Pacific Journal of Mathematics

COHERENT STATES, HOLOMORPHIC EXTENSIONS, AND HIGHEST WEIGHT REPRESENTATIONS

KARL-HERMANN NEEB

Volume 174 No. 2 June 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 174, No. 2, 1996

COHERENT STATES, HOLOMORPHIC EXTENSIONS, AND HIGHEST WEIGHT REPRESENTATIONS

KARL-HERMANN NEEB

Let G be a connected finite dimensional Lie group. In this paper we consider the problem of extending irreducible unitary representations of G to holomorphic representations of certain semigroups S containing G and a dense open submanifold on which the semigroup multiplication is holomorphic. We show that a necessary and sufficient condition for extendability is that the unitary representation of G is a highest weight representation. This result provides a direct bridge from representation theory to coadjoint orbits in $\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G. Namely the moment map associated naturally to a unitary representation maps the orbit of the highest weight ray (the coherent state orbit) to a coadjoint orbit in $\mathfrak{g}^*$ which has many interesting geometric properties such as certain convexity properties and an invariant complex structure.

In this paper we use the interplay between the orbit picture and representation theory to obtain a classification of all irreducible holomorphic representations of the semigroups S mentioned above and a classication of unitary highest weight representations of a rather general class of Lie groups. We also characterize the class of groups and semigroups having sufficiently many highest weight representations to separate the points.

0. Introduction.

A closed convex cone W in the Lie algebra $\mathfrak{g}$ is called *invariant* if it is invariant under the adjoint action. The starting point in the theory of holomorphic extensions of unitary representations was Ol'shanskiĭ's observation that if W is a pointed generating invariant cone in a simple Lie algebra $\mathfrak{g}$, where G is a corresponding linear connected group, and $G_{\mathbb{C}}$ its universal complexification, then the set $S_W = G\exp(iW)$ is a closed subsemigroup of $G_{\mathbb{C}}$ ([**Ol82**]). This theorem has been generalized by Hilgert and 'Olafsson to solvable groups ([**HiOl92**]) and the most general result of this type, due to Lawson ([**La94**], [**HiNe93**]), is that if $G_{\mathbb{C}}$ is a complex Lie group with an antiholomorphic involution inducing the complex conjugation on $\mathfrak{g}_{\mathbb{C}} = \mathbf{L}(G_{\mathbb{C}})$, then the set

$S_W = G\exp(iW)$ is a closed subsemigroup of $G_{\mathbb{C}}$. The class of semigroups obtained by this construction is not sufficient for many applications in representation theory. For instance Howe's oscillator semigroup (cf. [**How88**]) is a 2-fold covering of such a semigroup, but it cannot be embedded into any group. In [**Ne94d**] we have shown that given a Lie algebra $\mathfrak{g}$, a generating invariant convex cone $W \subseteq \mathfrak{g}$, and a discrete central subgroup D invariant under complex conjugation of the simply connected group corresponding to the Lie algebra $\mathfrak{g} + i(W \cap (-W))$, there exists a semigroup $S = \Gamma(\mathfrak{g}, W, D)$ called the *Ol'shanskiĭ semigroup* defined by this data. This semigroup is the quotient $\widetilde{S}/D$, where $\widetilde{S}$ is the universal covering semigroup of S (cf. [**Ne92**]) and $D \cong \pi_1(S)$ is a discrete central subgroup of $\widetilde{S}$. Moreover, the semigroup $\widetilde{S}$, also denoted $\Gamma(\mathfrak{g}, W)$, can be obtained as the universal covering semigroup of the subsemigroup $\langle \exp(\mathfrak{g} + iW) \rangle$ of the simply connected complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The semigroup S contains a dense open semigroup ideal called the interior $\operatorname{int} S$ which is a complex manifold on which the semigroup multiplication is holomorphic. Complex conjugation on the cone $\mathfrak{g} + iW$ integrates to an antiautomorphism $s \mapsto s^*$ of S which is antiholomorphic on $\operatorname{int} S$.

Let us write $B(\mathcal{H})$ for the C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$. A *holomorphic representation* of an Ol'shanskiĭ semigroup S is a weakly continuous monoid morphism $\pi : S \to B(\mathcal{H})$ such that π is holomorphic on the complex manifold $\operatorname{int}(S)$ and $\pi(s)^* = \pi(s^*)$ holds for all $s \in S$. One can think of representations of S as analytic continuations of unitary representations of the subgroup $\mathrm{U}(S) = \{s \in S : s^*s = \mathbf{1}\}$ of unitary elements in S.

If the cone W is pointed, then, according to the results in Chapter 9 of [**HiNe93**], holomorphic contraction representations of $S = \Gamma(\mathfrak{g}, W, D)$ are in one-to-one correspondence to unitary representations π of the connected Lie group G with fundamental group D satisfying the condition that the essentially selfadjoint operators $i \cdot d\pi(X)$ are negative for every $X \in W$. We note that this result is due to Ol'shanskiĭ in the case of simple groups ([**Ol82**]) and to Hilgert and 'Olafsson in the case of solvable groups ([**HiOl92**]). The case of simple Lie groups has also been considered in [**Sta86**] and for the case of the metaplectic representation we also refer to [**How88**], [**BK80**], [**Br85**] and [**Hi89**]. The main technique in [**Ne94d**] was to reduce the problem of studying holomorphic representations to the investigation of holomorphic representations by contractions.

One of the main results in [**Ne94d**] consists of a characterization of those unitary representations of a connected Lie group G which can be extended to holomorphic representations of an Ol'shanskiĭ semigroup $S = \Gamma(\mathfrak{g}, W, \pi_1(G))$ with $\mathrm{U}(S)_0 \cong G$.

From now on we assume that $\mathfrak{g}$ is a (CA) *Lie algebra*, i.e., the group of inner automorphisms of $\mathfrak{g}$ is closed in the group $\mathrm{Aut}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$. As we have seen in [**Ne94d**], this condition is a rather natural one since it entails that every connected group G with $\mathbf{L}(G) = \mathfrak{g}$ is a type I group.

Let us say that a subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is *compactly embedded* if the group generated by $e^{\mathrm{ad}\,\mathfrak{a}}$ has compact closure in $\mathrm{Aut}(\mathfrak{g})$. We assume that $\mathfrak{g}$ contains a compactly embedded Cartan algebra $\mathfrak{t}$, i.e., $\mathfrak{t}$ is compactly embedded and maximal abelian in $\mathfrak{g}$. Then there exists a unique maximal compactly embedded subalgebra $\mathfrak{k}$ containing $\mathfrak{t}$. We write $K = \exp \mathfrak{k}$ for the corresponding analytic subgroup of G.

In [**Ne94e**, Th. III.8] we have shown that for every irreducible representation $(\pi, \mathcal{H})$ of the Ol'shanskiĭ S the space $\mathcal{H}^K$ of K-finite vectors is a highest weight module of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and that $\pi(\mathrm{int}\, S)$ consists of trace class operators. In the same paper we applied these results to obtain a rather satisfactory disintegration and character theory for Ol'shanskiĭ semigroups.

The best known examples for representations which fit into this theory are the irreducible representations of compact Lie groups, the *holomorphic discrete series* representations of simple hermitean Lie groups, the *metaplectic representation* of the 2-fold cover $H_n \rtimes \mathrm{Mp}(n, \mathbb{R})$ of the *Jacobi group* $\mathrm{St}(n, \mathbb{R}) := H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$, where H_n denotes the $(2n+1)$-dimensional Heisenberg group, and the *oscillator representation* of the (2n + 2)-dimensional oscillator group. Other examples are the *ladder representations* of the subgroups of $\mathrm{Mp}(n, \mathbb{R})$ obtained by restriction of the metaplectic representation.

In this paper we combine the results from [**Ne94f**] on coadjoint orbits with the results from [**Ne94d, e**] on holomorphic representations. In the first section we consider the coadjoint orbits associated to unitary highest weight representations. We show that the orbit of the highest weight ray in the projective space is a *coherent state orbit*, i.e., a complex Kähler manifold. For reductive Lie groups we show that the unitary highest weight representations are precisely those which have such an orbit. This result is originally due to Lisiecki (cf. [**Li90**, 3.8]) but we note that our proof does not need the Bott-Borel-Weil Theorem. We also discuss the integrality conditions which are necessary for unitarizability.

In Section II we discuss the *metaplectic* or *Weil representation* of the two-fold covering $\mathrm{Mp}(n, \mathbb{R})$ of the symplectic group $\mathrm{Sp}(n, \mathbb{R})$. We are mainly interested in the extension of this representation to a unitary representation of the semidirect product $\mathrm{HMp}(n, \mathbb{R}) = H_n \rtimes \mathrm{Mp}(n, \mathbb{R})$. This representation plays a crucial role in Section III, where we use a theorem of Satake on tensor product decompositions of irreducible representations of semidirect products ([**Sa71**]) to obtain a classification of the unitary highest weight representations of a Lie group G. These representations are tensor products of suitably modified extended metaplectic representations and highest weight representations of reductive Lie groups which are classified by Enright, Howe, Wallach ([**EHW83**]) and Jakobsen ([**Jak83**]).

The first three sections of this paper only deal with Lie groups and their representations. In Section IV we turn to the characterization of those unitary highest weight representations which extend holomorphically to a given Ol'shanskiĭ semigroup. We prove in particular that every unitary highest weight representation extends holomorphically to a certain Ol'shanskiĭ semigroup.

Moreover we use the results of Section III to show that every irreducible holomorphic representation of an Ol'shanskiĭ semigroup is a highest weight representation which in turn leads to a classification of all irreducible holomorphic representations.

Section V contains a proof of the Gelfand-Raïkov Theorem for Ol'shanskiĭ semigroups which says that for every Ol'shanskiĭ's semigroup $S = \Gamma(\mathfrak{g}, W, D)$ the holomorphic representations separate the points if and only if $\mathfrak{g}$ is admissible, i.e., $\mathfrak{g} \oplus \mathbb{R}$ contains a pointed generating invariant cone, and $H(W) := W \cap (-W)$ is a compact Lie algebra. Moreover, if W is pointed, then even the holomorphic contraction representations separate the points.

1. Highest weight representations and coadjoint orbits.

1.1. Coherent state orbits. Let G be a simply connected Lie group and $\mathfrak{g}$ its Lie algebra. We assume that $\mathfrak{g}$ contains a compactly embedded Cartan algebra $\mathfrak{t}$. Let $\mathfrak{k}$ denote the uniquely determined maximal compactly embedded subalgebra containing $\mathfrak{t}$ ([**HHL89**, A.2.40]). We set $T := \exp \mathfrak{t}$ and $K := \exp \mathfrak{k}$.

Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a root decomposition as follows ([**HiNe93**, Ch. 7]). For a linear functional $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} :=\{X \in \mathfrak{g}_{\mathbb{C}} : (\forall Y \in \mathfrak{t}_{\mathbb{C}})[Y, X] = \alpha(Y)X\}$$

and

$$\Delta :=\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} : \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}.$$

Then

$$\mathfrak{g}_{\mathbb{C}} =\mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha},$$

$\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ for all $\alpha \in \Delta$ and $\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$, where $X \mapsto \overline{X}$ denotes complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. A root α is said to be *compact* if $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$. We write Δ_k for the set of compact roots and Δ_p for the set of non-compact roots. We say that $\mathfrak{g}$ has *cone potential* if $\left[X, \overline{X}\right] = 0$ for $X \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ implies that $X = 0$ (cf. [**HHL89**, Ch. III]). We write $\mathcal{W}_{\mathfrak{k}}$ for the *Weyl group* of the compact Lie algebra $\mathfrak{k}$. This is a finite group which acts on $\mathfrak{t}$.

Definition 1.1.

