{2n-2} is a $\mathbb{Z}/p\mathbb{Z}-$cohomology $\mathbb{C}P^{n-1}$. If $n = 2$ and $p = 2$, then there are two possibilities, either the fixed point set is S^2 and an isolated point or $\mathbb{R}P^2$ ([**2**, pp. 378-383]). If M^{2n} admits a G_p action fixing F^{2n-2} and an isolated point, then the action is said to be of Type II_0. An action of G_p on M^{2n} fixing F^{2n-2} is of Type II_0 if and only if F^{2n-2} is orientable ([**4**], Lemma 4.1). This means that $D_p(M^{2n})$ is the set of degrees arising from actions of Type II_0. The set $D_p(M^{2n})$ is related to a larger set of invariants which contains information about the tangent representation at the isolated fixed point. We will define this set in the next paragraph.

Suppose that p is an odd prime, g is a generator of G_p and $\lambda = \exp(2\pi i/p)$. If M^{2n} admits a G_p action of Type II_0 fixing F^{2n-2}, then the normal bundle of $F^{2n-2} \subset M^{2n}$ has a complex structure and the eigenvalue of the action of a generator g of G_p on the normal bundle of $F^{2n-2} \subset M^{2n}$ is λ if g is chosen properly. If pt is the isolated fixed point, then the tangent space $\tau_{pt}(M^{2n})$ may be thought of as a complex representation of G_p, and with the right choice of complex structure, the eigenvalues of the differential of g are contained in the set $\{\lambda^j : 1 \leq j \leq \mu\}$, where $\mu = (p-1)/2$. Let m_j be the multiplicity of the eigenvalue λ^j and let $DE_p(M^{2n})$ be the set of $(\mu+1)-$ tuples of integers such that $(d; m_1, m_2, \dots, m_\mu) \in DE_p(M^{2n})$ if M^{2n} admits a G_p action fixing a submanifold of degree d and having multiplicities $m_1, m_2, \dots, m_\mu$ at the isolated fixed point. Note that $m_1 + m_2 + \cdots + m_\mu = n$. If p is odd, then $D_p(M^{2n})$ is the image of $DE_p(M^{2n})$ under projection on the first factor. Trivially, $DE_3(M^{2n}) = D_3(M^{2n})$. There is evidence to support the following strengthened version of Conjecture 1.0.

Conjecture 1.2. *If $DE_p(M^{2n})$ is nonempty, then*

$$DE_p(M^{2n}) = \{(1; n, 0, \dots, 0)\}.$$

Conjecture 1.2 is equivalent to the conjecture that if M^{2n} admits a G_p action of Type II_0, then the degree of the fixed submanifold is 1 and the representation of G_p at the isolated fixed point is n times the representation of G_p at the normal bundle of the fixed submanifold. The conjecture is true if $n = 3$ or 4 and $p \geq 3$ ([**3**], Theorem $A(n = 3, p \geq 3; n = 4, p > 3)$, [**7**], Theorem 1.7 $(n = 4, p = 3)$). Theorem 1.1 in the case $n = 2m+1$ and p odd can be phrased in terms of $DE_p(M^{4m+2})$.

Theorem 1.3. *Suppose that* $(1; m_1, m_2, \dots, m_\mu) \in DE_p(M^{4m+2})$. *If* $(d; m_1', m_2', \dots, m_\mu')$ *is an element of* $DE_p(M^{4m+2})$, *then* $d = 1$.

We will prove a theorem similar to Theorem 1.3 about $DE_p(M^{4m}), p$ odd. Our result is not as strong as Theorem 1.3, but is strong enough to contain new information about $DE_p(\mathbb{C}P^{2m})$ and $D_3(\mathbb{C}P^{2m})$.

Theorem 1.4. *Suppose that* M^{4m} *is a homotopy* $\mathbb{C}P^{2m}$. *Assume that* $(1; m_1, m_2, \dots, m_\mu)$ *and* $(d; m_1, m_2, \dots, m_\mu)$ *are both elements of* $DE_p(M^{4m})$. *If* $m \not\equiv 0 \pmod 4$, *then* d *is odd. If* $m \not\equiv 0, 4,$ *or* $7 \pmod 8$, *then* $d = 1$.

Corollary 1.5. *Suppose that* $(d; 2m, 0, \dots, 0) \in DE_p(\mathbb{C}P^{2m})$. *If* $m \not\equiv 0 \pmod 4$, *then* d *is odd. If* $m \not\equiv 0, 4,$ *or* $7 \pmod 8$, *then* $d = 1$.

