

*Pacific  
Journal of  
Mathematics*

**INTERPOLATING BLASCHKE PRODUCTS GENERATE  $H^\infty$**

JOHN BRADY GARNETT AND ARTUR NICOLAU

## INTERPOLATING BLASCHKE PRODUCTS GENERATE $H^\infty$

JOHN GARNETT AND ARTUR NICOLAU

**The algebra of bounded analytic functions on the open unit disc is generated by the set of Blaschke products having simple zeros which form an interpolating sequence.**

Let  $H^\infty$  be the algebra of bounded analytic functions in the unit disc  $\mathbb{D}$  and set

$$\|f\| = \sup_{z \in \mathbb{D}} |f(z)|,$$

for  $f \in H^\infty$ . A *Blaschke product* is an  $H^\infty$  function of the form

$$B(z) = \prod_{\nu=1}^{\infty} \frac{-\bar{z}_\nu}{|z_\nu|} \frac{z - z_\nu}{1 - \bar{z}_\nu z}$$

with  $\sum(1 - |z_\nu|) < \infty$ . In [5] D.E. Marshall proved that  $H^\infty$  is the closed linear span of the Blaschke products: given  $f \in H^\infty$  and  $\varepsilon > 0$ , there are constants  $c_1, \dots, c_n$  and Blaschke products  $B_1, \dots, B_n$  such that

$$(1) \quad \|f + c_1 B_1 + \dots + c_n B_n\|_\infty < \varepsilon.$$

In fact, Marshall proved that the unit ball of  $H^\infty$  is the uniformly closed convex hull of the set of Blaschke products (including  $B \equiv 1$ ).

A Blaschke product  $B(z)$  is called an *interpolating Blaschke product* if

$$(2) \quad \inf_{\nu} (1 - |z_\nu|^2) |B'(z_\nu)| = \delta_B > 0,$$

because of the Carleson theorem that (2) holds if and only if every interpolation problem

$$f(z_\nu) = w_\nu, \quad \nu = 1, 2, \dots,$$

for  $\{w_\nu\} \in l^\infty$ , has a solution  $f \in H^\infty$ . Although the interpolating Blaschke products comprise a small subset of the set of all Blaschke products, they play a central role in the theory of  $H^\infty$ . See the last three chapters of [3]. The theorem in this paper helps explain why interpolating Blaschke products are so important in that theory.

**Theorem.**  $H^\infty$  is the closed linear span of the interpolating Blaschke products.

In other words, (1) is true with the additional proviso that each of  $B_1, \dots, B_n$  is an interpolating Blaschke product.

The theorem solves a problem posed in [3] and [4]. It is not known if the set of interpolating Blaschke products is norm dense in the set of all Blaschke products. It is also not known if the unit ball of  $H^\infty$  is the closed convex hull of the set of all interpolating Blaschke products.

Recently, Marshall and A. Stray [6] proved the theorem in the special case that  $f$  extends continuously to almost every point of  $\partial\mathbb{D}$ , and our proof closely follows their reasoning. In particular, the idea of comparing (11) and (12) and the argument deriving the theorem from Lemma 3 below are both due to them. We thank Violant Marti for making the drawings.

The *hyperbolic distance* between  $z \in \mathbb{D}$  and  $w \in \mathbb{D}$  is

$$\rho(z, w) = \log \left( \frac{1 + \left| \frac{z - w}{1 - \bar{w}z} \right|}{1 - \left| \frac{z - w}{1 - \bar{w}z} \right|} \right),$$

and the *hyperbolic derivative* of an analytic function  $f$  is

$$(1 - |z|^2) |f'(z)|.$$

The hyperbolic derivative is invariant under conformal changes in  $z \in \mathbb{D}$ .

The Blaschke product with zeros  $\{z_\nu\}$  is an interpolating Blaschke product if and only if the following conditions both hold:

$$(3) \quad \inf_{\nu \neq \mu} \rho(z_\mu, z_\nu) > 0$$

and

$$(4) \quad \sum_{z_\nu \in Q} (1 - |z_\nu|) < C\ell(Q)$$

for all  $Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \ell(Q), 1 - \ell(Q) < r < 1\}$ . See [1] or Chapter VII of [3].