(a) A subset $\Delta^+ \subseteq \Delta$ is called a *positive system* if there exists $X_0 \in i\mathfrak{t}$ such that $\Delta^+ = \{\alpha \in \Delta : \alpha(X_0) > 0\}$ and a set Σ is called a *parabolic system* if there exists $X_0 \in i\mathfrak{t}$ such that

$$\Sigma = \{\alpha \in \Delta : \alpha(X_0) \geq 0\}.$$

A positive system Δ^+ is said to be $\mathfrak{k}$-*adapted* if the set $\Delta_p^+ := \Delta^+ \cap \Delta_p$ of positive non-compact roots is invariant under the Weyl group $\mathcal{W}_{\mathfrak{k}}$. We say that $\mathfrak{g}$ is *quasihermitean* if there exists a $\mathfrak{k}$-adapted positive system (cf. [**Ne94e**, Prop. II.7]).

(b) For a $\mathfrak{g}_{\mathbb{C}}$-module V and $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$ we set

$$V^{\alpha} := \{v \in V : (\forall X \in \mathfrak{t}_{\mathbb{C}}) X.v = \alpha(X)v\}.$$

This space is called the *weight space of weight* α and α is said to be a *weight* of V if $V^{\alpha} \neq \{0\}$. We write $\mathcal{P}_V$ for the set of weights of V. Suppose that $\lambda \in \mathcal{P}_V$. An element $v \in V^{\lambda}$ is called *primitive* (with respect to Δ^+) if $\mathfrak{g}_{\mathbb{C}}^{\alpha}.v = 0$ holds for all $\alpha \in \Delta^+$.

A $\mathfrak{g}_{\mathbb{C}}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.

We recall that for each $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ there exists a, up to isomorphy, unique highest weight module $L(\lambda)$ of $\mathfrak{g}_{\mathbb{C}}$ (cf. [**Ne94e**, Prop. II.10]).

(c) In the following we write representations on Hilbert spaces as pairs $(\pi, \mathcal{H})$, where $\mathcal{H}$ is the representing Hilbert space and π the corresponding homomorphism to $B(\mathcal{H})$.

The irreducible highest weight module $L(\lambda)$ is said to be *unitarizable* if there exists a unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of the simply connected Lie group G with $\mathbf{L}(G) = \mathfrak{g}$ such that $L(\lambda)$ is isomorphic to the $\mathfrak{g}_{\mathbb{C}}$-module $\mathcal{H}^{K,\infty}$ of K-finite smooth vectors in $\mathcal{H}$ (cf. [**Ne94e**, Th. III.6]). In this case $(\pi, \mathcal{H})$ is called a *unitary highest weight representation* of G. Note that in this case $\mathcal{H}$ can be viewed as the completion of $L(\lambda) \cong \mathcal{H}^{K,\infty}$ with respect to the $\mathfrak{g}$-invariant hermitean scalar product on $L(\lambda)$.

For more details on highest weight representations we refer to Section II in [**Ne94e**].

Definition 1.2. Let $(\pi, \mathcal{H})$ be a unitary representation of the Lie group G. Then we have an action of the group G on the projective space $\mathbb{P}(\mathcal{H})$ given by $g.[v] := [g.v]$, where $q : \mathcal{H} \setminus \{0\} \to \mathbb{P}(\mathcal{H}), v \mapsto [v] := \mathbb{C}v$ denotes the quotient mapping.

We write $\mathcal{H}^\infty$ for the space of *smooth vectors* , i.e., the set of all elements $v \in \mathcal{H}$ such that the mapping $G \to \mathcal{H}, g \mapsto g.v$ is smooth. Then the Lie algebra $\mathfrak{g}$ acts on $\mathcal{H}^\infty$ by

$$d\pi(X).v := \frac{d}{dt}\Big|_{t=0} \pi(\exp tX).v$$

and since $\mathcal{H}^\infty$ is a complex vector space, this action extends to $\mathfrak{g}_\mathbb{C}$ and turns $\mathcal{H}^\infty$ in a $\mathfrak{g}_\mathbb{C}$-module. The mapping

$$\Psi : \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}^*, \qquad [v] \mapsto \left(X \mapsto \frac{1}{i} \frac{\langle d\pi(X).v, v \rangle}{\langle v, v \rangle} \right)$$

is called the *moment mapping* of this representation

For the following we recall the *coadjoint action* of a Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra by $g.\omega := \mathrm{Ad}^*(g).\omega = \omega \circ \mathrm{Ad}(g)^{-1}$. We write $\mathcal{O}_\omega$ for the coadjoint orbit $\mathrm{Ad}^*(G).\omega$. Moreover we identify $\mathfrak{t}^*$ with the subspace $[\mathfrak{t}, \mathfrak{g}]^\perp \subseteq \mathfrak{g}^*$, so that we consider every functional on $\mathfrak{t}$ as a functional on $\mathfrak{g}$.

Lemma 1.3. *The moment mapping Ψ has the following properties:*

(i) *Ψ is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$.*

(ii) *If $\mathfrak{t} \subseteq \mathfrak{g}$ is a compactly embedded Cartan algebra, $\lambda = i\omega \in \mathfrak{t}_\mathbb{C}^*$, and $\mathcal{H}^\lambda \subseteq \mathcal{H}^\infty$ the corresponding weight space, then $\Psi\left(\mathbb{P}(\mathcal{H}^\lambda)\right) = \omega$.*

(iii) *If $v_\lambda \in \mathcal{H}^\infty$ is a vector of weight $\lambda = i\omega$, then Ψ maps $G.[v_\lambda]$ onto the coadjoint orbit $\mathcal{O}_\omega$.*

(iv) *For $v \in \mathcal{H}^\infty$ and $X \in \mathfrak{g}$ we have*

$$d\Psi([v])dq(v)d\pi(X).v = -\Psi([v]) \circ \mathrm{ad}\, X.$$

(v) *Let $v \in \mathcal{H}^\infty$ with $\|v\| = 1$ und write $p_v(w) := w - \langle w, v\rangle v$ for the projection of $\mathcal{H}$ onto the subspace $v^\perp$. Then we have for $X \in \mathfrak{g}_\mathbb{C}$*

$$\begin{aligned}\Psi([v])\left(i\left[X, \overline{X}\right]\right) &= \|d\pi(X).v\|^2 - \left\|d\pi\left(\overline{X}\right).v\right\|^2 \\ &= \|p_v(d\pi(X).v)\|^2 - \left\|p_v\left(d\pi\left(\overline{X}\right).v\right)\right\|^2.\end{aligned}$$

Proof. (i) This follows by a direct computation from $\pi(G) \subseteq U(\mathcal{H})$.

(ii) For $X \in \mathfrak{t}$ and $v \in \mathcal{H}^\lambda$ we find that $d\pi(X).v = \lambda(X)v$ so that $\langle \Psi([v]), X\rangle = -i\lambda(X) = \omega(X)$. For $X \in \mathfrak{g}_{\mathbb{C}}^\alpha$ we have $d\pi(X).v \in \mathcal{H}^{\lambda+\alpha}$ which is orthogonal to $\mathcal{H}^\lambda$ and $d\pi(\overline{X}).v \in \mathcal{H}^{\lambda-\alpha}$ which is also orthogonal to $\mathcal{H}^\lambda$. Hence $\Psi([v])$ vanishes on $[\mathfrak{t}, \mathfrak{g}]$ and therefore $\Psi([v]) = \omega$.
(iii) follows by combining (i) and (ii).
(iv) Let $\gamma(t) := \pi(\exp tX).[v]$. Then $\gamma'(0) = dq(v)d\pi(X).v$ and therefore (i) implies that

$$\begin{aligned} d\Psi([v])dq(v)d\pi(X).v &= \frac{d}{dt}\Big|_{t=0} \Psi(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*(\exp tX).\Psi([v]) \\ &= \frac{d}{dt}\Big|_{t=0} \Psi([v]) \circ e^{-t\,\mathrm{ad}\,X} = -\Psi([v]) \circ \mathrm{ad}\,X. \end{aligned}$$

(v) For the first assertion we calculate

$$\begin{aligned} \Psi([v])\left(i\left[X,\overline{X}\right]\right) &= \left\langle d\pi\left(\left[X,\overline{X}\right]\right).v, v\right\rangle \\ &= \left\langle d\pi(X)d\pi\left(\overline{X}\right).v, v\right\rangle - \left\langle d\pi\left(\overline{X}\right)d\pi(X).v, v\right\rangle \\ &= -\left\langle d\pi\left(\overline{X}\right).v, d\pi\left(\overline{X}\right).v\right\rangle + \langle d\pi(X).v, d\pi(X).v\rangle \\ &= -\left\|d\pi\left(\overline{X}\right).v\right\|^2 + \|d\pi(X).v\|^2. \end{aligned}$$

For the second we note that $\|p_v(w)\|^2 = \|w\|^2 - |\langle w, v\rangle|^2$ so that

$$|\langle d\pi(X).v, v\rangle| = \left|-\left\langle v, d\pi\left(\overline{X}\right).v\right\rangle\right| = \left|\left\langle d\pi\left(\overline{X}\right).v, v\right\rangle\right|$$

implies that

$$\begin{aligned} \Psi([v])\left(i\left[X,\overline{X}\right]\right) &= \|d\pi(X).v\|^2 - \left\|d\pi\left(\overline{X}\right).v\right\|^2 \\ &= \|d\pi(X).v\|^2 - |\langle d\pi(X).v, v\rangle|^2 \\ &\quad - \left\|d\pi\left(\overline{X}\right).v\right\|^2 + \left|\left\langle d\pi\left(\overline{X}\right).v, v\right\rangle\right|^2 \\ &= \|p_v(d\pi(X).v)\|^2 - \left\|p_v\left(d\pi\left(\overline{X}\right).v\right)\right\|^2. \end{aligned}$$

□

For more details on the moment map and its convexity properties we refer to [**Ne94g**]. In general one cannot expect that an orbit $G.[v]$ is a symplectic submanifold of the infinite dimensional complex Kähler manifold $\mathbb{P}(\mathcal{H}^\infty)$ (cf. [**GS84**, p. 168]). The situation is considerably better if we consider orbits of highest weight vectors. The following lemma extends [**Li90**, Prop. 2.8].

Lemma 1.4. *Suppose that v_λ is a unit highest weight vector in $\mathcal{H}^K \subseteq \mathcal{H}^\infty$, $\lambda = i\omega$, and*

$$\mathfrak{b}_\omega := \{X \in \mathfrak{g}_\mathbb{C} : d\pi(X).v_\lambda \in \mathbb{C}v_\lambda\}.$$

Then the following assertions hold:

(i) *The tangent space of the orbit $G.[v_\lambda] \subseteq \mathbb{P}(\mathcal{H}^\infty)$ is a complex subspace.*

(ii) $\mathfrak{b}_\omega + \overline{\mathfrak{b}_\omega} = \mathfrak{g}_\mathbb{C}$.