Note that Corollary 1.5 follows immediately from Theorem 1.4 because $(1; n, 0, \dots, 0) \in DE_p(\mathbb{C}P^n)$ for arbitrary n. If $V_2^{4m-2} = \{[z_0, z_1, \dots, z_{2m+1}] \in \mathbb{C}P^{2m} : z_0^2 + z_1^2 + \cdots + z_{2m+1}^2 = 0\}$, then V_2^{4m-2} is a $\mathbb{Q}$-cohomology $\mathbb{C}P^{2m-1}$ ([**11**], p. 71) and so it can not be eliminated as a possible codimension-2 component of an action of Type II_0 on $\mathbb{C}P^{2m}$ on the basis of cohomological criteria ([**2**], pp. 378-383). If $d \in D_3(\mathbb{C}P^n)$, n arbitrary, then $d^2 \equiv 1 \pmod 9$, and so V_2^{4m-2} is not fixed by a G_3 action on $\mathbb{C}P^{2m}$ ([**8**], Corollaries D and E). If V_2^{4m-2} is fixed by a G_p action on $\mathbb{C}P^{2m}$ and $p \geq 5$, what can be said about the multiplicities of the eigenvalues of the action at the isolated fixed point? It follows from Corollary 1.5 that if $m \not\equiv 0 \pmod 4$, $p \geq 5$, and V_2^{4m-2} is fixed by a G_p action on $\mathbb{C}P^{2m}$, then the multiplicities at the isolated fixed point are exotic, that is $(m_1, m_2, \dots, m_\mu) \neq (2m, 0, \dots, 0)$. It is not known if $\mathbb{C}P^{2m}$ admits a G_p action of Type II_0 with exotic multiplicities at the isolated fixed point.

Theorem 1.6. *Suppose that* M^{4m} *is a homotopy* $\mathbb{C}P^{2m}$ *and that* $m \not\equiv 0, 4,$ *or* $7 \pmod 8$. *If* $1 \in D_3(M^{4m})$, *then* $D_3(M^{4m}) = \{1\}$.

Corollary 1.7. *If* $m \not\equiv 0, 4,$ *or* $7 \pmod 8$, *then* $D_3(\mathbb{C}P^{2m}) = \{1\}$.

Theorem 1.6 and Corollary 1.7 are immediate consequences of Theorem 1.4 and they add to our understanding of $D_3(M^{2n})$ in general and $D_3(\mathbb{C}P^n)$

in particular. Upper bounds for $D_3(M^{2n})$, n arbitrary, in terms of prime divisors are known ([**7**, Theorem 1.6]), but knowledge of these upper bounds produced results weaker than Corollary 1.7. We were only able to produce the result that $D_3(\mathbb{C}P^{10}) = \{1\}$ ([**7**], p. 177) using these methods and this result is contained in Corollary 1.7.

This paper is organized as follows. Section 2 contains a proof of Theorem 1.1 based on a congruence for the degree of the fixed submanifold which is valid if $p = 2$ or n is odd. Section 3 contains integrality results for the signatures of self-intersections of codimension-2 submanifolds of arbitrary $2n$-manifolds. In Section 4, we apply the results of Section 3 to the study of G_p actions of Type II_0. We show that the Atiyah-Singer $g-$Signature Formula for actions of Type II_0 reduces to a formula involving $(d; m_1, m_2, \dots, m_\mu)$, the signatures of the self-intersections of a submanifold of degree 1, and algebraic numbers $\alpha_j = (\lambda^j + 1)(\lambda^j - 1)^{-1}, 1 \leq j \leq \mu$. This formula is a special case of the Berend-Katz version of the Atiyah-Singer $g-$Signature Formula ([**1**], Theorem 2.2). Section 5 contains some combinatorial material which will be used in the proof of a theorem in Section 6 which contains Theorem 1.4 as a special case.

2. Degree one fixed submanifolds.

If M^{2n} is a cohomology $\mathbb{C}P^n$, let K_x^{2n-2} be an oriented submanifold dual to $x \in H^2(M; \mathbb{Z})$, a generator of the cohomology algebra, that is, K_x^{2n-2} is a submanifold of degree 1. Such a submanifold can always be found ([**10**], Théorèm II. 27). If n is a positive integer, let $f(n)$ be $n!$ divided by a maximal power of 2. Let $K_x^{(s)}$ be the s-fold transverse self-intersection of K_x in M^{2n}. The numerical congruences in the next theorem relate integers in the set $D_p(M^{2n})$ to the signatures of K_x and $K_x^{(2)}$ if $p = 2$ or n is odd.

Theorem 2.1. ([**4**, Theorem B]). *If* $d \in D_p(M^{2n})$, *then*

$$\pm f(n) \equiv f(n) d \operatorname{Sign} K_x (\operatorname{mod} \; d(1-d^2)), \text{ if } n \text{ is odd,} \tag{2.2}$$

$$\pm f(n) \equiv f(n) d^2 \operatorname{Sign} K_x^{(2)} (\operatorname{mod} \; 2d^2(1-d^2)), \text{ if } n \text{ is even and } p = 2. \tag{2.3}$$

Corollary 2.4. (1) *Suppose that* $1 \in D_p(M^{2n})$. *If* n *is odd, then* $\operatorname{Sign} K_x = \pm 1$. *If* n *is even and* $p = 2$, *then* $\operatorname{Sign} K_x^{(2)} = \pm 1$.

(2) *Suppose that* $D_p(M^{2n})$ *is not empty. If* n *is odd and* $\operatorname{Sign} K_x = \pm 1$, *or if* n *is even,* $p = 2$, *and* $\operatorname{Sign} K_x^{(2)} = \pm 1$, *then* $D_p(M^{2n}) = \{1\}$.

Proof. We begin by verifying statement (1). Suppose that $1 \in D_p(M^{2n})$. If n is odd, then it follows immediately from (2.2) that Sign $K_x = \pm 1$. If n is even and $p = 2$, then it follows immediately from (2.3) that Sign $K_x^{(2)} = \pm 1$.