**Lemma 1.** *Let  $B$  be a Blaschke product and let  $\{z_\nu\}$  be its zeros, counted with their multiplicities. Then the following are equivalent:*

- (a)  $B = B_1 \dots B_N$ , with each  $B_j$  an interpolating Blaschke product.
- (b) Condition (4) holds.
- (c) There exist positive constants  $\rho_0, \delta_0$  such that for each  $z_\nu$  there is  $w_\nu$  with

$$(5) \quad \rho(z_\nu, w_\nu) \leq \rho_0$$

and

$$(6) \quad (1 - |w_\nu|^2) |B'(w_\nu)| \geq \delta_0.$$

In [6] it is shown that if  $B$  satisfies one of these conditions, then  $B$  is the uniform limit of a sequence of interpolating Blaschke products.

*Proof of Lemma 1.* The equivalence between (a) and (b) is in [7]. Assume (c) holds, let

$$Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \ell(Q), 1 - \ell(Q) < r < 1\},$$

and set

$$T(Q) = \{re^{i\theta} \in Q : 1 - \ell(Q) < r < 1 - 2^{-1}\ell(Q)\}.$$

To prove (4), we may assume there exists  $z_\nu \in T(Q)$ . Let  $w_\nu$  satisfy (5) and (6). Then there exists  $a_\nu$  such that  $\rho(a_\nu, z_\nu) < \rho_0$  and  $|B(a_\nu)| \geq m = m(\rho_0, \delta_0) > 0$ . Then the inequalities

$$\begin{aligned} \log m^{-2} \geq \log |B(a_\nu)|^{-2} &\geq \sum_{z_\mu \in Q} \frac{(1 - |z_\mu|^2)(1 - |a_\nu|^2)}{|1 - \bar{a}_\nu z_\mu|^2} \\ &\geq \frac{A(\rho_0, \delta_0)}{\ell(Q)} \sum_{z_\mu \in Q} (1 - |z_\mu|) \end{aligned}$$

show that (4) holds.

If (a) holds, there exists  $C > 0$  such that

$$|B(z)| \geq C \prod_{j=1}^N \inf_{\{B_j(z_\nu)=0\}} \left| \frac{z - z_\nu}{1 - \bar{z}_\nu z} \right|.$$

Fix  $\delta_0 > 0$ . Given  $z_\nu$ , there exists  $\zeta_\nu$  such that  $\rho(z_\nu, \zeta_\nu) \leq \delta_0$  and  $|B(\zeta_\nu)| \geq m = m(\delta_0) > 0$ , and then the geodesic arc from  $z_\nu$  to  $\zeta_\nu$  contains a point  $w_\nu$  at which (6) holds. □

We write  $\mathcal{F}$  for the set of finite products of interpolating Blaschke products. By the remark following Lemma 1, it is enough to prove (1) with each  $B_j \in \mathcal{F}$ , and by Marshall's theorem it is also enough to prove (1) when  $f = B$  is a Blaschke product.

Fix a Blaschke product  $B$  and let  $0 < \alpha < \beta < 1$ ,  $M = 2^K > 1$ , and  $\delta < 1$  be constants which will be determined later. We may assume  $|B(0)| > \beta$ . Consider "squares" of the form

$$Q_{n,j} = \{re^{i\theta} : 2\pi j 2^{-n} \leq \theta < 2\pi(j+1)2^{-n}; 1 - 2^{-n} \leq r < 1\}$$

and their top halves

$$T(Q_{n,j}) = Q_{n,j} \cap \{re^{i\theta} : 1 - 2^{-n} \leq r < 1 - 2^{-n-1}\}.$$

Let  $G_1 = \{Q_1^{(1)}, Q_2^{(1)}, \dots\}$  be the set of maximal  $Q_{n,j}$  for which

$$\inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

The squares in  $G_1$  have disjoint interiors. Write  $S_{p,j}^{(1)}$ ,  $1 \leq p \leq M = 2^K$ , for  $2^K$  different  $Q_{n+K,j} \subset Q_k^{(1)} = Q_{n,j}$ . If  $M$  is fixed and  $1 - \beta$  is small, then by Harnack's inequality

$$(7) \quad \sup_{T(S_{p,j}^{(1)})} |B(z)| < \beta.$$

Now let  $H_1 = \{V_1^{(1)}, V_2^{(1)}, \dots\}$  be the set of maximal  $Q_{n,j}$  such that

$$V_k^{(1)} \subset Q_p^{(1)}$$

for some  $Q_p^{(1)}$  and

$$\inf_{T(V_k^{(1)})} |B(z)| > \beta.$$

Since  $|B|$  has nontangential limit 1 almost everywhere,

$$\sum_{V_k^{(1)} \subset Q_p^{(1)}} \ell(V_k^{(1)}) = \ell(Q_p^{(1)}).$$

If  $(1 - \beta)/(1 - \alpha)$  is small, then

$$l(V_k^{(1)}) < \frac{1}{M} l(Q_p^{(1)})$$

when  $V_k^{(1)} \subset Q_p^{(1)}$ , again by Harnack's inequality. Hence  $V_k^{(1)} \subset S_{p,j}^{(1)}$ , for some  $p, j$ , because of (7).