(iii) $\omega\left(i\left[\overline{X}, X\right]\right) \geq 0$ *for all $X \in \mathfrak{b}_\omega$, where equality holds if and only if $X \in \mathfrak{b}_\omega \cap \overline{\mathfrak{b}_\omega}$.*

(iv) *If $\mathfrak{g}^\omega = \{X \in \mathfrak{g} : \omega \circ \operatorname{ad} X = 0\}$ is the stabilizer algebra of ω, then*

$$(\mathfrak{g}^\omega)_\mathbb{C} = \mathfrak{b}_\omega \cap \overline{\mathfrak{b}_\omega}.$$

(v) *The moment mapping Ψ induces a bijection of $G.[v_\lambda]$ onto $\mathcal{O}_\omega$.*

Proof. (i), (ii) First we note that $\mathfrak{b}_\omega$ contains $\mathfrak{b}(\Delta^+) := \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\mathbb{C}^\alpha$. Hence

$$\mathfrak{b}_\omega + \overline{\mathfrak{b}_\omega} \supseteq \mathfrak{b}(\Delta^+) + \mathfrak{b}(-\Delta^+) = \mathfrak{g}_\mathbb{C}.$$

We conclude that $\mathfrak{g} + \mathfrak{b}_\omega = \mathfrak{g}_\mathbb{C}$. Thus $d\pi(\mathfrak{g}_\mathbb{C}).v_\lambda = d\pi(\mathfrak{g}).v_\lambda$ which is the tangent space of the orbit $G.v_\lambda$ in $\mathcal{H}^\infty$ is a complex vector space. The tangent space of the orbit in the projective space inherits this property.
(iii) First we note that polarization shows that

$$\begin{aligned}\omega_\mathbb{C}\left(i\left[X, \overline{Y}\right]\right) &= \left\langle d\pi\left(\left[X, \overline{Y}\right]\right).v_\lambda, v_\lambda\right\rangle \\ &= -\left\langle d\pi\left(\overline{Y}\right).v_\lambda, d\pi\left(\overline{X}\right).v_\lambda\right\rangle + \langle d\pi(X).v_\lambda, d\pi(Y).v_\lambda\rangle\end{aligned}$$

for all $X, Y \in \mathfrak{g}_\mathbb{C}$. This formula entails in particular that the orthogonality of the weight spaces in $\mathcal{H}^\infty$ implies the orthogonality of the root space in $\mathfrak{b}_\omega$ with respect to this form. Hence it suffices to check positivity on the root spaces and on $\mathfrak{t}_\mathbb{C}$. For $X \in \mathfrak{t}_\mathbb{C}$ we evidently have that $\omega\left(i\left[X, \overline{X}\right]\right) = 0$. For $X \in \mathfrak{b}_\omega \cap \mathfrak{g}_\mathbb{C}^\alpha$ we have $d\pi(X).v_\lambda = 0$ and therefore $\omega\left(i\left[X, \overline{X}\right]\right) = -\left\|d\pi\left(\overline{X}\right).v_\lambda\right\|^2 \leq 0$ by Lemma I.3(v) which even shows that equality holds if and only if $d\pi\left(\overline{X}\right).v_\lambda = 0$, i.e., if $X \in \mathfrak{b}_\omega \cap \overline{\mathfrak{b}_\omega}$.
(iv) Let $X \in \mathfrak{g}^\omega$. We use (ii) to find $Y \in \mathfrak{b}_\omega$ with $X = Y + \overline{Y}$. Then

$$0 = i\omega \circ \operatorname{ad} X\left(i\left(Y - \overline{Y}\right)\right) = i\omega\left(i\left[Y + \overline{Y}, Y - \overline{Y}\right]\right) = 2i\omega\left(i\left[\overline{Y}, Y\right]\right).$$

Hence, in view of (ii), $Y \in \mathfrak{b}_\omega \cap \overline{\mathfrak{b}_\omega}$ and therefore $X \in \mathfrak{b}_\omega \cap \overline{\mathfrak{b}_\omega}$.

If, conversely, $X \in \mathfrak{b}_\omega \cap \overline{\mathfrak{b}_\omega}$, then (iii) above implies that $i\omega_{\mathbb{C}}\left(i\left[X, \overline{Y}\right]\right) = -\omega_{\mathbb{C}}\left(\left[X, \overline{Y}\right]\right) = 0$ holds for all $Y \in \mathfrak{b}_\omega$. Since $X \in \mathfrak{g}$, we also obtain $\omega_{\mathbb{C}}([X, Y]) = 0$ for all $Y \in \mathfrak{b}_\omega$. Now (ii) entails that $\omega \circ \operatorname{ad} X = 0$, i.e., $X \in \mathfrak{g}^\omega$.

(v) In view of (iv), the Lie algebras of the stabilizer subgroup of $\omega \in \mathfrak{g}^*$ and $[v_\lambda] \in \mathbb{P}(\mathcal{H}^\infty)$ coincide. Hence the mapping from $G.[v_\lambda]$ onto $\mathcal{O}_\omega$ is a covering. Now the assertion is a consequence of the simple connectedness of the coadjoint orbit $\mathcal{O}_\omega$ ([**Ne94c**, Th. I.18]). □

Before we apply the results from [**Ne94c**, **f**], we recall some of the crucial definitions.

Definition 1.5. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\omega \in \mathfrak{g}^*$.

(a) A complex subalgebra $\mathfrak{b} \subseteq \mathfrak{g}_{\mathbb{C}}$ is called a *complex polarization in* ω if it satisfies the following conditions:
(P1) $\mathfrak{b} \cap \mathfrak{g} = \mathfrak{g}^\omega$, where $\mathfrak{g}^\omega$ is the stabilizer Lie algebra of ω,
(P2) $\mathfrak{b} + \overline{\mathfrak{b}} = \mathfrak{g}_{\mathbb{C}}$, and
(P3) $\omega_{\mathbb{C}}([\mathfrak{b}, \mathfrak{b}]) = \{0\}$, where $\omega_{\mathbb{C}}$ denotes the complex linear extension of ω to $\mathfrak{g}_{\mathbb{C}}$.

(b) A complex polarization $\mathfrak{b}$ in ω is said to be *positive* if $\omega\left(i\left[X, \overline{X}\right]\right) \geq 0$ holds for all $X \in \mathfrak{b}$.

(c) The stabilizer $\mathfrak{g}^\omega$ of ω is said to be *strictly reduced* if the ideal $\mathcal{O}_\omega^\perp \subseteq \mathfrak{g}^\omega$ is trivial, i.e., if the coadjoint orbit $\mathcal{O}_\omega$ spans $\mathfrak{g}^*$. It is called *reduced* if the largest ideal contained in it is central.

(d) We say that an element $\omega \in \mathfrak{g}^*$ is *admissible* if the coadjoint orbit $\mathcal{O}_\omega$ is closed and its convex hull contains no lines. We call $\omega \in \mathfrak{g}^*$ *strictly admissible* if there exists a closed invariant convex set $C \subseteq \mathfrak{g}^*$ which contains no lines and which contains the coadjoint orbit $\mathcal{O}_\omega$ in its algebraic interior. We say that $\mathcal{O}_\omega$ is (*strictly*) *admissible* if ω is (strictly) admissible. Note that strict admissibility implies admissibility (cf. [**HNP93**, Cor. 5.12]).

(e) A finite dimensional real Lie algebra $\mathfrak{g}$ is said to be *admissible* if the extended Lie algebra $\mathfrak{g} \oplus \mathbb{R}$ contains pointed generating invariant cones.

In the following proposition we will see that the positive system Δ^+ is automatically $\mathfrak{k}$-adapted whenever there exists a corresponding unitarizable highest weight module.

Proposition 1.6. *Let ω and $\mathfrak{b}_\omega$ be as in* Lemma I.4. *Then the following assertions hold:*

(i) $\mathfrak{b}_\omega \supseteq \mathfrak{b}(\Delta^+)$ *is a positive complex polarization in* $\omega \in \mathfrak{g}^*$.

(ii) $\mathcal{O}_\omega^\perp = \ker d\pi$.

(iii) *ω is strictly admissible.*
If, in addition, π has discrete kernel. Then we even have:
(iv) *$\mathfrak{g}^\omega$ is strictly reduced.*
(v) *Δ^+ is $\mathfrak{k}$-adapted.*
(vi) *$\Sigma_\omega := \{\alpha \in \Delta : \mathfrak{g}_\mathbb{C}^\alpha.v_\lambda = \{0\}\}$ is a parabolic system of roots.*
(vii) *$\mathfrak{g}$ is admissible.*

Proof. (i) This has been shown in Lemma I.4.
(ii) Let $\mathfrak{a} := \mathcal{O}_\omega^\perp$ and write A for the corresponding analytic normal subgroup of G. Then $X.v_\lambda = i\omega(X)v_\lambda = 0$ holds for all $X \in \mathfrak{a}$ and therefore $\pi(a).v_\lambda = v_\lambda$ for all $a \in A$. Let $\mathcal{H}^A$ denote the set of all A-fixed vectors in $\mathcal{H}$. Since A is a normal subgroup, this is a G-invariant subspace of $\mathcal{H}$. Hence its closedness implies that $\mathcal{H} = \mathcal{H}^A$ because the representation π is irreducible (cf. [**Ne94e**, Th. III.6]). We conclude that A acts trivially on $\mathcal{H}$, hence that $\mathfrak{a} \subseteq \ker d\pi$. On the other hand, Lemma I.3(iii) implies that $\mathcal{O}_\omega \subseteq \ker d\pi^\perp$, i.e., $\ker d\pi \subseteq \mathcal{O}_\omega^\perp = \mathfrak{a}$, whence $\mathfrak{a} = \ker d\pi$.

For the remaining assertions it suffices to prove them for the quotient algebra $\mathfrak{g}/\mathfrak{a}$, so that we may w.l.o.g. assume from now on that π has discrete kernel. We postpone the proof of (iii) for a while.
(iv) This is a direct consequence of (ii) and the assumption $\ker d\pi = 0$.
(v) First we use [**Ne94f**, Prop. III.4] to see that $\mathfrak{b}_\omega \cap \overline{\mathfrak{b}_\omega} \subseteq \mathfrak{k}_\mathbb{C}$. Then (i) implies that $\mathfrak{g}_\mathbb{C}^{-\alpha} \subseteq \overline{\mathfrak{b}}_\omega$ for $\alpha \in \Delta_p^+$. Hence $\mathfrak{g}$ has cone potential by [**Ne94f**, Prop. III.5]. Now [**Ne94f**, Th. IV.21(iv)] together with (i) and (iv) yield that Δ^+ is $\mathfrak{k}$-adapted.
(vi) We have already seen in the proof of (v) that $\Sigma_\omega \cap \Delta_p = \Delta_p^+$. So, in view of (v), it suffices to show that $\Sigma \cap \Delta_k$ is parabolic ([**Ne94e**, Lemma II.6]). Let $\mathfrak{p}^+ := \sum_{\alpha\in\Delta_p^+} \mathfrak{g}_\mathbb{C}^\alpha$. According to [**Ne94e**, Cor. II.12],

$$\mathcal{H}_+^K := \{v \in \mathcal{H}^K : \mathfrak{p}^+.v = \{0\}\}$$

is an irreducible highest weight module for $\mathfrak{k}_\mathbb{C}$, hence the assertion follows from the fact that $\{X \in \mathfrak{k}_\mathbb{C} : X.v \in \mathbb{C}v\}$ is a parabolic subalgebra of $\mathfrak{k}_\mathbb{C}$ (cf. [**Bou90**, Ch. 8; §3; no. 4; Prop. 11]).
(iii) According (i), (iv)–(vi) above, this follows from [**Ne94f**, Th. IV.23].
(vii) In view of (iv) and (iii), this follows from [**HNP93**, 5.15]. □

Definition 1.7. Let $(\pi, \mathcal{H})$ be a continuous unitary representation of the Lie group G. A *coherent state orbit* (CS-orbit) is an orbit of G in $\mathbb{P}(\mathcal{H}^\infty)$ which is a complex submanifold. A representation $(\pi, \mathcal{H})$ is called a *coherent state representation* (CS-representation) if it is irreducible, π has discrete kernel, and there exists $0 \neq v \in \mathcal{H}^\infty$ such that $G.[v]$ is a CS-orbit. For more details and the applications of coherent states we refer to [**Pe86**].

As a consequence of Lemma I.4(i) we record the following proposition.

Proposition 1.8. *Let G be a connected Lie group such that $\mathfrak{g}$ contains a compactly embedded Cartan algebra $\mathfrak{t}$. Then every highest weight representation of G is a CS-representation.*

Now we can use the results from [**Ne94f**] to obtain a new proof of the following result which is due to Lisiecki ([**Li90**, 3.8]). Note that we don't use the Bott-Borel-Weil Theorem in the proof.

