Pacific Journal of Mathematics

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Volume 173 No. 2 April 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 173, No. 2, 1996

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This paper aims to prove a Hörmander multiplier theorem for sub-Laplacians on nilpotent Lie groups. We investigate the holomorphic functional calculus of the sub-Laplacians, then we link the L^1 norm of the complex time heat kernels with the order of differentiability needed in the Hörmander multiplier theorem. As applications, we show that order $d/2+1$ suffices for homogeneous nilpotent groups of homogeneous dimension d, while for generalised Heisenberg groups with underlying space $\mathbf{R}^{2n+k}$ and homogeneous dimension $2n+2k$, we show that order $n+(k+5)/2$ for k odd and $n+3+k/2$ for k even is enough; this is strictly less than half of the homogeneous dimension when k is sufficiently large.

1. Introduction.

We begin with the classical Laplacian $(-\Delta)$ on the Euclidean space $\mathbf{R}^d$. The multiplier theorem of L. Hörmander [**Ho**] gives a sufficient condition on a function $m : \mathbf{R}^+ \to \mathbf{C}$ for the operator $m(-\Delta)$ to be bounded on $L^p(\mathbf{R}^d)$ whenever $1 < p < \infty$, namely, when m satisfies the condition that

$$\lambda^k \left| m^{(k)}(\lambda) \right| \leq c \qquad \forall \lambda \in \mathbf{R}^+ \tag{1}$$

for $0 \leq k \leq s = [d/2]+1$ where c is a constant and $[d/2]$ is the integral part of $d/2$.

By using fractional differentiation, the value of s in condition (1) can be improved slightly but it is known that for $(-\Delta)$ on $\mathbf{R}^d$, the value cannot be improved beyond $s = d/2$. We call s the order of the Hörmander multiplier theorem.

A lot of work has been done to obtain results of this type for other operators. E.M. Stein [**St**] proved a general result for a large class of operators, but only when the function m is of Laplace transform type, a rather restrictive condition. This was later improved by M. Cowling [**Co**], using the transference method and interpolation. For the sub-Laplacian L on a homogeneous nilpotent Lie group G of homogeneous dimension d, the following results are known. A. Hulanicki and Stein proved a Hörmander multiplier theorem for L with order $3d/2+2$ when G is a stratified group; this was reported by G.B. Folland and Stein [**FS**]. L. De Michele and G. Mauceri [**DM**] improved Hulanicki and Stein's results and obtained order $d/2+1$. Recently, M. Christ [**Ch**] investigated the problem carefully, and proved a Hörmander multiplier theorem with order $d/2$ when G is a homogeneous nilpotent group. His principal result was then reproved and extended by Mauceri and S. Meda [**MM**].

All the above results rely on certain estimates on the heat kernels, L^2 information derived from the spectral theorem, and the Calderón–Zygmund operator theory. However, the factors controlling the order s were to some extent hidden by the complexity of the proofs.

One open question is whether the condition $s \geq d/2$ is necessary as in the Euclidean case [**Ch**]. Another natural question is to decide what factors control the order s. It seemed that $s = d/2$ is the optimal value [**Ch**], but recently D. Müller and Stein (conference announcement) showed that for the Heisenberg group of homogeneous dimension $2n+2$, the order can be lowered to $n+1/2$.

In this paper, we show that the order s is controlled by the behaviour of the $L^1(G)$ norm of the heat kernels for complex time (Theorem 2). As a corollary, we obtain $s = d/2+1$ for homogeneous nilpotent groups (Theorem 3). Although this order is not optimal, our proof is different from and much easier than the previous proofs. Further, if G is the generalised Heisenberg group of homogeneous dimension $2n+2k$, with underlying manifold $\mathbf{R}^{2n+k}$, then we obtain a Hörmander multiplier theorem with order $s = n+k/2+\beta$ where $\beta = 5/2$ for k odd and $\beta = 3$ for k even (Theorem 4). This order is strictly less than half the homogeneous dimension when k is sufficiently large.