Next let  $G_2 = \{Q_1^{(2)}, Q_2^{(2)}, \dots\}$  be the set of maximal  $Q_{n,j}$  such that

$$Q_{n,j} \subset V_k^{(1)} \in H_1$$

and

$$\inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

If  $(1 - \beta)/(1 - \alpha)$  is small, then

$$(8) \quad \sum_{Q_j^{(2)} \subset V_k^{(1)}} \ell(Q_j^{(2)}) < \varepsilon \ell(V_k^{(1)})$$

(see [3, p. 334]).

We form the  $S_{p,k}^{(2)}$  as before and continue, obtaining  $Q_j^{(m)}$ ,  $S_{p,j}^{(m)}$  and  $V_k^{(m+1)}$  with

$$Q_j^{(m)} \supset S_{p,j}^{(m)} \supset V_k^{(m+1)}.$$

See Figure 1. Then  $B(z)$  has zeros only in

$$\bigcup_{m,j} \left( Q_j^{(m)} \setminus \bigcup_{V_k^{(m+1)} \subset Q_j^{(m)}} V_k^{(m+1)} \right).$$

In fact, if  $1 - \alpha$  is small enough, all zeros from

$$Q_j^{(m)} \setminus \bigcup_{V_k^{(m+1)} \subset Q_j^{(m)}} V_k^{(m+1)}$$

fall into

$$\bigcup_{p=1}^M R_{p,j}^{(m)} = \bigcup_{p=1}^M \left( S_{p,j}^{(m)} \setminus \bigcup_{V_k^{(m+1)} \subset S_{p,j}^{(m)}} V_k^{(m+1)} \right),$$

and we require  $1 - \alpha$  to be that small.

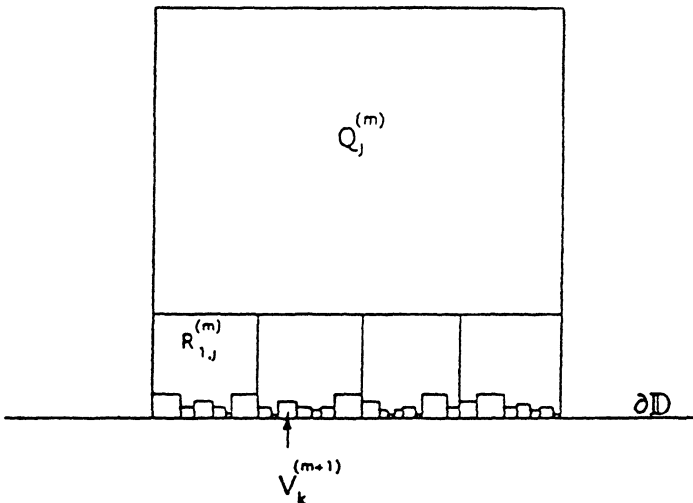


Figure 1.

Now factor

$$B = B_1 B_2 \cdots B_M,$$

where for fixed  $p, B_p$  has zeros only in  $\bigcup_{m,j} R_{p,j}^{(m)}$ . Fix  $p$ , set

$$\Gamma_{p,j}^{(m)} = \partial R_{p,j}^{(m)} \setminus \partial S_{p,j}^{(m)},$$

and mark points  $z_\nu^* = z_\nu^*(m, p, j)$  on  $\Gamma_{p,j}^{(m)}$  with

$$(9) \quad \rho(z_\nu^*, z_{\nu+1}^*) = \delta.$$

Let  $B_p^*$  be the Blaschke product with zeros  $\bigcup_{m,j} z_\nu^*(m, p, j)$ . Then by (3), (4), (8) and (9),  $B_p^*$  is an interpolating Blaschke product.