Theorem 1.9. *Let G be a connected reductive Lie group and $(\pi, \mathcal{H})$ an irreducible continuous unitary representation with discrete kernel. Then $(\pi, \mathcal{H})$ is a CS-representation if and only if it is a highest weight representation. In this case the uniquely determined CS-orbit is the highest weight orbit.*

Proof. If $(\pi, \mathcal{H})$ is a highest weight representation, then we have just seen in Proposition 1.8 that it is a CS-representation.

Suppose, conversely, that $(\pi, \mathcal{H})$ is a CS-representation. Pick a unit vector $v \in \mathcal{H}^\infty$ such that $G.[v]$ is a CS-orbit, set $\mathfrak{b} := \{X \in \mathfrak{g}_\mathbb{C} : d\pi(X).v \in \mathbb{C}v\}$ and $\omega := \Psi([v])$. Then the fact that $G.[v]$ is a complex orbit shows that $\mathfrak{b} + \overline{\mathfrak{b}} = \mathfrak{g}_\mathbb{C}$ and by Lemma I.3(v), we also have that

$$\omega\left(i\left[\overline{X}, X\right]\right) = \left\|p_v\left(d\pi\left(\overline{X}\right).v\right)\right\|^2 \geq 0$$

for all $X \in \mathfrak{b}$, where equality holds if and only if $d\pi\left(\overline{X}\right).v \in \mathbb{C}v$, i.e., $\overline{X} \in \mathfrak{b}$. If $X \in \mathfrak{g} \cap \mathfrak{b}$, then $\omega(i[X, X]) = 0$ entails that $d\pi(X).v = 0$. Hence $X \in \mathfrak{g}^\omega$ (Lemma 1.3(iv)), and therefore $\mathfrak{b} \cap \mathfrak{g} \subseteq \mathfrak{g}^\omega$. If on the other hand $X \in \mathfrak{g}^\omega$, then there exists $Y \in \mathfrak{b}$ with $X = Y + \overline{Y}$ and so

$$0 = \omega\left(i\left[Y - \overline{Y}, Y + \overline{Y}\right]\right) = 2\omega\left(i\left[Y, \overline{Y}\right]\right).$$

We conclude that $d\pi\left(\overline{Y}\right).v \in \mathbb{C}v$. Hence $\overline{Y} \in \mathfrak{b}$ and $X = Y + \overline{Y} \in \mathfrak{b} \cap \mathfrak{g}$. This shows that $\mathfrak{b} \cap \mathfrak{g} = \mathfrak{g}^\omega$.

Moreover

$$\mathfrak{b} \to \mathbb{C}, \quad X \mapsto \frac{1}{i}\langle d\pi(X).v, v\rangle = \Psi([v])(X)$$

is a homomorphisms of Lie algebras, so that $\mathfrak{b}$ is isotropic for $\Psi([v])$. Now [**Ne94f**, Lemma I.7] shows that $\mathfrak{b}$ is a positive complex polarization in ω.

To see that $\mathfrak{g}^\omega$ is strictly reduced, let $\mathfrak{a} := \mathcal{O}_\omega^\perp$ and A the corresponding analytic normal subgroup. Then A fixes v pointwise and since the closed space $\mathcal{H}^A$ of A-fixed vectors is G-invariant, the irreducibility of the representation shows that A acts trivially on $\mathcal{H}$. Therefore $\mathfrak{a} = \{0\}$ is a consequence of the discrete kernel assumption. Thus we have shown that $\mathfrak{g}^\omega$ is strictly reduced.

Since $\mathfrak{g}$ is reductive, [**Ne94f**, Lemma IV.15] applies and we find a compactly embedded Cartan algebra $\mathfrak{t} \subseteq \mathfrak{g}^\omega$. With [**Ne94f**, Prop. IV.16] we even see that $\mathfrak{b}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. This proves that v is a primitive element in $\mathcal{H}^\infty$ with respect to a positive system Δ^+ such that $\mathfrak{b}$ contains all the positive root spaces. Now the first assertion follows from Theorem III.6 in [**Ne94e**].

The preceding argument also shows that every CS-orbit is a highest weight orbit. Since the space of primitive elements for an irreducible representations is one-dimensional, the uniqueness of the CS-orbit follows. □

Remark 1.10. In [**Li91**] Lisiecki announces some results which can be used to obtain a generalization of Theorem 1.9 to admissible Lie groups, i.e., to those for which $\mathfrak{g}$ is an admissible Lie algebra.

We may w.l.o.g. assume that G is not reductive. We start with a choice of a compactly embedded Cartan algebra $\mathfrak{t}$, write $\mathfrak{n}$ for the nilradical of $\mathfrak{g}$ and pick a $\mathfrak{t}$-invariant reductive subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ is a semidirect product decomposition (cf. [**HiNe93**, Prop. 7.3]). Note that $\mathfrak{n} = [\mathfrak{t}, \mathfrak{n}] + \mathfrak{z}(\mathfrak{g})$ and that $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{t}_\mathfrak{l}$, where $\mathfrak{t}_\mathfrak{l} := \mathfrak{t} \cap \mathfrak{l}$. Since $\mathfrak{g}$ is not reductive, $\mathfrak{n} \neq \mathfrak{z}(\mathfrak{g})$, and since $\mathfrak{g}^\omega$ is strictly reduced and $\mathfrak{g}$ is admissible, $\dim \mathfrak{z}(\mathfrak{g}) = 1$. Moreover $\mathfrak{n} \cong \mathfrak{h}_n$, where $\mathfrak{h}_n$ is the Heisenberg algebra of dimension $2n + 1$. We also recall that G is unimodular (cf. [**HiNe93**, Prop. 7.3]).

According to the fact that π is irreducible, there exists a non-zero functional $\alpha \in \mathfrak{z}(\mathfrak{g})^* \subseteq \mathfrak{g}^*$ such that $\pi(\exp X) = e^{i\alpha(X)}\mathbf{1}$ for all $X \in \mathfrak{z}(\mathfrak{g})$.

In view of Theorem 1 in [**Li91**], the semidirect product $G = N \rtimes L$ is a subgroup of the group $H_n \rtimes \mathrm{Sp}(n, \mathbb{R})^\sim$, where H_n is the $(2n+1)$-dimensional Heisenberg group and the homomorphism $L \to \mathrm{Sp}(n, \mathbb{R})$ is obtained by the action of L on $[\mathfrak{t}, \mathfrak{n}]$ which in turn corresponds to the N-orbit of $[v]$ in $\mathbb{P}(\mathcal{H}^\infty)$ which is also a complex manifold (cf. [**Li91**]).

Moreover, the embedding $\mathfrak{s} \subseteq \mathfrak{sp}(n, \mathbb{R})$ can be chosen in such a way that a maximal compactly embedded subalgebra $\mathfrak{k}_l \subseteq \mathfrak{l}$ is mapped into a maximal compactly embedded subalgebra of $\mathfrak{sp}(n, \mathbb{R})$.

On the other hand Theorem 2 in [**Li91**] shows that the moment mapping Ψ maps a CS-orbit onto a coadjoint Kähler orbit meeting the space $\mathfrak{t}^*$, where $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) + \mathfrak{t}_\mathfrak{l}$ is a compactly embedded Cartan algebra of $\mathfrak{g}$. Now the arguments from the proof of Theorem I.9 carry over to the general case because the result in [**Ne94f**] apply to coadjoint Kähle orbits meeting $\mathfrak{t}^*$.

We note that the main point in these results is to use the Kähler structure on the CS-orbit to obtain results on the structure of the group G. Here the main ingredient is the paper [**DoNa88**] which contains the proof of the

Fundamental Conjecture for Homogeneous Kähler Manifolds. The main difficulty in the application of the result in [**DoNa88**] to our setting is that they use different types of groups which act transitively on the Kähler manifold under consideration.

So far we have encountered several properties of those functionals $\omega \in \mathfrak{t}^*$ for which $i\omega$ arises as a highest weight of a unitary highest weight representations of G. In the next subsection we discuss another condition which has to be satisfied by these functionals and in Section III we will see how to obtain a description of all highest weights of unitary representations.

1.2. The integrality condition. As in the preceding sections, $\mathfrak{g}$ denotes a finite dimensional real Lie algebra, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan algebra, and $\mathfrak{k} \supseteq \mathfrak{t}$ a maximal compactly embedded subalgebra. We fix a *simply connected* Lie group G with $\mathbf{L}(G) = \mathfrak{g}$.

Definition 1.11.

(a) Let $\omega \in \mathfrak{g}^*$ and $\mathcal{O}_\omega$ be the corresponding coadjoint orbit. We say that ω, or the orbit $\mathcal{O}_\omega$, is *integral*, if the homomorphism

$$\sigma : \mathfrak{g}^\omega \to \mathbb{C}, \quad X \mapsto i\omega(X)$$

integrates to a unitary character $\chi : G^\omega \to \mathbb{C}^*$.

(b) Let $\alpha \in \Delta_k$ be a compact root. We choose elements $Y_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ such that $\overline{Y_\alpha} = Y_{-\alpha}$ and $\check{\alpha} := [\overline{Y}_\alpha, Y_\alpha]$ satisfies $\alpha(\check{\alpha}) = 2$ (cf. [**HiNe93**, Th. 7.4]).

Lemma 1.12. *The lattice* $\mathfrak{t}_e := \exp^{-1}(\mathbf{1}) \cap \mathfrak{t}$ *is given by the subgroup*

$$\operatorname{span}_{\mathbb{Z}}\{2\pi i\check{\alpha} : \alpha \in \Delta_k^+\}$$

of $\mathfrak{t}$.

Proof. Since G is simply connected, the same holds for the subgroup $K := \langle \exp \mathfrak{k} \rangle$ because $\mathfrak{k}$ is maximal compactly embedded ([**HiNe91**, III.7.20]). We conclude that $K \cong \exp Z(\mathfrak{k}) \times [K, K]$, where $\exp Z(\mathfrak{k}) = Z(K)_0$ is a vector group. Hence $\mathfrak{t}_e \subseteq \mathfrak{t} \cap [\mathfrak{k}, \mathfrak{k}]$ and therefore we may assume that $\mathfrak{g}$ is compact and simple. Then K is compact and simply connected. Now the assertion follows from [**Bou82**, Ch. 9; §4; no. 6; Prop. 11]. ☐

Lemma 1.13. *Let* $\omega \in \mathfrak{t}^*$. *Then* $\mathfrak{k}^\omega$ *is maximal compactly embedded in* $\mathfrak{g}^\omega$

Proof. Since $\mathfrak{t} \subseteq \mathfrak{g}^\omega$, this follows from [**Ne94f**, Lemma II.7(iv)]. ☐

Proposition 1.14. *For* $\omega \in \mathfrak{t}^* \subseteq \mathfrak{k}^*$ *the following are equivalent:*

(1) ω *is integral.*

(2) $\omega|_{[\mathfrak{k},\mathfrak{k}]}$ *is integral.*
(3) $\omega(\mathfrak{t}_e) \subseteq 2\pi\mathbb{Z}$.
(4) $i\omega(\check{\alpha}) \in \mathbb{Z}$ *for all* $\alpha \in \Delta_k^+$.
(5) *There exists a positive system* Δ^+ *such that* $\lambda = i\omega$ *is a highest weight of a finite dimensional irreducible representation of* K.

Proof. (1) $\Rightarrow$ (2): trivial.
(2) $\Rightarrow$ (3): Let $\chi : [K,K]^\omega \to \mathbb{C}^*$ be a character with $d\chi(\mathbf{1}) = i\omega$ and $X \in \mathfrak{t}_e$. Then $X \in [\mathfrak{k},\mathfrak{k}] \cap \mathfrak{t} \subseteq \mathfrak{k}^\omega$ and therefore $1 = \chi(\mathbf{1}) = \chi(\exp X) = e^{i\omega(X)}$ entails that $\omega(X) \in 2\pi\mathbb{Z}$.
(3) $\Rightarrow$ (1): Suppose that $\omega(\mathfrak{t}_e) \subseteq 2\pi\mathbb{Z}$. Further, let $\widetilde{G^\omega}$ denote the universal covering group of G^ω. Since ω vanishes on the commutator algebra of $\mathfrak{g}^\omega$, there exists a character $\widetilde{\chi} : \widetilde{G^\omega} \to \mathbb{C}^*$ with $d\widetilde{\chi}(\mathbf{1}) = i\omega$.