The author would like to thank Alan McIntosh and Jennifer Randall for several helpful conversations, and especially Michael Cowling who suggested the project and gave some valuable advice.

2. H_∞ functional calculus.

The references for this section are the papers of A. McIntosh [**Mc**] and Cowling, I. Doust, McIntosh, and A. Yagi [**CDMY**].

Definition. A closed operator L in a Banach space X is said to be of type ω, $0 \leq \omega < \pi$, if its spectrum is a subset of the closed sector $S_\omega = \{z \in C \mid |\arg z| \leq \omega\} \cup \{0\}$, and the resolvents $(L-\lambda I)^{-1}$ satisfy the inequality

$$\left\|(L-\lambda I)^{-1}\right\| \leq c_\mu \left|\lambda\right|^{-1}$$

when $|\arg \lambda| \geq \mu > \omega$.

For $\mu > \omega$, let $H_\infty(S_\mu^0)$ be the usual space of bounded holomorphic functions in the open sector S_μ^0, which is just the interior of S_μ. Further, define

$$\Psi(S_\mu^0) = \left\{ m \in H_\infty(S_\mu^0) \mid \exists s > 0, c > 0 \text{ such that } |m(z)| \leq \frac{c|z|^s}{1+|z|^{2s}} \right\}.$$

Suppose that $\omega < \theta < \mu$. Let γ be the contour defined by the function

$$\gamma(t) = \begin{cases} te^{i\theta} & \text{if } 0 \leq t < \infty \\ -te^{-i\theta} & \text{if } -\infty < t \leq 0. \end{cases}$$

We adopt the definitions of H_∞ functional calculus of [**Mc**], as follows. For $m \in \Psi(S_\mu^0)$, then

$$m(L) = \frac{1}{2\pi i} \int_\gamma (L - \lambda I)^{-1} m(\lambda)\, d\lambda.$$

The above integral is absolutely convergent in the norm topology and $m(L)$ is a bounded linear operator which is independent of the choice of θ. For $m \in H_\infty(S_\mu^0)$, we define

$$m(L) = \frac{1}{2\pi i}(I+L)^2 L^{-1} \int_\gamma (L - \lambda I)^{-1} \frac{\lambda m(\lambda)}{(1+\lambda)^2}\, d\lambda$$

when L is a one to one operator of type ω with dense domain and dense range. This definition is consistent with the previous one when $m \in \Psi(S_\mu^0)$.

We now define $\Lambda_{\infty,1}^\alpha(\mathbf{R}^+)$ to be the class of all bounded measurable functions $m : \mathbf{R}^+ \to \mathbf{C}$ such that $\|m\|_{\Lambda_{\infty,1}^\alpha} < \infty$, where

$$\|m\|_{\Lambda_{\infty,1}^\alpha} = \|m\|_\infty + \sum_{n \in \mathbf{Z}} 2^{|n|\alpha} \|(m \circ \exp) * \phi_n^\vee\|_\infty ;$$

in this definition, for all ξ in $\mathbf{R}$,

$$\phi_0(\xi) = (2 - 2|\xi|)_+ - (1 - 2|\xi|)_+,$$
$$\phi_1(\xi) = (1 - 2|\xi - 1|)_+ + \left(\frac{1}{2} - \left|\xi - \frac{3}{2}\right|\right)_+,$$

and

$$\phi_{n\epsilon}(\xi) = \phi_1(2^{-n}\epsilon\xi),$$

when $n = 1, 2, 3, \ldots$ and $\epsilon = \pm 1$; here $\phi^\vee$ denotes the inverse Fourier transform of ϕ. It is not hard to check, using Fourier analysis, that if condition

(1) holds when $k = 0, 1, 2, ..., s$, then $m \in \Lambda^{\alpha}_{\infty,1}(\mathbf{R}^+)$ when $\alpha < s$. It was observed by Coifman that there were similarities between having functional calculus for bounded analytic functions in all sectors and Hörmander-type theorems. The following theorem is proved in [**CDMY**] (Theorem 4.10).