Our next step is the verification of statement (2). Suppose that $d \in D_p(M^{2n})$. If n is odd and Sign $K_x = \pm 1$, then it follows from (2.2) that $f(n)(\pm 1 \pm d) \equiv 0 \pmod{d(1-d^2)}$. If $d \neq 1$, this implies that either $d(1+d)$ or $d(1-d)$ divides $f(n)$. Neither divisibility condition is possible since $f(n)$ is odd and so $d = 1$. If n is even, $p = 2$, and Sign $K_x^{(2)} = \pm 1$, then it follows from (2.3) that $f(n)(\pm 1 \pm d^2) \equiv 0 \pmod{2d^2(1-d^2)}$. This congruence can only hold if the signs of 1 and d^2 are not the same because (2.3) implies that d^2 divides $f(n)$ and so in particular d is odd. Therefore $f(n)(1+d^2) \not\equiv 0 \pmod{2d^2(1-d^2)}$ because $1 + d^2 \not\equiv 0 \pmod 8$ and $1 - d^2 \equiv 0 \pmod 8$. This means $f(n)(1-d^2) \equiv 0 \pmod{2d^2(1-d^2)}$. The assumption $d \neq 1$ leads to the contradiction that $f(n)$ is even after dividing by $1 - d^2$ and so $d = 1$. □

Proof of Theorem 1.1. We begin by verifying statement (1). Suppose that $p = 2$ and $1 \in D_2(M^{2n})$. If n is odd, then it follows from statement (1) in Corollary 2.4 that Sign $K_x = \pm 1$ and so (2) in Corollary 2.4 implies that $D_2(M^{2n}) = \{1\}$. If n is even, then statement (1) in Corollary 2.4 implies that Sign $K_x^{(2)} = \pm 1$ and so $D_2(M^{2n}) = \{1\}$ by (2) in Corollary 2.4.

The verification of statement (2) proceeds as follows. Suppose that n is odd, p and q are primes such that $1 \in D_p(M^{2n})$ and $D_q(M^{2n})$ is not empty. Statement (1) in Corollary 2.4 implies that Sign $K_x = \pm 1$ and so (2) in Corollary 2.4 implies that $D_q(M^{2n}) = \{1\}$.

Statement (3) follows from (2.2) and (2.3) together with the fact $f(n)$ is odd. If n is odd and $d \in D_p(M^{2n})$, then (2.2) implies that d divides $f(n)$. It n is even, $p = 2$, and $d \in D_2(M^{2n})$, then it follows from (2.3) that d^2 divides $f(n)$. □

Note that if we take $p = q$ in statement (2) of Theorem 1.1 we obtain an assertion contained in the abstract of this paper. If n is odd and p is any prime, then $1 \in D_p(M^{2n})$ implies that $D_p(M^{2n}) = \{1\}$.

The rest of this paper is devoted to the study of $D_p(M^{2n})$ in the case n even and p odd. The hypothesis $(d; m_1, m_2, \ldots, m_\mu) \in DE_p(M^{2n})$ in this case leads to an equation similar to (2.2) and (2.3). This equation involves an integrality formula for the signatures of self-intersections of codimension-2 submanifolds.

3. An integrality theorem for signatures of self-intersections of codimension-2 submanifolds.

If M^{2n} is an arbitrary smooth, closed, oriented $2n$-manifold and $K^{2n-2} \subset M^{2n}$ is a closed, oriented submanifold, let $K^{(s)}$ denote the s-fold self-intersection of K in M. The dimension of $K^{(s)}$ is $2(n-s)$. If K is dual to a cohomology class $y \in H^2(M;\mathbb{Z})$, then we will use the notation K_y as in Section 2. If z is a complex number and d is a nonnegative integer, let

$$T_d(z) = [(1+z)^d - (1-z)^d]/[(1+z)^d + (1-z)^d]. \tag{3.1}$$

If $r_i(d)$ is the coefficient of z^{2i+1} in the Maclaurin series for $T_d(z)$, then $r_i(d)$ is a polynomial in d with rational coefficients and if $n-s$ is even, then Sign $K_{dy}^{(s)}$ can be expanded in terms of Sign $K_y^{(2k+s)}, 0 \leq k \leq (n-s)/2$, and certain combinations of the polynomials $r_i(d)$ ([4], formula (3.7)). The polynomials $r_i(d)$ factor in such a way that the signature expansion leads to a numerical congruence involving Sign $K_{dy}^{(s)}$ and Sign $K_y^{(s)}$ ([4], formula (3.8)). In this paper, we will combine the expansion and the congruence in a single integrality formula which will yield more detailed information (see formula (3.12)). Let $\mathbb{N}$ be the set of nonnegative integers and let $\mathbb{Q}$ be the set of rational numbers.

Definition 3.2. *If $k, s \in \mathbb{N} \setminus \{0\}$, then the function $R_{k,s} : \mathbb{N} \longrightarrow \mathbb{Q}$ is defined by*

$$R_{k,s}(d) = \sum_{i_1+i_2+\cdots+i_s=k} r_{i_1}(d) r_{i_2}(d) \ldots r_{i_s}(d). \tag{3.3}$$

The notation in (3.3) means that every possible choice of nonnegative integers $i_1, i_2, \ldots, i_s$ with $i_1 + i_2 + \cdots + i_s = k$ occurs in the summation. For example, $R_{k,1}(d) = r_k(d)$.