**Lemma 2.**  $|B_p^*| \leq \delta^{1/4}$  on  $\bigcup_{m,j} R_{p,j}^{(m)}$ .

*Proof.* Clearly  $|B_p^*| < \delta$  on  $\bigcup_{m,j} \Gamma_{p,j}^{(m)}$ . Fix one  $R_{p,j}^{(m)}$ . Then for any  $\varepsilon > 0$ , the harmonic measure

$$\omega \left( z, \Gamma_{p,j}^{(m)}, \mathbb{D} \setminus \bigcup \left\{ \overline{V_k^{(m+1)} \subset S_{p,j}^{(m)}} \right\} \right) > \frac{1}{4} - \varepsilon$$

for all  $z \in R_{p,j}^{(m)}$ , provided  $(1 - \beta)/(1 - \alpha)$  is small. Since  $\log |B_p^*(z)|$  is harmonic, that shows  $|B_p^*| \leq \delta^{1/4}$  on  $R_{p,j}^{(m)}$ .  $\square$

**Lemma 3.** *There exist  $A = A(\alpha, \beta, \delta, M)$  and  $\eta = \eta(\alpha, \beta, \delta, M) > 0$  so that if*

$$(10) \quad \inf_{\xi \in \bigcup_{m,j} R_{p,j}^{(m)}} \rho(z, \xi) > A$$

and if

$$|B_p B_p^*(z)| = \delta^{1/8},$$

then

$$(1 - |z|^2) \left| (B_p B_p^*)'(z) \right| \geq \eta.$$

*Proof.* We have

$$(11) \quad \frac{1}{4} \log \frac{1}{\delta} = \log |B_p B_p^*(z)|^{-2} \sim \sum_\nu \frac{(1 - |z|^2)(1 - |z_\nu|^2)}{|1 - \bar{z}_\nu z|^2},$$

where  $\{z_\nu\}$  is the zero set of  $B_p B_p^*$ . On the other hand,

$$(12) \quad (1 - |z|^2) \frac{(B_p B_p^*)'(z)}{B_p B_p^*(z)} = \bar{z} \sum_\nu \frac{(1 - |z|^2)(1 - |z_\nu|^2)}{|1 - \bar{z}_\nu z|^2} \left( \frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right).$$

By (10) there is  $A'$  so that if  $|z - z_\nu| < A'(1 - |z|)$ , then  $z_\nu \in R_{p,j}^{(m)}$  where  $\ell(S_{p,j}^{(m)}) < 1 - |z|$ . See Figure 2.

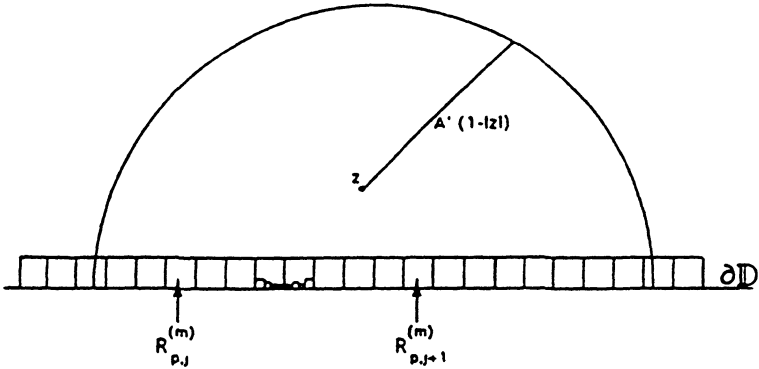


Figure 2.

If  $(1 - \alpha)$  is small compared to  $1/M$ , then  $\inf_{T(S_{p,j}^{(m)})} |B(z)| \geq C(\alpha) > 0$  and

$$\sum_{\{z_n \in R_{p,j}^{(m)}; B(z_n)=0\}} (1 - |z|^2) \leq C_1(\alpha) \ell(S_{p,j}^{(m)}),$$

where  $C_1(\alpha)$  tends to 0 if  $\alpha$  tends to 1. Therefore

$$\begin{aligned} \sum_{|z_\nu - z| < A'(1 - |z|)} \frac{(1 - |z_\nu|^2)(1 - |z|^2)}{|1 - \bar{z}_\nu z|^2} &\leq \frac{1}{1 - |z|^2} \sum_{|z_\nu - z| < A'(1 - |z|)} (1 - |z_\nu|^2) \\ &\leq \frac{1}{\delta M} (1 + \varepsilon + \varepsilon^2 + \dots) \\ &\quad + \frac{C_1(\alpha)}{M} (1 + \varepsilon + \varepsilon^2 + \dots). \end{aligned}$$