Theorem 1. *Suppose that L is a one-one operator of type 0 in $L^p(X)$, $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *L admits a bounded $H_\infty(S^0_\mu)$ functional calculus for all positive μ and there exist positive constants C and α such that*

$$(2) \qquad \|m(L)\| \leq C\mu^{-\alpha}\,\|m\|_{H_\infty(S^0_\mu)} \qquad \forall m \in H_\infty(S_\mu) \quad \forall \mu > 0;$$

(ii) *L admits a bounded $\Lambda^{\alpha}_{\infty,1}(\mathbf{R}^+)$ functional calculus.*

In this paper, we prove that the H_∞ functional calculus of the sub-Laplacian on a homogeneous nilpotent group satisfies (2), hence there is a Hörmander type functional calculus. Note that to establish the existence of the H_∞ functional calculus, we just need to prove (2) for m in $\Psi(S_\mu)$, for the extension to m in $H_\infty(S_\mu)$ then follows from the Convergence Lemma in [**CDMY**] (Lemma 2.1).

In the rest of this paper, the constants C and c may vary from line to line.

3. The L^1 norms of the heat kernels and the Hörmander multiplier theorem.

Let $\underline{\mathbf{g}}$ be a finite dimensional nilpotent Lie algebra. Assume that

$$\underline{\mathbf{g}} = \oplus_{i=1}^{m} \underline{\mathbf{g}}_i$$

as a vector space, where $[\underline{\mathbf{g}}_i, \underline{\mathbf{g}}_j] \subseteq \underline{\mathbf{g}}_{i+j}$ for all i, j, and $\underline{\mathbf{g}}_1$ generates $\underline{\mathbf{g}}$ as a Lie algebra.

Let G be the associated connected, simply connected Lie group. Then G has homogeneous dimension d given by the formula

$$d = \sum_{j=1}^{m} j \dim(\underline{\mathbf{g}}_j),$$

where $\dim(\underline{\mathbf{g}}_j)$ denotes the dimension of $\underline{\mathbf{g}}_j$.

Consider any finite subset $\{X_k\}$ of $\underline{\mathbf{g}}_1$ which spans $\underline{\mathbf{g}}_1$. Each X_k can be identified with a unique left invariant vector field on G. Define

$$L = -\sum_k X_k^2;$$

then L is a left invariant second order differential operator. We define $L^p(G)$ with respect to Haar measure (and denote the corresponding norms by $\|\cdot\|_p$), then L is non-negative self-adjoint on $L^2(G)$ and it admits a spectral resolution

$$L = \int_0^\infty \lambda \, dP_\lambda.$$

For any bounded Borel function on $[0, \infty)$, we can define

$$m(L) = \int_0^\infty m(\lambda) \, dP_\lambda$$

which is bounded on $L^2(G)$, and the corresponding operator norm, which we denote by $\|m(L)\|_{2\to 2}$, satisfies $\|m(L)\|_{2\to 2} = \|m\|_\infty$.

Note that the operators $m(L)$ given by the spectral theorem and in Section 2 are identical when both definitions are applicable.

We need the following lemma which gives the upper bounds on the heat kernel and its derivatives.

Lemma . *Let h_z be the kernel of e^{-zL},* Re $z > 0$, *and* $\arg z = \theta$. *Then the following estimates hold:*

$$|h_z(x)| \le C\,(|z|\cos\theta)^{-\frac{d}{2}} \exp\left\{-c\cos\theta\frac{\|x\|^2}{|z|}\right\} \tag{3}$$

$$|X_i h_z(x)| \le C\,(|z|\cos\theta)^{-\frac{d+1}{2}} \exp\left\{-c\cos\theta\frac{\|x\|^2}{|z|}\right\}. \tag{4}$$

Proof. The following estimates on the heat kernel $h_t(x)$ and its derivatives for $t > 0$ are well known (e.g. see Saloff-Coste [**Sa**] and its references):

$$|h_t(x)| \le C\,t^{-\frac{d}{2}} \exp\left\{-c\frac{\|x\|^2}{t}\right\}$$

$$|X_i h_t(x)| \le C\,t^{-\frac{d+1}{2}} \exp\left\{-c\frac{\|x\|^2}{t}\right\}.$$

The required estimates then follow by interpolation as in Theorem 3.4.8 of Davies [**Da**]. □

We now represent the operator $m(L)$, using the semigroup e^{-zL}.