Lemma 3.4. *There exists a polynomial $c_{k,s}(d^2)$ with integer coefficients such that*

$$f(2k+s)R_{k,s}(d) = c_{k,s}(d^2)d^s(1-d^2). \tag{3.5}$$

Proof. We know ([4], Lemma 3.14) that $r_k(d) = d(1-d^2)q_k(d^2)$ where $f(2k+1)q_k(d^2)$ is a polynomial with integer coefficients and so (3.5) holds if $s = 1$ with $c_{k,1}(d^2) = f(2k+1)q_k(d^2)$. It follows from this fact, (3.3) and the fact that $\Pi_{j=1}^s f(2i_j+1)$ divides $f(2k+s)$ if $i_1 + i_2 + \cdots + i_s = k$ that (3.5) holds for $s > 1$ with $c_{k,s}(d^2)$ equal to a sum of products of the polynomials $c_{i_j,1}(d^2) = f(2i_j+1)q_{i_j}(d^2), 1 \leq j \leq s$. □

Note that (3.5) is a more precise formulation of (3.16) in [4]. The polynomials $q_k(d^2)$ involved in the construction of $c_{k,s}(d^2)$ in (3.5) are quite complicated ([7], Table 2.16). We record a recursion formula for these polynomials which we will use later. This formula follows from the factorization

$r_k(d^2) = d(1-d^2)q_k(d^2)$ and a recursion formula for $r_k(d^2)$ ([4], Lemma 3.14 and formula (3.18)). If $k \geq 2$ and $d \neq 1$, then

$$q_k(d^2) = \frac{\binom{d}{2k+1} - d\binom{d}{2k}}{d(1-d^2)} - \sum_{i=1}^{k-1} q_{k-i}(d^2)\binom{d}{2i}. \tag{3.6}$$

Our next step is to define an important integral multiple of $c_{k,s}(d^2)$, a polynomial associated with this multiple and a cohomology class $y \in H^2(M;\mathbb{Z})$. Note that $f(2k+s)$ divides $f(n)$ if $0 \leq k \leq (n-s)/2$.

Definition 3.7. If $n-s$ is even and $0 \leq k \leq (n-s)/2$, then

$$\hat{c}_{k,s}(d^2) = f(n)f(2k+s)^{-1}c_{k,s}(d^2). \tag{3.8}$$

Definition 3.9. If $K_y^{2n-2} \subset M^{2n}$ is as above and $n-s$ is a positive even integer, then

$$\delta_s(d^2, y) = \sum_{k=1}^{(n-s)/2} \hat{c}_{k,s}(d^2)\operatorname{Sign} K_y^{(2k+s)}. \tag{3.10}$$

We set $\delta_n(d^2, y) = 0$.

Theorem 3.11. *Suppose that $y \in H^2(M;\mathbb{Z})$ and that $K_y^{2n-2} \subset M^{2n}$ is dual to y. If $n-s$ is even, then*

$$f(n)\operatorname{Sign} K_{dy}^{(s)} = f(n)d^s \operatorname{Sign} K_y^{(s)} + d^s(1-d^2)\delta_s(d^2, y). \tag{3.12}$$

Proof. Formula (3.12) follows from the expansion

$$\operatorname{Sign} K_{dy}^{(s)} = d^s \operatorname{Sign} K_y^{(s)} + \sum_{k=1}^{(n-s)/2} R_{k,s}(d)\operatorname{Sign} K_y^{(2k+s)}, \tag{3.13}$$

([4], formula (3.7)) by multiplying both sides of the expansion by $f(n)$ and using Lemma 3.4 together with Defintions 3.7 and 3.9. ☐

Formula (3.12) is the integrality formula for the signature Sign $K_{dy}^{(s)}$ promised at the beginning of this section. It has some advantages over (3.13) for some applications because every term in (3.12) is an integer. A congruence for $f(n)$ Sign $K_{dy}^{(s)}$ can be read off immediately from (3.12).

Corollary 3.14. *Suppose that $y \in H^2(M;\mathbb{Z})$ and that $K_y^{2n-2} \subset M^{2n}$ is dual to y. If $n-s$ is even, then*

$$f(n)\operatorname{Sign} K_{dy}^{(s)} \equiv f(n)d^s \operatorname{Sign} K_y^{(s)} \pmod{d^s(1-d^2)}. \tag{3.15}$$

Formula (3.15) is the same as (3.8) in [**4**]. Our next step is to make a combinatorial analysis of $\delta_s(d^2, y)$ to obtain additional information from formula (3.12).

Lemma 3.16. *If* $k, s \in \mathbb{N} \setminus \{0\}$, *then*

$$c_{k,s}(d^2) \equiv sf(2k+s)q_k(d^2) \pmod{(1-d^2)}. \tag{3.17}$$

Proof. Formula (3.17) holds for $s = 1$ since $c_{k,1}(d^2) = f(2k+1)q_k(d^2)$ (see the proof of Lemma 3.4.). If $s \geq 2$, it follows from (3.3) and the factorization $r_k(d) = d(1-d^2)q_k(d^2)$ ([4], Lemma 3.14) that

$$R_{k,s}(d) = sd^s(1-d^2)q_k(d^2) + \sum r_{i_1}(d)r_{i_2}(d)\dots r_{i_s}(d), \tag{3.18}$$

where the summation is taken over all partitions $i_1, i_2, \dots, i_s$ of k with at least two i_j positive. It follows from the factorization $r_{i_j}(d) = d(1-d^2)q_{i_j}(d^2)$, the facts that $\Pi_{j=1}^s f(2i_j+1)$ divides $f(2k+s)$ if $i_1 + i_2 + \dots + i_s = k$ and that at least two i_j are positive in the summation in (3.18) that

$$f(2k+s)R_{k,s}(d) \equiv sd^s(1-d^2)f(2k+s)q_k(d^2) \pmod{d^s(1-d^2)^2}. \tag{3.19}$$