Since $\pi_1(G^\omega) \subseteq Z(\widetilde{G^\omega}) \subseteq \exp_{\widetilde{G^\omega}} \mathfrak{t}$ ([**HiNe91**, III.7.11]), it follows that

$$\pi_1(G^\omega) \subseteq \exp \mathfrak{t}_e$$

because the homomorphism $\widetilde{G^\omega} \to G$ factors over G^ω. Hence $\omega(\mathfrak{t}_e) \subseteq 2\pi\mathbb{Z}$ entails that the kernel of $\widetilde{\chi}$ contains $\pi_1(G^\omega)$. Thus $\widetilde{\chi}$ factors to a character χ of G^ω.
(3) $\Leftrightarrow$ (4): Lemma 1.12.
(4) $\Leftrightarrow$ (5): We choose Δ^+ such that $i\omega(\check{\alpha}) \geq 0$ holds for all $\alpha \in \Delta^+$. Then the assertion follows from [**Bou90**, Ch. 8; §7; no. 2; Th. 1]. □

Remark 1.15. If $\omega \in \mathfrak{t}^*$ is such that the dimension of $\mathcal{O}_\omega$ is maximal, i.e., $\mathfrak{t} = \mathfrak{g}^\omega$ (cf. [**Ne94f**, Lemma II.4]), then $\omega(i\check{\alpha}) \neq 0$ holds for all $\alpha \in \Delta_k$. If, in addition, ω is integral, then $i\omega(\check{\alpha}) \in \mathbb{Z} \setminus \{0\}$.

Note that for every positive system Δ^+ every element $\omega \in \mathfrak{t}^*$ is conjugate under the Weyl group $\mathcal{W}_\mathfrak{k}$ to an element ω' with $i\omega'(\check{\alpha}) \geq 0$ for all $\alpha \in \Delta_k^+$.

Proposition 1.16. *If* $\lambda = i\omega \in i\mathfrak{t}^*$ *is a highest weight of a unitary highest weight representation* $(\pi, \mathcal{H})$ *of* G*, then* ω *is integral.*

Proof. Let v_λ denote a highest weight vector. Then $\mathbb{C}v_\lambda$ is invariant under the group G^ω because this group fixes the element $[v_\lambda]$ in $\mathbb{P}(\mathcal{H})$ (Lemma 1.4(v)). Moreover, the corresponding character of G^ω is given by $\chi(\exp X) = e^{i\omega(X)}$, hence ω is integral. □

We subsume the results obtained so far in the following theorem. It contains a collection of necessary conditions for a coadjoint orbit to arise as a highest weight orbit for a unitarizable highest weight module. We recall the definition of the cones

$$C_{\min} := C_{\min}(\Delta_p^+) := \operatorname{cone}\left\{ i\left[\overline{X}, X\right] : X \in \mathfrak{g}_\mathbb{C}^\alpha, \alpha \in \Delta_p^+ \right\} \subseteq \mathfrak{t},$$

where $\operatorname{cone}(E)$ denotes the smallest closed convex cone containing E and

$$C_{\max} := C_{\max}(\Delta_p^+) := \{X \in \mathfrak{t} : (\forall \alpha \in \Delta_p^+) i\alpha(X) \geq 0\}.$$

For a set E in a vector space V we put $E^\star := \{\alpha \in V^* : (\forall e \in E)\alpha(e) \geq 0\}$. Note that $E^\star$ is always a closed convex cone which is the dual cone of E if E is a cone.

Theorem 1.17. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an irreducible representation of the Lie group G with discrete kernel such that $\mathcal{H}_\lambda^K$ is a highest weight module of $\mathfrak{g}_{\mathbb{C}}$ with highest weight $\lambda = i\omega$ with respect to a positive system Δ^+. Then $\mathfrak{g}$ is admissible, the subalgebra $\mathfrak{g}^\omega$ is strictly reduced, ω is integral, Δ^+ is $\mathfrak{k}$-adapted, $\omega \in \operatorname{int} C_{\min}^\star$, and there exists a parabolic subset Σ such that $\mathfrak{p}_\Sigma := \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_{\mathbb{C}}^\alpha$ is a positive complex polarization in ω.*

Proof. This follows from Propositions 1.6, 1.16, and [**Ne94f**, Th. IV.21]. □

Proposition 1.18. *If π (as in* Theorem 1.17) *is not assumed to have discrete kernel but the Lie algebra $\mathfrak{g}$ is supposed to be quasihermitean, then the following conclusions hold:*

(i) $\omega \in C_{\min}^\star$.

(ii) *There exists a $\mathfrak{k}$-adapted positive system Δ^+ such that $\mathcal{H}_\lambda^K$ is a highest weight module with respect to Δ^+.*

Proof. (i) This follows from Lemma 1.4(iii).
(ii) Put $\mathfrak{a} := \ker d\pi_\lambda$, $\mathfrak{h} := \mathfrak{g}/\mathfrak{a}$ and write $q : \mathfrak{g} \to \mathfrak{h}$ for the quotient map. Let $\mathfrak{t}' := q(\mathfrak{t})$ denote the Cartan algebra of $\mathfrak{h}$. Then we can identify $\Delta_1 := \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{t}'_{\mathbb{C}})$ with the subset $\Delta_1 := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^\alpha \not\subseteq \mathfrak{a}_{\mathbb{C}}\}$. According to Theorem 1.17, the positive system $\Delta_1^+ = \Delta_1 \cap \Delta^+$ is $\mathfrak{k}$-adapted.

Note that $q(\mathfrak{z}(\mathfrak{k})) = \mathfrak{z}(q(\mathfrak{k}))$ because $\mathfrak{k}$ is a reductive Lie algebra. Since Δ_1^+ is $\mathfrak{k}$-adapted, we find $X_1 \in i\mathfrak{z}(q(\mathfrak{k}))$ such that $\alpha(X_1) > 0$ for all $\alpha \in \Delta_{p,1}^+$. Pick $X_1' \in i\mathfrak{z}(\mathfrak{k})$ with $q(X_1') = X_1$ and $X_1'' \in i\mathfrak{z}(\mathfrak{k})$ so near to X_1' that no non-compact root vanishes on X_1''.

The positive system Δ_1^+ can be defined by an element X_0' which is arbitrarily near to X_1''. Then we have

$$\Delta_p^+ = \{\alpha \in \Delta_p : \alpha(X_0') > 0\} = \{\alpha \in \Delta_p : \alpha(X_1'') > 0\}$$

and this set is $\mathcal{W}_{\mathfrak{k}}$-invariant because X_0'' is fixed by $\mathcal{W}_{\mathfrak{k}}$.

To see that this positive system is compatible with the highest weight module $L(\lambda)$, we note that for $\alpha \in \Delta_p^+$ we always have $q(\mathfrak{g}_{\mathbb{C}}^\alpha) = \mathfrak{h}_{\mathbb{C}}^\alpha$, a space which might be $\{0\}$. Therefore $\mathfrak{g}_{\mathbb{C}}^\alpha.v_\lambda = \mathfrak{h}_{\mathbb{C}}^\alpha.v_\lambda = \{0\}$ for $\alpha \in \Delta_p^+$. Then one only has to choose the system Δ_k^+ such that λ is dominant integral. □

2. The Metaplectic Representation.

Let H_n denote the $(2n+1)$-dimensional Heisenberg group, $\mathrm{Sp}(n,\mathbb{R})$ the symplectic group on $\mathbb{R}^{2n}$ and $\mathrm{Sp}(n,\mathbb{R})^\sim$ its universal covering group. In this section we discuss a representation of the simply connected covering $\mathrm{St}(n,\mathbb{R})^\sim = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})^\sim$ of the Jacobi group which is called the *metaplectic representation*. It will be crucial in the next section. The most convenient model for this representation is the *Fock model* which we describe in the following subsection.

2.1. The Fock model. Let V denote a finite dimensional complex Hilbert space and μ_V the Lebesgue measure on V determined by any isometry with $\mathbb{C}^n$. We write $\mathrm{Hol}(V)$ for the space of holomorphic functions on V and define the *Fock space*

$$\mathcal{F}_V := \left\{ f \in \mathrm{Hol}(V) : \int_V |f(z)|^2 e^{-\|z\|^2}\, d\mu_V(z) < \infty \right\}.$$

We recall that $\mathcal{F}_V$ is a Hilbert space with respect to the scalar product

$$\langle f, h\rangle := \frac{1}{\pi^{\dim \mathcal{H}}} \int_V f(z)\overline{h(z)} e^{-\|z\|^2}\, d\mu(z).$$

Let $n = \dim \mathcal{H}$ and fix an orthonormal basis $e_1, \ldots, e_n$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ we write $z^\alpha := z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$, $|\alpha| := \sum_{j=1}^n \alpha_j$ and $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_n!$. Then the functions

$$\zeta_\alpha\left(\textstyle\sum_j z_j e_j\right) = \frac{1}{\sqrt{\alpha!}} z^\alpha$$

form an orthonormal basis of the Hilbert space $\mathcal{F}_V$ (cf. [**Fo89**, p. 40]). The Hilbert space $\mathcal{F}_V$ is a reproducing kernel Hilbert space since the functions $E_z(w) := e^{\langle w, z\rangle}$ are contained in $\mathcal{F}_V$ and satisfy $f(z) = \langle f, E_z\rangle$ for all $z \in V$ (cf. [**Fo89**, pp. 41-42]). Note that

$$\|E_z\|^2 = \sum_\alpha \frac{1}{\alpha!} |z^\alpha|^2 = e^{\|z\|^2}.$$

We consider the *annihilation operators* $A_j := \frac{\partial}{\partial z_j}$ and the *creation operators* $A_j^* := z_j$, where z_j means the multiplication operator. Then

$$[A_j, A_k^*] = \delta_{jk}\mathbf{1}, \quad A_j\zeta_\alpha = \sqrt{\alpha_j}\zeta_{\alpha-\varepsilon_j}, \quad \text{and} \quad A_j^*\zeta_\alpha = \sqrt{\alpha_j + 1}\zeta_{\alpha+\varepsilon_j},$$

where $\varepsilon_1, \ldots, \varepsilon_n$ denote the canonical basis vectors in $\mathbb{R}^n$. Hence the base vectors in $\mathcal{F}_V$ can be written as

$$\zeta_\alpha = \frac{1}{\sqrt{\alpha!}} (A_1^*)^{\alpha_1} \cdot \ldots \cdot (A_n^*)^{\alpha_n}.\zeta_0,$$

where $\zeta_0 = 1$ denotes the constant function.

To connect our general setup to the notation of Folland's book, we have to identify V with $\mathbb{C}^n \cong \mathbb{R}^{2n}$, where the complex structure on $\mathbb{R}^{2n}$ is given by $I.(x,y) = (-y,x)$ and the scalar product on the Hilbert space $\mathbb{C}^n$ is $\langle z, w\rangle = \pi z^\top \overline{w} = \pi \sum_{j=1}^n z_j \overline{w}_j$.

2.2. The Schrödinger representation. Let us return to the general setting. We define the *Heisenberg group* H_V as the set $V \times \mathbb{R}$ endowed with the multiplication

$$(v,t)(v',t') = (v+v', t+t' - \operatorname{Im}\langle v, v'\rangle).$$

Then the Lie algebra of H_V which is called the *Heisenberg algebra*, is $\mathfrak{h}_V := V \times \mathbb{R}$ with

$$[(v,t),(v',t')] = (0, -2\operatorname{Im}\langle v, v'\rangle).$$

Then the exponential function of H_V is simply the identity and the multiplication on H_V is the corresponding Campbell-Hausdorff multiplication. Note that $[(v,0),(iv,0)] = -\operatorname{Im}\langle v, iv\rangle = \operatorname{Im} i\langle v,v\rangle = \langle v,v\rangle \geq 0$. For $V = \mathbb{C}^n$ we write H_n resp. $\mathfrak{h}_n$ instead of H_V resp. $\mathfrak{h}_V$.