As in Section 2, for $m \in \Psi(S_\delta)$, we choose the contour $\gamma = \gamma_- + \gamma_+$, where

$$\gamma_+(t) = te^{i\mu} \quad \text{if } 0 \le t < \infty$$
$$\gamma_-(t) = -te^{-i\mu} \quad \text{if } -\infty < t \le 0$$

with $\delta > \mu$, and write

$$m(L) = \frac{1}{2\pi i} \int_{\gamma} (L - \lambda I)^{-1} m(\lambda)\, d\lambda.$$

Assume $\lambda \in \gamma_+$; then we have

$$(L - \lambda I)^{-1} = \int_{\Gamma_+} e^{\lambda z} e^{-zL} dz$$

where the curve Γ_+ is defined by $\Gamma_+(t) = te^{i\theta}$ for $t \geq 0$ and $\theta = (\pi - \mu)/2$. Therefore

$$\begin{aligned} m_+(L) &= \frac{1}{2\pi i} \int_{\gamma_+} \left[\int_{\Gamma_+} e^{\lambda z} e^{-zL}\, dz \right] m(\lambda)\, d\lambda \\ &= \int_{\Gamma_+} \left[\frac{1}{2\pi i} \int_{\gamma_+} e^{\lambda z} m(\lambda)\, d\lambda \right] e^{-zL}\, dz, \end{aligned}$$

by a change in the order of integration. Define Γ_- similarly: $\Gamma_-(t) = te^{-i\theta}$ for $t \geq 0$. A similar argument shows that

$$\begin{aligned} m_-(L) &= \frac{1}{2\pi i} \int_{\gamma_-} \left[\int_{\Gamma_-} e^{\lambda z} e^{-zL}\, dz \right] m(\lambda)\, d\lambda \\ &= \int_{\Gamma_-} \left[\frac{1}{2\pi i} \int_{\gamma_-} e^{\lambda z} m(\lambda)\, d\lambda \right] e^{-zL}\, dz, \end{aligned}$$

and therefore

$$m(L) = \int_{\Gamma_+} e^{-zL} n_+(z)\, dz + \int_{\Gamma_-} e^{-zL} n_-(z)\, dz,$$

where

$$n_\pm(z) = \frac{1}{2\pi i} \int_{\gamma_\pm} e^{\lambda z} m(\lambda)\, d\lambda,$$

which implies the bound

$$|n_\pm(z)| \leq \frac{1}{2\pi} \|m\|_\infty (\cos\theta)^{-1} |z|^{-1}. \tag{5}$$

Consequently, the kernel of $K_m(x)$ of $m(L)$ is given by

$$K_m(x) = \int_{\Gamma_+} h_z(x)\, n_+(z)\, dz + \int_{\Gamma_-} h_z(x)\, n_-(z)\, dz. \tag{6}$$

We now state our main theorem.

Theorem 2. *Let h_z be the kernel of e^{-zL}, $\operatorname{Re} z > 0$, $\arg z = \theta$. Assume that for some $\ell > 0$ the $L^1(G)$ norm of the complex time heat kernel h_z satisfies*

$$\|h_z\|_1 \leq C(\cos\theta)^{-\ell}.$$

Then the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1,\infty)$ if the function m satisfies the Hörmander condition (1) *of order $s = \ell + 1$.*

Proof. We denote the Haar measure by dx and the control distance associated to the sub-Laplacian L by d. We write $d(e,x) = \|x\|$, where e is the identity element of G.

Our plan of proof is to prove that L has a bounded holomorphic functional calculus as in (i) of Theorem 1 with $\alpha = l+1$. Then Theorem 2 follows from Theorem 1.