Formula (3.17) now follows from (3.19) and (3.5). ☐

Corollary 3.20. *If* $n - s$ *is even, then*

$$\delta_s(d^2, y) \equiv s \sum_{k=1}^{(n-s)/2} f(n)q_k(d^2)\operatorname{Sign} K_y^{(2k+s)} (\operatorname{mod}(1-d^2)), \tag{3.21}$$

where the summation is taken to be zero if $n = s$.

Proof. Immediate from (3.8), (3.10) and Lemma 3.16. ☐

Theorem 3.22. *Suppose that* $y \in H^2(M; \mathbb{Z})$ *and that* $K_y^{2n-2} \subset M^{2n}$ *is dual to* y. *If* $n - s$ *is even, then*

$$f(n)\operatorname{Sign} K_{dy}^{(s)} \equiv \tag{3.23}$$

$$f(n)d^s \operatorname{Sign} K_y^{(s)} + sd^s(1-d^2) \sum_{k=1}^{(n-s)/2} f(n)q_k(d^2)\operatorname{Sign} K_y^{(2k+s)} (\operatorname{mod} d^s(1-d^2)^2).$$

Proof. Immediate from (3.12) and (3.21). ☐

Note that (3.23) contains (3.15) in this paper and (3.9) in [**4**] as special cases. This type of sharpened congruence will be important when we return to cohomology projective spaces with G_p actions of Type II_0 in the next section.

4. The Atiyah-Singer g-Signature Theorem for G_p actions of type II_0.

In this section, we return to the main theme of this paper, cohomology projective space with G_p actions of Type II_0. If M^{2n} is a cohomology $\mathbb{C}P^n$ and $x \in H^2(M;\mathbb{Z})$ is a generator of the cohomolgy algebra, then (3.12) is an integrality formula for the signature of the s-fold self-intersection of a submanifold of degree d in terms of d, the signature of $K_x^{(s)}$ and the polynomial $\delta_s(d^2,x)$. Our next step is to introduce two polynomial functions of a complex variable associated with M^{2n} and d.

Definition 4.1. If M^{2n} is cohomology $\mathbb{C}P^n$ and $d \in \mathbb{N}$, then

$$P(z) = \begin{cases} \sum_{k=1}^{m} \operatorname{Sign} K_x^{(2k)} z^{2k-2}, & n = 2m, \\ \sum_{k=1}^{m} \operatorname{Sign} K_x^{(2k+1)} z^{2k-2}, & n = 2m+1. \end{cases} \tag{4.2}$$

$$Q_d(z) = \begin{cases} \sum_{k=1}^{m-1} \delta_{2k}(d^2,x) z^{2k-2}, & n = 2m, \\ \sum_{k=1}^{m-1} \delta_{2k+1}(d^2,x) z^{2k-2}, & n = 2m+1. \end{cases} \tag{4.3}$$

We state the Atiyah-Singer g-Signature Formula for G_p actions of Type II_0 in terms of d, $P(z)$, $Q_d(z)$, and the complex numbers $\alpha_j = (\lambda^j+1)(\lambda^j-1)^{-1}, 1 \leq j \leq \mu$.

Theorem 4.4. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$ and that p is an odd prime. If $(d;m_1,m_2,\ldots,m_\mu) \in DE_p(M^{2n})$, then*

$$(4.5)\quad f(n)\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_\mu^{m_\mu} =$$
$$\begin{cases} \pm f(n) \pm f(n)d^2(\alpha_1^2-1)P(d\alpha_1) + d^2(1-d^2)(\alpha_1^2-1)Q_d(d\alpha_1), & n \text{ even}, \\ \pm f(n)\alpha_1 \pm f(n)d^3(\alpha_1^3-\alpha_1)P(d\alpha_1) + d^3(1-d^2)(\alpha_1^3-\alpha_1)Q_d(d\alpha_1), & n \text{ odd}. \end{cases}$$

Before we turn to the proof of Theorem 4.4, we accept it and deduce some consequences. The next corollary is an immediate consequence of (4.5) and the fact that the coefficients of $P(z)$ and $Q_d(z)$ are rational integers.

Corollary 4.6. *If p is an odd prime, then the Atiyah-Singer g-Signature Formula for a G_p action of Type II_0 on a cohomology $\mathbb{C}P^n$ is an equation in the ring of complex numbers $\mathbb{Z}[\alpha_1,\alpha_2,\ldots,\alpha_\mu]$.*

Corollary 4.6 is a special case of a theorem of Berend and Katz which exhibits the general Atiyah-Singer g-Signature Formula as a formula in a ring of complex numbers ([**1**], Theorem 2.2). Formula (4.5) expresses the signature formula in a form useful for our calculations. Earlier efforts showed no clear pattern for arbitrary n ([**5**], p. 573). The term in (4.5) involving $Q_d(d\alpha_1)$ is the modulus of a congruence in the ring $\mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_\mu]$ ([**7**], Theorem 4.2). Knowledge of this modulus will enable us to obtain more information about d.