The *Schrödinger representation* $(\rho, \mathcal{F}_V)$ of H_V on $\mathcal{F}_V$ is defined by

$$(\rho(v,t).f)(z) = e^{it - \frac{1}{2}\|v\|^2 - \langle z, v\rangle} f(z+v).$$

It is well known that this prescription defines an irreducible unitary representation of H_V on $\mathcal{F}_V$ (cf. [**Fo89**]).

The basic result on the representation theory of H_V is the following theorem:

Theorem 2.1 (Stone-von Neumann). *Let π be a unitary representation of the Heisenberg group H_V on a Hilbert space $\mathcal{H}$ such that $\pi(0,t) = e^{it}\mathbf{1}$ for all $t \in \mathbb{R}$. Then $\mathcal{H}$ decomposes as a direct sum of invariant subspaces $\widehat{\bigoplus}_{j\in J}\mathcal{H}_j$ such that the representations $(\pi_j, \mathcal{H}_j)$ are unitarily equivalent to $(\rho, \mathcal{F}_V)$. In particular if π is irreducible, then $(\pi, \mathcal{H})$ is equivalent to $(\rho, \mathcal{F}_V)$.*

Proof. [**Fo89**, p. 35] $\square$

Since the imaginary part $J(v,w) := \operatorname{Im}\langle w, v\rangle$ defines a symplectic structure on each Hilbert space V, it makes sense to consider the symplectic group $\operatorname{Sp}(V)$. We form the semidirect product $\operatorname{HSp}(V) := H_V \rtimes \operatorname{Sp}(V)$, where $\operatorname{Sp}(V)$ acts on $H_V \cong V \times \mathbb{R}$ by $A.(x,t) := (A.x, t)$. Note that these are automorphisms of the group H_V. Set $T_A.(x,t) := (A.x,t)$. For $V = \mathbb{C}^n$ we write $\operatorname{HSp}(n) := H_n \rtimes \operatorname{Sp}(n, \mathbb{R})$.

We consider the Schrödinger representation ρ on $\mathcal{F}_V$. Pick $A \in \mathrm{Sp}(V)$. Then T_A is an automorphism of H_V fixing the center pointwise. Hence $\rho \circ T_A$ is an irreducible representation of H_V on $\mathcal{F}_V$. According to the Stone-von Neumann Theorem (Theorem 2.1), there exists a unitary operator $\nu'(A)$ on $\mathcal{F}_V$ such that

$$(\rho \circ T_A)(g) = \nu'(A)\rho(g)\nu'(A)^{-1} \qquad \forall g \in H_V.$$

Definition 2.2. Since $\mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{O}(2n, \mathbb{R}) \cong \mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$, it follows that $\pi_1(\mathrm{Sp}(n, \mathbb{R})) \cong \mathbb{Z}$. Hence there exists a uniquely determined 2-fold covering group $\mathrm{Mp}(n, \mathbb{R})$ of $\mathrm{Sp}(n, \mathbb{R})$. We set $\mathrm{HMp}(n) := H_n \rtimes \mathrm{Mp}(n, \mathbb{R})$. In the general setting we write $\mathrm{HMp}(V) := H_V \rtimes \mathrm{Mp}(V)$.

Theorem 2.3. *There exists a unique continuous unitary representation ν of $\mathrm{Mp}(V)$ on $\mathcal{F}_V$ such that*

$$(\rho \circ T_A)(g) = \nu\left(\widetilde{A}\right)\rho(g)\nu\left(\widetilde{A}\right)^{-1}$$

holds for all $g \in H_V$ and $\widetilde{A}$ in $\mathrm{Mp}(V)$ lying over $A \in \mathrm{Sp}(V)$. Hence this representation can be used to define a continuous unitary representation $\widetilde{\nu}$ of $\mathrm{HMp}(V)$ by $\widetilde{\nu}(h, a) := \rho(h)\nu(a)$.

Proof. The first part is [**Fo89**, p. 185] and the second part is obvious. $\square$

Definition 2.4. The representation $(\nu, \mathcal{F}_V)$ of $\mathrm{Mp}(V)$ is called the *metaplectic representation* and $(\widetilde{\nu}, \mathcal{F}_V)$ is called the *extended metaplectic representation.*

For the purpose of explicit calculations, it is more convenient to use a different realization of the group $\mathrm{Sp}(n, \mathbb{R})$, namely the group

$$\mathrm{Sp}_c(n, \mathbb{R}) := \left\{ \mathcal{A} = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in \mathrm{Gl}(2n, \mathbb{C}) : \mathcal{A} \in \mathrm{U}(n, n) \right\}$$

(cf. [**Fo89**, p. 175]). Then

$$K := \left\{ \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & \overline{P} \end{pmatrix} : P \in \mathrm{U}(n) \right\}$$

is a maximal compact subgroup and

$$T_K := \left\{ \begin{pmatrix} D & \mathbf{0} \\ \mathbf{0} & D^{-1} \end{pmatrix} : D \in \mathrm{U}(n) \text{ diagonal} \right\}$$

is a maximal torus. We have

$$\nu\begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & \overline{P} \end{pmatrix}.f(z) = (\det{}^{-\frac{1}{2}} P) f(P^{-1}z),$$

where $\det^{-\frac{1}{2}}(P)$ means the corresponding function on the twofold covering of $\mathrm{U}(n)$ (cf. [**Fo89**, p. 184]). The same formula applies to elements in the maximal torus T_K. For $X \in \mathfrak{u}(n) \cong \mathfrak{k}$ this means that

$$\left(\nu(\exp X).f\right)(z) = e^{-\frac{1}{2}\operatorname{tr} X} f\left(\exp(-X).z\right).$$

In the abstract setting one can directly check that for $g \in U(V)$ the operator $\left(\nu'(g).f\right)(z) := f(g^{-1}.z)$ satisfies $\nu'(g)\rho(v,t)\nu'(g^{-1}) = \rho(g.v,t)$. The determinant factor is necessary to extend the representation to the big group $\mathrm{Sp}(V)$.

We fix the basis $e_j := E_{jj}$, $j = 1, \ldots, n$ in $i\mathfrak{t}_k \subseteq i\mathfrak{u}(n)$. Then we have for $X = \sum_{j=1}^n x_j e_j$ the formula

$$d\nu(X) = -\frac{1}{2}\left(\sum_{j=1}^n x_j\right)\mathbf{1} - \sum_{j=1}^n x_j z_j \frac{\partial}{\partial z_j}.$$

Extending the Cartan algebra $\mathfrak{t}_k$ to the Cartan algebra $\mathfrak{t} := \mathfrak{t}_k + Z(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{hsp}(n)$, we obtain

$$d\widetilde{\nu}(X) = \left(x_0 - \frac{1}{2}\sum_{j=1}^n x_j\right)\mathbf{1} - \sum_{j=1}^n x_j z_j \frac{\partial}{\partial z_j}, \tag{2.1}$$

where $e_0 = (0, 0, -i) \in i\mathfrak{h}_n$.

The compact roots Δ_k are the roots of the compact Lie algebra $\mathfrak{u}(n)$, hence

$$\Delta_k^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}.$$

The non-compact positive roots are given by

$$\Delta_p^+ = \{\varepsilon_k + \varepsilon_j : 1 \leq k, j \leq n\}$$

(cf. [**Ne94a**, IV]). In the Lie algebra $\mathfrak{hsp}(n, \mathbb{R})$ we also have roots corresponding to the root spaces contained in $(\mathfrak{h}_n)_\mathbb{C}$ which are given by

$$\Delta_r^+ = \{\varepsilon_j : 1 \leq j \leq n\}.$$

The following proposition is an immediate consequence of (2.1).

Proposition 2.5. *With respect to the positive system Δ^+, the metaplectic representation $\widetilde{\nu}$ of* $\mathrm{HMp}(n)$ *is a highest weight representation with highest weight*

$$\lambda_\nu = \left(1, -\frac{1}{2}, \dots, -\frac{1}{2}\right) = \varepsilon_0 - \frac{1}{2}\sum_{j=1}^{n} \varepsilon_j = \varepsilon_0 - \rho(\Delta_r^+),$$

where $\rho(\Delta_r) = \frac{1}{2}\sum_{\alpha\in\Delta_r^+} m(\alpha)\alpha$ and $m(\alpha) = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^\alpha$.

Proof. Let $\mathfrak{t}$ be the compactly embedded Cartan algebra from the preceding subsection. We claim that the positive system Δ^+ described above is chosen such that $C_{\min,z} = \mathbb{R}^+$, i.e., that $i\left[\overline{Y}, Y\right] \geq 0$ for all

$$Y \in (\mathbb{C}^n)^+ := \{X - iIX : X \in \mathbb{R}^{2n}\}$$

where $I(x,y) = (-y,x)$. To see this, one only has to note that

$$\begin{aligned} i[X + iIX, X - iIX] = 2i[X, -iIX] &= 2[X, IX] \\ &= -4\,\mathrm{Im}\langle X, IX\rangle = 4\|X\|^2 \geq 0. \end{aligned}$$

Let $\lambda_\nu = \varepsilon_0 - \rho(\Delta_r^+)$ denote the highest weight of the extended metaplectic representation $\widetilde{\nu}$. Then $\lambda_\nu \in iC_{\min}^\star$ because $-i\varepsilon_0(0,1) = -i\varepsilon_0(ie_0) = 1 > 0$ and $-\rho(\Delta_r^+) \in iC_{\max}^\star \subseteq iC_{\min}^\star$ (cf. [**Ne93a**, III.20]). $\square$

3. The classification of the irreducible representations.

In this section we will see how one can reduce the classification of the unitary highest weight representations of a general Lie groups G, where $\mathfrak{g}$ contains a compactly embedded Cartan algebra, to the case where G is reductive. Note that if $\mathfrak{g}$ does not contain a compactly embedded Cartan algebra, then it makes no sense to consider unitary highest weight representations. The reductive case in turn splits into the trivial abelian case, the compact case which is classical, and the case of hermitean simple Lie algebras, where the classification of the unitary highest weight representations is due to Enright, Howe, and Wallach ([**EHW83**]) and Jakobsen ([**Jak83**]).

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan algebra, and $\Delta^+ \subseteq \Delta$ a $\mathfrak{k}$-adapted positive system of roots. We assume that $\mathfrak{g}$ has cone potential. We also fix a $\mathfrak{t}$-invariant semidirect decomposition $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$, such that $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{t}_{\mathfrak{l}}$, where $\mathfrak{t}_{\mathfrak{l}} = \mathfrak{t} \cap \mathfrak{l}$ holds (cf. [**HiNe93**, Prop. 7.3]) and set $\Delta_r := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{n}_{\mathbb{C}}\}$. We set

$$C_{\min,z} := C_{\min} \cap \mathfrak{z}(\mathfrak{g}) = \mathrm{cone}\left\{i\left[\overline{X}_\alpha, X_\alpha\right] : X_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Delta_r^+\right\}.$$

Pick $\alpha \in C^\star_{\min,\mathfrak{z}} \subseteq Z(\mathfrak{g})^\star$. We consider $\mathfrak{g}$ as a direct sum of $\mathfrak{m} := [\mathfrak{t}, \mathfrak{n}]$, $Z(\mathfrak{g})$, and $\mathfrak{l}$, and accordingly we write the elements of $\mathfrak{g}$ as triples $(Y, Z, X) \in \mathfrak{m} \times Z(\mathfrak{g}) \times \mathfrak{l}$.