Let $m \in \Psi(S_\mu^0)$. To apply Calderón–Zygmund operator theory, we first prove the following estimate

$$(7) \qquad I = \int_{\|x\|\geq 2\|y\|} \left|K_m(x) - K_m(y^{-1}x)\right| dx \leq C\|m\|_\infty (\cos\theta)^{-(l+1+\epsilon)}.$$

Using (5) and (6), and changing the order of integration, we have

$$(8) \qquad I \leq C\|m\|_\infty (\cos\theta)^{-1} \int_\Gamma \int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx\, |z|^{-1}\, d|z|,$$

where $\int_\Gamma$ is short for $\int_{\Gamma_+} + \int_{\Gamma_-}$. We write

(9)

$$\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx$$
$$= \left(\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx\right)^\alpha \left(\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx\right)^{1-\alpha},$$

where α will be specified later. We estimate the second factor:

(10)

$$\left(\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx\right)^{1-\alpha} \leq (2\|h_z\|_1)^{1-\alpha} \leq C(\cos\theta)^{-\ell(1-\alpha)}.$$

To estimate the first factor, we use the upper bound on $X_i h_z$ in the lemma to obtain

$$(11) \qquad \int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx$$
$$\leq C\|y\| \int_{\|x\|\geq\|y\|} (|z|\cos\theta)^{-\frac{d+1}{2}} e^{-c\cos\theta\|x\|^2/|z|}\, dx,$$

where the constant c in the right hand side of (11) is half the constant c in the right hand side of (4). We now use polar coordinates in G, and deduce that

$$\begin{aligned}
&\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx \\
&\leq C\,\|y\|\,(|z|\cos\theta)^{-\frac{d+1}{2}} \int_{\|y\|}^{\infty} e^{-c\cos\theta r^2/|z|} r^{d-1} dr \\
&= C\,\|y\|\,(|z|\cos\theta)^{-\frac{d+1}{2}} \left(\frac{|z|}{c\cos\theta}\right)^{\frac{d}{2}} \int_{c\|y\|^2\cos\theta/|z|}^{\infty} e^{-s} s^{\frac{d}{2}-1} ds \\
&\leq C\left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{1}{2}} (\cos\theta)^{-d-1} e^{-c\|y\|^2\cos\theta/|z|}\left[1 + \left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{d}{2}-1}\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
(12)\qquad &\int_\Gamma \left(\int_{\|x\|\geq 2\|y\|} \left|h_z(x) - h_z(y^{-1}x)\right| dx\right)^{\alpha} |z|^{-1}\, d|z| \\
&\leq C\int_\Gamma \left(\left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{1}{2}} (\cos\theta)^{-d-1} e^{-c\|y\|^2\cos\theta/|z|}\right. \\
&\qquad \left.\cdot\left[1 + \left(\frac{\|y\|^2\cos\theta}{|z|}\right)^{\frac{d}{2}-1}\right]\right)^{\alpha} |z|^{-1}\, d|z| \\
&\leq C\int_0^{\infty} \left(t^{\frac{1}{2}}(\cos\theta)^{-d-1} e^{-ct}\left[1 + t^{\frac{d}{2}-1}\right]\right)^{\alpha} t^{-1} dt \\
&\leq c_\alpha\,(\cos\theta)^{-\alpha(d+1)},
\end{aligned}$$

where c_α becomes large as $\alpha \to 0$. We combine the inequalities (8) to (10) and (12), to get

(13)

$$I = \int_{\|x\|\geq 2\|y\|} \left|K_m(x) - K_m(y^{-1}x)\right| dx \leq c_\alpha\,\|m\|_\infty\,(\cos\theta)^{-1-\ell(1-\alpha)-\alpha(d+1)}.$$

By choosing α in (13) sufficiently small, interpolation shows that for any p, $1 < p \leq 2$, there exists a constant $c_{\epsilon,p}$ for any $\epsilon > 0$ such that

$$\|m(L)\|_{L^p(G)} \leq c_{\epsilon,p}\,\|m\|_\infty\,(\cos\theta)^{-\ell-1-\epsilon}.$$

We now fix p, $1 < p \leq 2$. To get rid of ϵ, we choose $p_1 = \frac{p+1}{2}$ and ϵ sufficiently small in the estimate of $\|m(L)\|_{L^{p_1}(G)}$, then interpolation between p_1 and 2 gives us the desired estimate.