It is worth recording (4.5) in the case $p = 3$. Then $\mu = 1$, $\alpha_1 = -i/\sqrt{3}$, and $3^{[n/2]-1}P(\frac{di}{\sqrt{3}})$ and $3^{[n/2]-1}Q_d(\frac{di}{\sqrt{3}})$ are rational integers. If we define a numerical function $a(n) = f(n)[3^{[n/2]} + (-1)^{[n/2]-1}]/4$, then (4.5) is an equation of rational integers involving $a(n)$.

Corollary 4.7. *If $d \in D_3(M^{2n})$, then*

(4.8)
$$\pm a(n) = \begin{cases} f(2m)d^2 3^{m-1}P(\frac{di}{\sqrt{3}}) + d^2(1-d^2)3^{m-1}Q_d(\frac{di}{\sqrt{3}}), & n = 2m, \\ f(2m+1)d^3 3^{m-1}P(\frac{di}{\sqrt{3}}) + d^3(1-d^2)3^{m-1}Q_d(\frac{di}{\sqrt{3}}), & n = 2m+1. \end{cases}$$

Formula (4.8) contains information about the modulus of a congruence of rational integers for $D_3(M^{2n})$ ([**7**], Theorem 5.1) as well as the fact that $d \in D_3(M^{2n})$ implies d^2 divides $a(n)$ if n is even and d^3 divides $a(n)$ if n is odd. This divisibility condition was used to obtain upper bounds for $D_3(M^{2n})$ in terms of prime divisors ([**7**], Table 5.4).

The proof of Theorem 4.4 is based on (3.12) and a formula of Berend and Katz for the contribution $L_\theta(\nu)L(F)[F]$ to the signature formula of the normal bundle ν of a codimension-2 submanifold F of an arbitrary $2n$−manifold with a smooth, orientation preserving G_p action fixing F. This contribution is the product of a nonstable characteristic class, $L_\theta(\nu)$, depending on $\theta = 2\pi/p$, and $L(F)$, the total Hirzebruch L-class of F, evaluated on the fundamental class of F. We choose the generator of G_p so that the eigenvalue of the action of G_p on ν is λ.

Proposition 4.9. ([**1**, Formula (8.1)]). *If M^{2n} is an arbitrary $2n$-manifold which admits a smooth, orientation preserving G_p action fixing a codimension-2 submanifold F, then*

(4.10)
$$L_\theta(\nu)L(F)[F] = \begin{cases} -(\alpha_1^2 - 1)\sum_{k=1}^{m} \alpha_1^{2k-2}\operatorname{Sign} F^{(2k)}, & n = 2m, \\ \alpha_1 \operatorname{Sign} F + (\alpha_1^3 - \alpha_1)\sum_{k=1}^{m} \alpha_1^{2k-2}\operatorname{Sign} F^{(2k+1)}, & n = 2m+1. \end{cases}$$

Corollary 4.11. (Atiyah-Singer G-Signature Formula). *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$ and that p is an odd prime. If M^{2n} admits a smooth G_p action of Type II_0 fixing $F^{2n-2} \subset M^{2n}$ and having eigenvalue multiplicities $m_1, m_2, \ldots, m_\mu$ at the isolated fixed point, then the g-signature of the action is given by*

$$\text{Sign}(g, M) = \pm L_\theta(\nu) L(F)[F] \pm \alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_\mu^{m_\mu}. \tag{4.12}$$

Proof of Theorem 4.4. Formula (4.5) follows by multiplying both sides of (4.12) by $f(n)$ and then using (3.12) and (4.10) together with the facts that Sign $(g, M) = \pm 1$ if n is even and Sign $(g, M) = 0$ if n is odd, which follow immediately from the fact that M^{2n} is a cohomology $\mathbb{C}P^n$, and Sign $F = \pm 1$ if n is odd ([**4**], Lemma 4.1). □

In our next lemma, we begin our study of the effect on $DE_p(M^{2n})$ of the presence of a fixed submanifold of degree one in the case n even. We will see that if a submanifold of degree other than one is fixed, then under certain conditions an equation holds which leads to a contradiction in some cases.

Lemma 4.13. *Suppose that M^{4m} is a cohomology $\mathbb{C}P^{2m}$ and that $(1; m_1, m_2, \ldots, m_\mu)$ is an element of $DE_p(M^{4m})$. If $(d; m_1', m_2', \ldots, m_\mu') \in DE_p(M^{4m}), d \neq 1$, and $\alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_\mu^{m_\mu} = \pm \alpha_1^{m_1'} \alpha_2^{m_2'} \ldots \alpha_\mu^{m_\mu'}$, then*

$$d^2 Q_d(d\alpha_1) = f(2m) \sum_{k=1}^{m} \text{Sign}\, K_x^{(2k)} (1 + d^2 + \cdots + d^{2k-2}) \alpha_1^{2k-2}. \tag{4.14}$$