Take  $M$  so large (and consequently  $1 - \alpha$  so small) that

$$\sum_{|z_\nu - z| < A'(1 - |z|)} \frac{(1 - |z_\nu|^2)(1 - |z|^2)}{|1 - \bar{z}_\nu z|^2} < \frac{1}{16} \log \frac{1}{\delta}.$$

If  $|z - z_\nu| > A'(1 - |z|)$  then

$$\left| \arg \left( \frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| < c(A')$$



where  $c(A') \rightarrow 0$  as  $A' \rightarrow \infty$ . Hence

$$\begin{aligned} & \left| \sum_{|z-z_\nu| \geq A'(1-|z|)} \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left( \frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| \\ & \geq \cos^{-1}(c(A')) \sum_{|z-z_\nu| \geq A'(1-|z|)} \left| \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left( \frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right|. \end{aligned}$$

Consequently,

$$\begin{aligned} & (1-|z|^2) \left| (B_p B_p^*)'(z) \right| \\ & \geq \delta^{1/8} \left( \left| \sum_{|z-z_\nu| \geq A'(1-|z|)} \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left( \frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| \right. \\ & \quad \left. - \sum_{|z-z_\nu| < A'(1-|z|)} \left| \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left( \frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| \right) \\ & \geq \delta^{1/8} \left( \cos^{-1}(c(A')) \frac{11}{16} \log(1/\delta) - \frac{1}{16} \log(1/\delta) \right), \end{aligned}$$

and if  $A'$  is large, that proves the lemma.  $\square$

With Lemma 3, the remainder of the proof is just like in the Marshall-Stray paper [6]. There is  $\gamma$ ,  $|\gamma| = \delta^{1/8}$ , so that

$$\frac{B_p B_p^* - \gamma}{1 - \bar{\gamma} B_p B_p^*} = C_p$$

is a Blaschke product, by a theorem of Frostman [2]. Suppose  $C_p(z) = 0$ . Then

$$|B_p B_p^*(z)| = \delta^{1/8}$$

and

$$(1-|z|^2) |C_p'(z)| = \frac{(1-|z|^2)}{1-|\gamma|^2} |(B_p B_p^*)'(z)|.$$

Thus by Lemma 3

$$(1-|z|^2) |C_p'(z)| \geq \frac{\eta}{1-\delta^{1/4}}$$

if (10) holds. But if (10) fails, then there is  $\xi \in \bigcup_{m,j} R_{p,j}^{(m)}$  with  $\rho(z, \xi) < A$ . By Lemma 2,  $|B_p B_p^*(\xi)| \leq \delta^{1/4}$ . Somewhere along the hyperbolic geodesic from  $z$  to  $\xi$  there is a point  $w$  with

$$(1-|w|^2) |(B_p B_p^*)'(w)| > \eta' > 0$$

and  $\rho(z, w) < A$ . So by Lemma 1,  $C_p$  is a finite product of interpolating Blaschke products and  $B_p B_p^* \in \mathcal{F}$ .

For  $\sigma$  very small, replace  $B_p^*$  by

$$\tilde{B}_p^* = \frac{B_p^* - \sigma}{1 - \sigma B_p^*},$$

which is again an interpolating Blaschke product by [3, p. 404]. Repeating the above argument with  $\tilde{B}_p^*$ , we see that

$$\tilde{C}_p = \frac{B_p \tilde{B}_p^* - \tilde{\gamma}}{1 - \tilde{\gamma} B_p \tilde{B}_p^*}$$

is also a finite product of interpolating Blaschke products for some  $\tilde{\gamma}$ . Thus also  $B_p \tilde{B}_p^* \in \mathcal{F}$ . But then since

$$B_p \tilde{B}_p^* = -\sigma B_p + (1 - |\sigma|^2) B_p B_p^* + \dots,$$

we conclude that  $B_p \in \mathcal{F}$ . □

## References

- [1] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math., **80** (1958), 921-930.
- [2] O. Frostman, *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Medd. Lunds. Univ. Mat. Sem., **3** (1935), 1-118.
- [3] J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
- [4] P. Jones, *Ratios of interpolating Blaschke products*, Pacific J. Math., **95(2)** (1981), 311-321.
- [5] D. Marshall, *Blaschke products generate  $H^\infty$* , Bull. Amer. Math. Soc., **82** (1976), 494-496.
- [6] D. Marshall and A. Stray, *Interpolating Blaschke products*, Pacific J. Math., **173** (1996), 491-499.
- [7] G. Mc.Donald and C. Sundberg, *Toeplitz operators on the disc*, Indiana U. Math. J., **28** (1979), 595-611.