The case $p > 2$ follows from duality. □

4. Hörmander multiplier theorems for sub-Laplacians on Lie groups.

4.1. Nilpotent Lie groups. Theorem 2 reduces the difficult task of controlling the kernel K_m of the operator $m(L)$ as in (7) to finding the $L^1(G)$ norms of the complex heat kernels h_z. The obvious next question is how large the norms $\|h_z\|_1$ are.

To obtain a sharp estimate on $\|h_z\|_1$ in the general setting of nilpotent Lie groups might be difficult but we can get a useful upper bound on $\|h_z\|_1$ without much difficulty. That result is the content of the following theorem.

Theorem 3. *Let L be a sub-Laplacian on a homogeneous nilpotent Lie group G of homogeneous dimension d, as in Section 3. Then for each $\epsilon > 0$, there exists $c_\epsilon > 0$ such that the $L^1(G)$ norms of the complex heat kernels satisfy*

$$\|h_z\|_1 \le c_\epsilon (\cos \arg z)^{-\frac{d}{2}-\epsilon}.$$

Consequently, the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1, \infty)$ if m satisfies the Hörmander condition (1) up to order $s = \frac{d}{2} + 1$.

Proof. We first estimate the $L^2(G)$ norms of the complex heat kernels as follows. Let $z = t + iv$ and denote the norm of the operator e^{-zL} from $L^2(G)$ to $L^\infty(G)$ by $\|e^{-zL}\|_{2\to\infty}$. We then have

$$\|h_z\|_2 = \|e^{-zL}\|_{2\to\infty}.$$

By spectral theory, e^{-ivL} is an isometry on $L^2(G)$, so

$$\|e^{-zL}\|_{2\to\infty} = \|e^{-tL}\|_{2\to\infty}.$$

We conclude that

$$\|h_z\|_2 = \|h_t\|_2 = Ct^{-\frac{d}{4}} = C(\mathrm{Re}\ z)^{-\frac{d}{4}}. \tag{14}$$

The middle equality holds by homogeneity.

We observe that by homogeneity, $\|h_z\|_1 = \|h_{z/|z|}\|_1$, hence we can assume $|z| = 1$.

To estimate $\|h_z\|_1$, we denote $\cos \arg z$ by σ, choose $\beta = \frac{1}{2} + \upsilon$ and break G into two parts:

$$G_1 = \{x \in G \mid \|x\| < \sigma^{-\beta}\}$$
$$G_2 = \{x \in G \mid \|x\| \ge \sigma^{-\beta}\}.$$

We then have

$$\int_{G_1} |h_z(x)|\,dx \le (\text{vol } G_1)^{\frac{1}{2}} \left(\int_{G_1} |h_z(x)|^2\,dx\right)^{\frac{1}{2}} \le (\text{vol } G_1)^{\frac{1}{2}} \left(\int_{G} |h_z(x)|^2\,dx\right)^{\frac{1}{2}} \le C\,\sigma^{-\frac{d}{2}-\frac{dv}{2}}. \tag{15}$$

To estimate $\int_{G_2} |h_z(x)|\,dx$, we use the estimate (3) of the lemma, and then integrate in polar coordinates. It turns out that

$$\int_{G_2} |h_z(x)|\,dx \le C \int_{\sigma^{-\beta}}^{\infty} \sigma^{-\frac{d}{2}} \exp\{-c\sigma r^2\} r^{d-1} dr = C\sigma^{-d} \int_{\sigma^{1-2\beta}}^{\infty} \exp\{-cs\} s^{\frac{d}{2}-1} ds \le c_{d,v} \tag{16}$$

where $c_{d,v}$ depends only on d and v. It follows from (15) and (16) that by choosing $v = \frac{2\epsilon}{d}$, there exists c_ϵ such that

$$\|h_z\| \le c_\epsilon\, \sigma^{-\frac{d}{2}-\epsilon}.$$

To complete the proof, we apply Theorem 2, and then interpolate to get rid of ϵ (as in the proof of Theorem 2). □