Now we use the assumption that $\alpha \in C^\star_{\min}$. Let $\mathfrak{m}^+ := \bigoplus_{\beta\in\Delta_r^+} \mathfrak{g}_{\mathbb{C}}^{\beta}$. The mapping $\mathfrak{m}^+ \to \mathfrak{m}, X \mapsto X + \overline{X}$ is a linear isomorphism. Hence we obtain a complex structure I on $\mathfrak{m}$ by the prescription $I\left(X + \overline{X}\right) := i\left(X - \overline{X}\right)$. This is the complex structure on $\mathfrak{m}^+$ transported to $\mathfrak{m}$. Then

$$\mathfrak{m}^{\pm} = \{Y \mp iIY : Y \in \mathfrak{m}\}$$

and we define $\langle Y, Y'\rangle := \frac{1}{2}\alpha([Y, IY'])$.

This is a positive semidefinite real bilinear form on $\mathfrak{m}$. In fact, the positivity follows from

$$\begin{aligned}\left\langle X+\overline{X}, X+\overline{X}\right\rangle &= \frac{1}{2}\alpha\left(\left[X+\overline{X}, i\left(X-\overline{X}\right)\right]\right)\\ &= \alpha\left(-i\left[X,\overline{X}\right]\right) = \alpha\left(i\left[\overline{X},X\right]\right) \geq 0\end{aligned}$$

since $\alpha \in C^\star_{\min}$ and the symmetry can be seen as follows:

$$\begin{aligned}2\langle Y', Y\rangle = \alpha([Y', IY]) &= \alpha\left(\left[X' + \overline{X}', i\left(X - \overline{X}\right)\right]\right)\\ &= \alpha\left(i\left[\overline{X}', X\right] - i\left[X', \overline{X}\right]\right) = \alpha\left(i\left[\overline{X}, X'\right] - i\left[X, \overline{X}'\right]\right)\\ &= \alpha\left(\left[X + \overline{X}, i\left(X' - \overline{X}'\right)\right]\right) = \alpha([Y, IY']) = 2\langle Y, Y'\rangle.\end{aligned}$$

We also note that I leaves this form invariant because

$$\langle IY, IY'\rangle = -\alpha([IY, Y']) = \alpha([Y', IY]) = \langle Y', Y\rangle = \langle Y, Y'\rangle.$$

Let $\mathfrak{m}^\perp := \{X \in \mathfrak{m} : \langle X, V\rangle = \{0\}\}$. Then $\mathfrak{m}^\perp$ is a complex subspace and the form on $\mathfrak{m}$ induces a positive definite form on the complex vector space $\mathfrak{m}_\alpha := \mathfrak{m}/\mathfrak{m}^\perp$. In view of the invariance of the form under I, we even obtain a Hilbert space structure on $\mathfrak{m}_\alpha$ such that $\langle\cdot,\cdot\rangle$ is the real part of the scalar product.

For $Y \in \mathfrak{m}$ we write $[Y]$ for the corresponding element of $\mathfrak{m}_\alpha$. Then

$$-2\,\mathrm{Im}\langle[Y], [Y']\rangle = -2\,\mathrm{Re}\langle Y, IY'\rangle = \alpha([Y, Y'])$$

and since the brackets in $\mathfrak{g}$ can be computed as

$$[(Y, Z, X), (Y', Z', X')] = ([X, Y'] - [X', Y], [Y, Y'], [X, X']),$$

it is clear that the assignment

$$\beta(Y, Z, X) = ([Y], \alpha(Z), \mathrm{ad}\, X) \tag{3.1}$$

defines a homomorphism $\mathfrak{g} \to \mathfrak{h}_{\mathfrak{m}_\alpha} \rtimes \mathfrak{sp}(\mathfrak{m}_\alpha)$.

Let $\widetilde{\nu}$ denote the extended metaplectic representation of $\mathrm{HMp}(\mathfrak{m}_\alpha)$ on the Fock space $\mathcal{F}_{\mathfrak{m}_\alpha}$, G a simply connected Lie group with $\mathbf{L}(G) = \mathfrak{g}$, and $\widetilde{\beta} : G \to \mathrm{HMp}(\mathfrak{m}_\alpha)$ the Lie group homomorphism with $d\widetilde{\beta}(\mathbf{1}) = \beta$. We consider the representation $\nu_\alpha := \nu \circ \widetilde{\beta}$ of the group G. That this representation is irreducible follows immediately from the fact that the Schrödinger representation of the Heisenberg group $H_{\mathfrak{m}_\alpha}$ is irreducible and that, whenever $\alpha \neq \{0\}$, the group $\widetilde{\beta}(N) = H_{\mathfrak{m}_\alpha}$ is contained in the image of $\widetilde{\beta}$.

We claim that it is a highest weigth representation with respect to Δ^+. To see this, let $X \in \operatorname{int} C_{\max} \cap \mathfrak{l}$. Then we have in particular that $i\delta(X) \geq 0$ for all $\delta \in \Delta_r^+$. Using Proposition 2.5, we see that all $\mathfrak{t}_{\mathfrak{l}}$-weights on $\mathcal{F}_{\mathfrak{m}_\alpha}^K$ are contained in the set

$$-\rho\left(\widetilde{\Delta}_r^+\right) - \sum_{\delta \in \Delta_r^+} \mathbb{N}_0 \delta,$$

where $\widetilde{\Delta}_r^+ = \left\{\beta \in \Delta_r^+ : \mathfrak{g}_{\mathbb{C}}^\beta \not\subseteq \mathfrak{m}_{\mathbb{C}}^\perp\right\}$ is the set of positive roots contributing to $\mathfrak{m}_\alpha$ and

$$\rho\left(\widetilde{\Delta}_r^+\right)(X) = \frac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{m}_\alpha} X = \frac{1}{2}\sum_{\beta \in \widetilde{\Delta}_r^+} m_\alpha(\beta)\beta,$$

where $m_\alpha(\beta) = \dim \mathfrak{g}_{\mathbb{C}}^\beta - \dim\left(\mathfrak{m}_{\mathbb{C}}^\perp \cap \mathfrak{g}_{\mathbb{C}}^\beta\right)$.

It follows in particular that the operator $d\nu_\alpha(X)$ on $\mathcal{F}_{\mathfrak{m}_\alpha}$ is bounded from above and that ν_α is a highest weight representation ([**Ne94e**, Th. III.6]).

We record the results of the preceding discussion in the following proposition.

Proposition 3.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with cone potential, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan algebra, and Δ^+ a $\mathfrak{k}$-adapted positive system of roots. Fix a $\mathfrak{t}$-invariant semidirect decomposition $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$, such that $\mathfrak{t} = Z(\mathfrak{g}) \oplus \mathfrak{t}_{\mathfrak{l}}$ with $\mathfrak{t}_{\mathfrak{l}} = \mathfrak{t} \cap \mathfrak{l}$. Then there exists for every $\alpha \in C_{\min,z}^\star$ a highest weight representation $(\nu_\alpha, \mathcal{F}_{\mathfrak{m}_\alpha})$ of the simply connected group G on the Fock space $\mathcal{F}_{\mathfrak{m}_\alpha}$ with respect to Δ^+. The highest weight is given by*

$$\lambda_\alpha = \left(i\alpha, -\rho\left(\widetilde{\Delta}_r^+\right)\right), \quad \textit{where} \quad \rho\left(\Delta_r^+\right)(X) = \frac{1}{2}\operatorname{tr}\operatorname{ad}_{\mathfrak{m}_\alpha} X.$$

Let $G \cong N \rtimes L$ be a simply connected Lie group as above and χ a unitary character of N, i.e., $\chi(X, Z) = e^{i\alpha(Z)}$ with $\alpha \in Z(\mathfrak{g})^*$. We say that χ is *admissible* for Δ^+ if $\alpha \in C_{\min,z}^\star$ and that a unitary representation $(\pi, \mathcal{H})$ of G is *of type* α if $\pi(z) = \chi(z)\mathbf{1}$ for all $z \in Z(N)$. We write ν_α for the irreducible unitary representation of G on the Fock space $\mathcal{F}_{\mathfrak{m}_\alpha}$ described in Proposition 3.1. The following result is an extension of a theorem in [**Sa71**].

Theorem 3.2 (Satake's Theorem). *Let $\alpha \in \operatorname{int} C^\star_{\mathrm{min,z}}$ and $(\mathcal{H}_l, \pi_l)$ be a continuous unitary representation of L. Then the representation*

$$(\pi, \mathcal{H}) := (\pi_l \otimes \nu_\alpha, \mathcal{H}_l \widehat{\otimes} \mathcal{F}_{\mathfrak{m}_\alpha})$$

is a continuous unitary representation of G of type α, where π_l is considered as a representation of G by $\pi_l(n, l) := \pi_l(l)$. Conversely, all continuous unitary representations of G of type α can be obtained that way.

(i) *π is irreducible if and only if π_l is irreducible.*

(ii) *If π is irreducible, then the space $\mathcal{H}^K$ of K-finite vectors is dense in $\mathcal{H}$.*

(iii) *The representation $(\pi, \mathcal{H})$ is a highest weight representation with respect to Δ^+ if and only if $(\pi_l, \mathcal{H}_l)$ is a highest weight representation with respect to $\Delta_s^+ = \Delta_s^+(\mathfrak{l}_\mathbb{C}, (\mathfrak{t}_\mathfrak{l})_\mathbb{C})$.*

Proof. That we obtain a representation of type α by $\pi_l \otimes \nu_\alpha$ is clear. Now let $(\pi, \mathcal{H})$ be a continuous unitary representation of G of type α. Then we use the Stone-von Neumann Theorem (Theorem 2.1) to decompose $\mathcal{H}$ into a discrete direct sum of irreducible representations of N. Hence $(\pi/_N, \mathcal{H})$ is a factor representation and can be represented as $\mathcal{H} \cong \mathcal{H}_l \widehat{\otimes} \mathcal{F}_{\mathfrak{m}_\alpha}$ (cf. [**Wal92**, p. 331]), where $\pi(n) = \mathbf{1} \otimes \nu_\alpha(n)$ and the commutant of $\pi(N)$ agrees with $B(\mathcal{H}_l) \otimes \mathbf{1}$. For $g \in L$ we set

$$\pi'(g) := \pi(g) \circ (\mathbf{1} \otimes \nu_\alpha(g)^{-1}).$$

Then

$$\begin{aligned}
\pi'(g)\pi(n) &= \pi(g)(\mathbf{1} \otimes \nu_\alpha(g)^{-1})\pi(n) \\
&= \pi(g)(\mathbf{1} \otimes \nu_\alpha(g)^{-1}\nu_\alpha(n)) \\
&= \pi(g)(\mathbf{1} \otimes \nu_\alpha(g)^{-1}\nu_\alpha(n)\nu_\alpha(g))(\mathbf{1} \otimes \nu_\alpha(g)^{-1}) \\
&= \pi(g)(\mathbf{1} \otimes \nu_\alpha(g^{-1}ng))(\mathbf{1} \otimes \nu_\alpha(g)^{-1}) \\
&= \pi(g)\pi(g^{-1}ng)(\mathbf{1} \otimes \nu_\alpha(g)^{-1}) \\
&= \pi(n)\pi(g)(\mathbf{1} \otimes \nu_\alpha(g)^{-1}) \\
&= \pi(n)\pi'(g).
\end{aligned}$$

We conclude that $\pi'(g) = \pi_l(g) \otimes \mathbf{1}$ holds for $\pi_l(g) \in B(\mathcal{H}_l)$. We claim that π_l defines a continuous unitary representation on $\mathcal{H}_l$. The continuity follows immediately from the continuity on $\mathcal{H}$ because $\mathcal{H}_l \cong \mathcal{H}_l \otimes v$ holds for every unit vector v in $\mathcal{F}_{\mathfrak{m}_\alpha}$.