Proof. It follows from the hypotheses and (4.5) that

$$\begin{aligned} &f(2m) \alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_\mu^{m_\mu} \\ &\quad = \begin{cases} \pm f(2m) \pm f(2m) d^2 (\alpha_1^2 - 1) P(d\alpha_1) + d^2 (1 - d^2)(\alpha_1^2 - 1) Q_d(d\alpha_1), \\ \pm f(2m) \pm f(2m)(\alpha_1^2 - 1) P(\alpha_1). \end{cases} \end{aligned} \tag{4.15}$$

The choice of signs must be the same in the corresponding terms in the two lines in (4.5) as indicated. To see this, suppose that the signs of $f(2m)$ are not the same. Consider the image of the difference of the two lines under a homomorphism $\eta : \mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_\mu] \longrightarrow \mathbb{Z}/4\mathbb{Z}$ such that $\eta(1) = 1$ and $\eta(\alpha_j) = \pm 1, 1 \leq j \leq \mu$ ([**1**], Lemma 7.8). If the signs of $f(2m)$ are not the same, the contradiction $2 \equiv 0 \pmod 4$ is obtained. Therefore the signs of $f(2m)$ are the same in both lines. It follows that the same choice of orientation is used in computing Sign (g, M) in the two lines and so the signs of the terms involving $P(z)$ in the two lines are the same since

$\operatorname{Sign} K_x^{(2k)} = \{\tanh^{2k} xL(M)\}[M]$ ([**7**], Lemma 2.1). Formula (4.14) now follows by taking the difference of the two lines in (4.15) and dividing by $1 - d^2 \neq 0$ and $\alpha_1^2 - 1$. □

Definition 4.16. If $d, m \in \mathbb{N} \setminus \{0\}$, then

$$b(d^2, m) = \sum_{k=0}^{m-1} (m-k) d^{2k}, \tag{4.17}$$

$$\Delta(d^2, m) = \sum_{k=1}^{m-1} \left(\sum_{l=1}^{m-k} \hat{c}_{l,2k}(d^2) \right) d^{2k}. \tag{4.18}$$

The next step is to apply one of the homomorphisms

$$\eta : \mathbb{Z}[\alpha_1, \alpha_2, \ldots, \alpha_\mu] \longrightarrow \mathbb{Z}/4\mathbb{Z}$$

described above to (4.14) and obtain an equation which leads to a numerical congruence used in Section 6.

Proposition 4.19. *Suppose that M^{4m} is a homotopy $\mathbb{C}P^{2m}$ and that $(1; m_1, m_2, \ldots, m_\mu)$ is an element of $DE_p(M^{4m})$. If $(d; m_1', m_2', \ldots, m_\mu') \in DE_p(M^{4m})$, $d \neq 1$, and $\alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_\mu^{m_\mu} = \pm \alpha_1^{m_1'} \alpha_2^{m_2'} \ldots \alpha_\mu^{m_\mu'}$ then*

$$\Delta(d^2, m) \equiv \pm b(d^2, m) \pmod 4. \tag{4.20}$$

Proof. Since M^{4m} is a homotopy $\mathbb{C}P^{2m}$, it follows that $\operatorname{Sign} K_x^{(2k)} \equiv 1 \pmod 8, 1 \leq k \leq m$. In fact, $\operatorname{Sign} K_x^{(2k)} = 1 + 8\sigma_{2(m-k)}$, where $\sigma_{2(m-k)}$ is a Sullivan splitting invariant [**9**]. It follows from (3.10) and (4.3), that if M^{4m} is a homotopy $\mathbb{C}P^{2m}$, then $d^2 Q_d(d) \equiv \Delta(d^2, m) \pmod 4$. Formula (4.20) follows from this observation and the equation in $\mathbb{Z}/4\mathbb{Z}$ which results when any of the homomorphisms η described above is applied to (4.14). □

5. Combinatorics.

The purpose of this section is to analyze both sides of (4.20) and to determine the set of values of m and d for which (4.20) holds. We begin with the right side of (4.20).

Lemma 5.1. *If $d, m \in \mathbb{N} \setminus \{0\}$, then*

$$d \text{ odd} \Longrightarrow b(d^2, m) \equiv \begin{cases} 0 \pmod 4, m \equiv 0, 7 \pmod 8, \\ 1 \pmod 4, m \equiv 1, 6 \pmod 8, \\ 2 \pmod 4, m \equiv 3, 4 \pmod 8, \\ 3 \pmod 4, m \equiv 2, 5 \pmod 8. \end{cases} \tag{5.2}$$

$$(5.3) \qquad d \text{ even } \Longrightarrow b(d^2,m) \equiv m \pmod 4.$$

Proof. If follows from (4.17) that if d is odd, then $b(d^2,m) \equiv m(m+1)/2 \pmod 4$. Formula (5.2) follows from this fact and (5.3) follows immediately from (4.17). □

The analysis of the left hand side of (4.20) is more difficult because $\Delta(d^2,m)$ involves the numerical functions $\hat{c}_{l,2k}(d^2), 1 \le l \le m-k, 1 \le k \le m-1$ (formula (3.8)). The first step in the determination of $\Delta(d^2,m) \pmod 4$ involves formula (3.17).