4.2. Generalised Heisenberg groups. In the proof of Theorem 3, estimate (14) shows that the $L^2(G)$ norm of the complex heat kernels is a multiple of $(\cos \arg z)^{-d/4}$. If we use this estimate to obtain an upper bound for the $L^1(G)$ norm of the complex heat kernels, we have the power $d/2$. This is the reason why our Theorem 3 as well as previously known proofs which utilise the $L^2(G)$ estimate only obtain order $s \ge d/2$.

To improve the order beyond half the homogeneous dimension, we need a sharper estimate on the $L^1(G)$ norm of the complex heat kernels. This can be done for the generalised Heisenberg groups (or H-type groups).

We now give a brief definition of generalised Heisenberg groups. For more details, see the thesis of J. Randall [**Ra1**] and its references.

Let $\underline{g}$ be a 2-step nilpotent Lie algebra with an inner product. Let ζ be the centre of $\underline{g}$ and ϑ the orthogonal complement of ζ in $\underline{g}$. For $v \in \vartheta$, let $f_\vartheta = (\ker \text{ad}_{\ v}) \cap \vartheta$, and denote by ϑ_v the orthogonal complement of f_ϑ in ϑ. Then $\underline{g}$ is called an H-type algebra or a generalised Heisenberg algebra if $\text{ad}_{\ v} : \vartheta_v \to \zeta$ is a surjective isometry for every unit vector $v \in \vartheta$.

The connected simply connected Lie group G, associated with $\underline{\mathbf{g}}$ is called an H-type or generalised Heisenberg group.

For the generalised Heisenberg algebra $\underline{\mathbf{g}} = \vartheta \oplus \zeta$, let $\dim(\vartheta) = 2n$ and $\dim(\zeta) = k$; then G is a stratified group with dilations $\gamma_r(v, \xi) = (rv, r^2\xi)$, for $(v, \xi) \in \vartheta \oplus \zeta$, and homogeneous dimension $d = 2n + 2k$.

We can also define the sub-Laplacian L on G. The heat kernel $h_z(x)$ has an explicit representation which can be used to estimate its $L^1(G)$ norm, (see [**Ra1**]). Our next theorem is

Theorem 4. *The $L^1(G)$ norm for h_z satisfies the following estimate:*

$$\|h_z\|_1 \le \frac{c}{(\cos\arg z)^{n+\ell}} \quad \textit{where } \ell = \begin{cases} \frac{k+3}{2} & \textit{for } k \textit{ odd} \\ \frac{k}{2} + 2 & \textit{for } k \textit{ even.} \end{cases}$$

Hence the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1, \infty)$, if m satisfies the Hörmander condition (1) *up to order*

$$s = \begin{cases} n + \frac{k+5}{2} & \textit{for } k \textit{ odd} \\ n + \frac{k}{2} + 3 & \textit{for } k \textit{ even.} \end{cases}$$

Proof. The estimate on the $L^1(G)$ norm of the complex time heat kernels, which uses the explicit representation of the heat kernels, is the main result of [**Ra2**].

The second part of this theorem is a consequence of Theorem 2. □

NOTE:

(a) The order s obtained in this theorem is strictly less than half of the homogeneous dimension when k is sufficiently large.

(b) The Hörmander multiplier result in Theorem 4 can be obtained by direct estimate on the kernel K_m of the operator $m(L)$, using the explicit representation of the complex heat kernels [**D2**].

(c) After this paper was written up, it came to the author's knowledge that, by using the real variable method, W. Hebisch was successful in proving that on a product of generalised Heisenberg groups Hörmander type multiplier theorem for the sub-Laplacian is true with the order $s = \frac{D}{2} + \epsilon$, $\epsilon > 0$, where D is the euclidean dimension of the group [**He**].

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Received May 10, 1993 and revised August 4, 1993. The author was supported by the Australian Research Council.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 173 No. 2 April 1996