Received October 1, 1993 and revised November 3, 1993. The first author was partially supported by NSF grant DMS 91-04446 and the second author was partially supported by DGICYT grant PB89-0311, Spain.

*E-mail address:* jbg@math.ucla.edu

AND

UNIVERSITAT AUTÒNOMA DE BARCELONA  
08193 BELLATERRA, SPAIN

*E-mail address:* artur@manwe.mat.uab.es

<b>Peng Lin and Richard Rochberg</b> , Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights .....	127
<b>Donald E. Marshall and Arne Stray</b> , Interpolating Blaschke products ..	491
<b>Kathy D. Merrill and Lynne H. Walling</b> , On quadratic reciprocity over function fields .....	147
<b>Takahiko Nakazi and Masahiro Yamada</b> , $(A_2)$ -conditions and Carleson inequalities in Bergman spaces .....	151
<b>C. Ott</b> , A note on a paper of E. Boasso and A. Larotonda .....	173
<b>Victor Patrangenaru</b> , Classifying 3 and 4 dimensional homogeneous Riemannian manifolds by Cartan triples .....	511
<b>Carlo Pensavalle and Tim Steger</b> , Tensor products with anisotropic principal series representations of free groups .....	181
<b>Ying Shen</b> , On Ricci deformation of a Riemannian metric on manifold with boundary .....	203
<b>Albert Jeu-Liang Sheu</b> , The Weyl quantization of Poisson $SU(2)$ .....	223
<b>Alexandra Shlapentokh</b> , Polynomials with a given discriminant over fields of algebraic functions of positive characteristic .....	533
<b>Eric Stade and D.I. Wallace</b> , Weyl's law for $SL(3, \mathbb{Z}) \backslash SL(3, \mathbb{R}) / SO(3, \mathbb{R})$	241
<b>Christopher W. Stark</b> , Resolutions modeled on ternary trees .....	557
<b>Per Tomter</b> , Minimal hyperspheres in two-point homogeneous spaces .....	263
<b>Jun Tomiyama</b> , Topological Full groups and structure of normalizers in transformation group $C^*$ -algebras .....	571
<b>Nik Weaver</b> , Subalgebras of little Lipschitz algebras .....	283

# PACIFIC JOURNAL OF MATHEMATICS

Volume 173    No. 2    April 1996

---

A mean value inequality with applications to Bergman space operators	295
PATRICK ROBERT AHERN and ZELJKO CUCKOVIC	
$H^p$ -estimates of holomorphic division formulas	307
MATS ANDERSSON and HASSE CARLSSON	
Group structure and maximal division for cubic recursions with a double root	337
CHRISTIAN JEAN-CLAUDE BALLOT	
The Weil representation and Gauss sums	357
ANTONIA WILSON BLUHER	
Duality for the quantum $E(2)$ group	375
ALFONS VAN DAELE and S. L. WORONOWICZ	
Cohomology complex projective space with degree one codimension-two fixed submanifolds	387
KARL HEINZ DOVERMANN and ROBERT D. LITTLE	
On the mapping intersection problem	403
ALEXANDER DRANISHNIKOV	
From the $L^1$ norms of the complex heat kernels to a Hörmander multiplier theorem for sub-Laplacians on nilpotent Lie groups	413
XUAN THINH DUONG	
Isoperimetric inequalities for automorphism groups of free groups	425
ALLEN E. HATCHER and KAREN VOGTMANN	
Approximation by normal elements with finite spectra in $C^*$ -algebras of real rank zero	443
HUAXIN LIN	
Interpolating Blaschke products	491
DONALD EDDY MARSHALL and ARNE STRAY	
Interpolating Blaschke products generate $H^\infty$	501
JOHN BRADY GARNETT and ARTUR NICOLAU	
Classifying 3- and 4-dimensional homogeneous Riemannian manifolds by Cartan triples	511
VICTOR PATRANGENARU	
Polynomials with a given discriminant over fields of algebraic functions of positive characteristic	533
ALEXANDRA SHLAPENTOKH	
Resolutions modeled on ternary trees	557
CHRISTOPHER W. STARK	
Topological full groups and structure of normalizers in transformation group $C^*$ -algebras	571
JUN TOMIYAMA	