Lemma 5.4. *If $d, m \in \mathbb{N} \setminus \{0\}$, then*

$$(5.5) \qquad \Delta(d^2,m) \equiv \begin{cases} 2\sum\limits_{k=1}^{m-1} \binom{m+1-k}{2} f(2m)q_k(1) \pmod 4, d \text{ odd}, \\ 0 \pmod 4, d \text{ even}. \end{cases}$$

Proof. Formula (5.5) in the case of d even follows immediately from (4.18). To see that (5.5) holds in the case d odd, note that it follows from (3.8) and (3.17) that if $1 \le l \le m-k, 1 \le k \le m-1$, and d is arbitrary, then $f(2m)q_l(d^2)$ is an integer and

$$(5.6) \qquad \hat{c}_{l,2k}(d^2) \equiv 2kf(2m)q_l(d^2) \pmod{(1-d^2)}.$$

Formula (5.5) in the case d odd follows by inserting the mod 4 information provided by (5.6) into (4.18), reversing the order of summation, noting that the sum of the first $m-l$ integers is $\binom{m+1-l}{2}$, and changing the final summation index from l to k. □

It is clear from (5.5) that we can determine $\Delta(d^2,m)$ mod 4 if we can determine the parity of the integers $f(2m)q_k(1), 1 \le k \le m-1$. Recall that $q_k(d^2)$ is a rational number such that $f(2k+1)q_k(d^2)$ is an integer ([**4**], Lemma 3.14) and there is a recursion formula for $q_k(d^2)$ if $k \ge 2$ and $d \ne 1$ (see formula (3.6)). The parity of $f(2m)q_k(1)$ is the same as the parity of $f(2k+1)q_k(1), 1 \le k \le m-1$, and so the information we need is in the next lemma.

Lemma 5.7. *If $k \ge 1$, then $f(2k+1)q_k(1)$ is odd.*

Proof. Note that if d is odd, then $f(2k+1)q_k(1) \equiv f(2k+1)q_k(d^2) \pmod 8$ since $d^2 \equiv 1 \pmod 8$. It follows from (3.6) that if $k \ge 2$, then $f(2k+1)q_k(9) = -f(2k+1)\binom{3}{2}q_{k-1}(9)$. Since $q_1(d^2) = 1/3$ for any d ([**4**], Table

3.19), it follows by induction that if $k \geq 2$, then $f(2k+1)q_k(9)$ is odd and so $f(2k+1)q_k(1)$ is odd if $k \geq 2$ by the above remark. The proof of the lemma is complete since $f(3)q_1(1) = 1$. ☐

Lemma 5.8. *If $d, m \in \mathbb{N} \setminus \{0\}$, then*

$$d \text{ odd } \Longrightarrow \Delta(d^2, m) \equiv \begin{cases} 2 \pmod 4, & m \equiv 2 \pmod 4, \\ 0 \pmod 4, & m \not\equiv 2 \pmod 4. \end{cases} \tag{5.9}$$

$$d \text{ even } \Longrightarrow \Delta(d^2, m) \equiv 0 \pmod 4. \tag{5.10}$$

Proof. Formula (5.10) is just (5.5) in Lemma 5.4 in the case d even. To establish (5.9), note that it follows from the fact that $f(2m)q_k(1)$ is odd proven in Lemma 5.7 and (5.5) in the case d odd that (5.9) is equivalent to

$$\sum_{k=1}^{m-1} \binom{m+1-k}{2} \equiv \begin{cases} 1 \pmod 2, m \equiv 2 \pmod 4, \\ 0 \pmod 2, m \not\equiv 2 \pmod 4. \end{cases} \tag{5.11}$$

To see that (5.11) holds, note that

$$\sum_{k=1}^{m-1} \binom{m+1-k}{2} = \sum_{k=1}^{m-1} (m-k)k = \frac{(m+1)(m)(m-1)}{6}. \tag{5.12}$$

The first equality in (5.12) follows by noting that $\binom{m+1-k}{2}$ is the sum of the integers from 1 to $m-k$ and hence that $\sum_{k=1}^{m-1} \binom{m+1-k}{2}$ is the sum $(m-1)(1) + (m-2)(2) + \cdots + (1)(m-1)$. The second equality in (5.12) follows from well known summation formulas. Formula (5.11) follows immediately from (5.12). ☐

6. Proof of Theorem 1.4.

The purpose of this section is to state and prove a theorem which contains Thoerem 1.4 as a special case.

Theorem 6.1. *Suppose that M^{4m} is a homotopy $\mathbb{C}P^{2m}$. Assume that $(1; m_1, m_2, \ldots, m_\mu)$ and $(d; m'_1, m'_2, \ldots, m'_\mu)$ are both elements of $DE_p(M^{4m})$, and that $\alpha_1^{m_1}\alpha_2^{m_2}\ldots\alpha_\mu^{m_\mu} = \pm\alpha_1^{m'_1}\alpha_2^{m'_2}\ldots\alpha_\mu^{m'_\mu}$. If $m \not\equiv 0 \pmod 4$, then d is odd. If $m \not\equiv 0, 4$, or $7 \pmod 8$, then $d = 1$.*

Proof. Suppose that the hypotheses of the theorem are satisfied amd that d is even. It follows from (4.20), (5.3), and (5.10) that $m \equiv 0 \pmod 4$ and so, if $m \not\equiv 0 \pmod 4$, d is odd. If $m \not\equiv 0, 4$, or $7 \pmod 8$ and $d \neq 1$, then (4.20) does not hold in view of (5.2) and (5.9). ☐

The proof of Theorem 1.4 is now complete since Theorem 6.1 clearly contains Theorem 1.4 as a special case.

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