We calculate

$$\begin{aligned}(\pi_l(g)\pi_l(g'))\otimes\mathbf{1} &= \pi'(g)\pi'(g')\\ &= \pi(g)(\mathbf{1}\otimes\nu_\alpha(g)^{-1})\pi'(g')\\ &= \pi(g)\pi'(g')(\mathbf{1}\otimes\nu_\alpha(g)^{-1})\\ &= \pi(g)\pi(g')(\mathbf{1}\otimes\nu_\alpha(g')^{-1})(\mathbf{1}\otimes\nu_\alpha(g)^{-1})\\ &= \pi(gg')(\mathbf{1}\otimes\nu_\alpha(gg')^{-1})\\ &= \pi'(gg') = \pi_l(gg')\otimes\mathbf{1}.\end{aligned}$$

Hence π_l is a representation of L. Finally, for $ng \in G$ with $n \in N$ and $g \in L$ we have that

$$\begin{aligned}\pi(ng) &= \pi(n)\pi(g)\\ &= (\mathbf{1}\otimes\nu_\alpha(n))(\pi_l(g)\otimes\mathbf{1})(\mathbf{1}\otimes\nu_\alpha(g))\\ &= (\mathbf{1}\otimes\nu_\alpha(ng))(\pi_l(g)\otimes\mathbf{1})\\ &= \pi_l(g)\otimes\nu_\alpha(ng).\end{aligned}$$

Thus $\pi = \pi_l\otimes\nu_\alpha$, where π_l is extended to a representation of G by $\pi_l(ng) := \pi_l(g)$.

(i) Since the commutant of $\pi(N)$ is $B(\mathcal{H}_l)\otimes\mathbf{1}$, the commutant of $\pi(G)$ is the commutant of $\pi_l(L)$ in $B(\mathcal{H}_l)$ which is trivial if and only if π, or equivalently, π_l is irreducible.

(ii) We know from the explicit description of the representation ν_α that the space $\mathcal{F}^K_{\mathfrak{m}_\alpha}$ of K-finite vectors contains all the polynomials on $\mathfrak{m}_\alpha$, hence a dense subspace. On the other hand it follows from [**Ne94e**, Corollary III.7] that under the assumption that π and therefore π_l is irreducible, the space $\mathcal{H}_l$ contains a dense set of K-finite vectors because the reductive group L is a (CA) Lie group. We conclude that $\mathcal{H}_l^K\otimes\mathcal{F}^K_{\mathfrak{m}_\alpha}$ which obviously consists of K-finite vectors is dense in $\mathcal{H}$.

(iii) "$\Rightarrow$": Suppose that $(\pi_l,\mathcal{H}_l)$ is a highest weight representation with respect to Δ_s^+ and pick $X \in \operatorname{int} C_{\max}(\Delta^+) \subseteq \operatorname{int} C_{\max}(\Delta_s^+)$. Then the operators $i\cdot d\pi_l(X)$ and $i\cdot d\nu_\alpha(X)$ are bounded from above ([**Ne94e**, Th. III.6]). Pick unit vectors $v\in\mathcal{H}_l^\infty$ and $w\in\mathcal{F}^\infty_{\mathfrak{m}_\alpha}$. Then

$$\langle i\cdot d\pi(X)(v\otimes w), v\otimes w\rangle = \langle i\cdot d\pi_l(X).v,v\rangle + \langle id\nu_\alpha(X).w,w\rangle \leq C$$

for a certain constant $C > 0$. This implies that $i\cdot d\pi(X)$ is bounded from above on $\mathcal{H}$ (cf. [**We76**, p. 278]). Hence $(\pi,\mathcal{H})$ is a highest weight representation with respect to Δ^+ because the space $\mathcal{H}^K$ is dense in $\mathcal{H}$ ([**Ne94e**, Th. III.6]).

"$\Leftarrow$": If, conversely, $(\pi, \mathcal{H})$ is a highest weight representation with respect to Δ^+, then $i \cdot d\pi(X) \leq C\mathbf{1}$ by [**Ne94e**, Th. III.6]. Pick a fixed unit vector $w \in \mathcal{F}^\infty_{\mathfrak{m}_\alpha}$ and let v be a unit vector in $\mathcal{H}_l^\infty$. Then

$$\begin{aligned}\langle i \cdot d\pi_l(X).v, v\rangle = \langle i \cdot d\pi(X)(v \otimes w), v \otimes w\rangle - \langle i \cdot d\nu_\alpha(X).w, w\rangle \\ \leq C - \langle i \cdot d\nu_\alpha(X).w, w\rangle.\end{aligned}$$

Hence $i \cdot d\pi_l(X)$ is bounded from above and the assertion follows from Theorem III.6 in [**Ne94e**]. □

3.1. Unitarizable highest weight modules. For the remainder of this section G denotes a simply connected Lie group and $\mathfrak{g}$ its Lie algebra. We assume that $\mathfrak{g}$ has cone potential and that there exists a $\mathfrak{k}$-adapted positive system Δ^+.

Theorem 3.3. *Let $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ be a $\mathfrak{t}$-invariant semidirect decomposition, where $\mathfrak{n}$ is the nilradical and $\mathfrak{l}$ is reductive. Further let Δ^+ be a $\mathfrak{k}$-adapted positive system, $\omega \in \operatorname{int} C^\star_{\min}$, and $\alpha := \omega \mid_{Z(\mathfrak{g})}$. Then the highest weight module $L(\lambda)$ for $\lambda = i\omega$ is unitarizable if and only if the highest weight module $L_{\mathfrak{l}}(\lambda_l)$ for $\lambda_l := \lambda \mid_{i\mathfrak{t}_{\mathfrak{l}}} + \rho\left(\widetilde{\Delta}^+_r\right) \in i\mathfrak{t}^*_{\mathfrak{l}}$ is unitarizable. Here $\widetilde{\Delta}^+_r = \left\{\beta \in \Delta^+_r : \mathfrak{g}^\beta_{\mathbb{C}} \not\subseteq \mathfrak{m}^\perp_{\mathbb{C}}\right\}$ and $\rho\left(\widetilde{\Delta}^+_r\right) = \frac{1}{2}\sum_{\beta \in \Delta^+_r} m_\alpha(\beta)\beta$, where $m_\alpha(\beta) := \dim \mathfrak{g}^\beta_{\mathbb{C}} - \dim\left(\mathfrak{g}^\beta_{\mathbb{C}} \cap \mathfrak{m}^\perp_{\mathbb{C}}\right)$.*

Proof. Suppose that $L(\lambda)$ is unitarizable, i.e., that it occurs as $\mathcal{H}^K_\lambda$ for an irreducible unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G. Then Satake's Theorem (Theorem 3.2) shows that $(\pi_\lambda, \mathcal{H}_\lambda)$ decomposes as a tensor product $(\pi_l \otimes \nu_\alpha, \mathcal{H}_l \widehat{\otimes} \mathcal{F}_{\mathfrak{m}_\alpha})$, where $(\pi_l, \mathcal{H}_l)$ is a unitary highest weight representation of the reductive group L. Since the highest weight of the tensor product of two highest weight representations is the sum of the two highest weights, we conclude that $\lambda_l := \lambda|_{\mathfrak{l}_{\mathbb{C}}} + \rho\left(\widetilde{\Delta}^+_r\right) \in (\mathfrak{t}_{\mathfrak{l}})^*_{\mathbb{C}}$ is the highest weight of $\mathcal{H}_l^{K \cap L}$.

If, conversely, the $\mathfrak{l}_{\mathbb{C}}$-module $L_{\mathfrak{l}}(\lambda_l)$ is unitarizable and $(\pi_l, \mathcal{H}_l)$ is the corresponding unitary representation, then Satake's Theorem (Theorem 3.2) shows that $(\pi_l \otimes \nu_\alpha, \mathcal{H}_l \widehat{\otimes} \mathcal{F}_{\mathfrak{m}_\alpha})$ defines an irreducible unitary representation of G such that $\mathcal{H}^K$ is a highest weight module. Since λ is the sum of the highest weights λ_l and λ_α, it follows that it is the highest weight of $\mathcal{H}^K$. □

Remark 3.4. If $\mathfrak{l}$ is a reductive Lie algebra, then every irreducible highest weight module $L(\lambda)$ for $\lambda \in i\mathfrak{t}^*$ has a tensor product decomposition

$$L(\lambda) = L(\lambda_0) \otimes L(\lambda_1) \otimes \ldots \otimes L(\lambda_n),$$

according to the decomposition $\mathfrak{l} = Z(\mathfrak{l}) \oplus \mathfrak{l}_1 \oplus \ldots \oplus \mathfrak{l}_n$ into simple factors, where $\lambda_j := \lambda \mid_{(\mathfrak{l}_j)_{\mathbb{C}}}$ ([**Ja79**, p. 39]). The classification of unitarizable highest weight modules for a compact simple Lie algebra is the same as the classification of finite dimensional irreducible modules of a complex simple Lie algebra (cf. [**Bou90**, Ch. 8; §7]). For the case of hermitean simple Lie algebras the situation is much more involved because the unitarizable highest weight modules are infinite dimensional in this case. Nevertheless there exists a complete classification due to Enright, Howe and Wallach ([**EHW83**]) and Jakobsen ([**Jak83**]).

In view of the results for simple Lie algebras mentioned in Remark 3.4, the preceding theorem provides a complete classification of all irreducible highest weight representations of an admissible Lie group G. For the following remark we recall that in particular every admissible Lie algebra has cone potential and is quasihermitean (cf. [**Ne93a**, III.15]).

Remark 3.5. Let $\mathfrak{g}$ be a quasihermitean Lie algebra with cone potential, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan algebra and $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ a $\mathfrak{t}$-invariant semidirect decomposition as before. Let $\lambda \in i\mathfrak{t}^*$. If $L(\lambda)$ is unitarizable, then there exists a $\mathfrak{k}$-adapted positive system such that λ is dominant integral for Δ_k^+ and $\lambda \in iC_{\min}^\star$, i.e., $\lambda\left(\left[\overline{X}_\beta, X_\beta\right]\right) \geq 0$ for $X_\beta \in \mathfrak{g}_{\mathbb{C}}^\beta$ (cf. Proposition 1.18). If, conversely, $\lambda \in iC_{\min}^\star$ is dominant integral for Δ_k^+ and $\alpha := -i\lambda \mid_{Z(\mathfrak{g})}$, then the unitarizability of $L(\lambda)$ can be desided by looking at the functional $\lambda_l := \lambda \mid_{i\mathfrak{t}_\mathfrak{l}} + \rho\left(\widetilde{\Delta}_r^+\right)$, where $\widetilde{\Delta}_r^+$ only depends on α.

Corollary 3.6. *Suppose that $\mathfrak{g}$ is a Lie algebra with cone potential which is compact modulo its radical, i.e., $\mathfrak{l}$ is a compact Lie algebra. Let $\lambda \in i\mathfrak{t}$. Then $L(\lambda)$ is unitarizable if and only if there exists a $\mathfrak{k}$-adapted positive system such that $\lambda \in iC_{\min}^\star$ and λ is dominant integral with respect to Δ_k^+.*

If, in addition, $\mathfrak{g}$ is solvable, then $L(\lambda)$ is unitarizable if and only if there exists a positve system Δ^+ such that $\lambda \in iC_{\min}^\star$.

Proof. In view of Remark 3.5, we only have to recall that the unitarizability condition for a compact Lie algebra is exactly the condition of being dominant integral.

If $\mathfrak{g}$ is solvable, then $\mathfrak{k} = \mathfrak{t}$ is abelian, so that $\mathcal{W}_\mathfrak{k}$ is trivial and every positive system is $\mathfrak{k}$-adapted. Moreover, $\Delta_k = \emptyset$ so that the assertion follows from the first part. □

If we want to use the solution to the unitarzability problem as a solution to the classification problem for unitary highest weight representations, then one has to check that different highest weights lead to non-equivalent unitary representations:

Proposition 3.7. *Let $\mathfrak{g}$ be a Lie algebra containing a compactly embedded Cartan algebra and $L(\lambda)$, $L(\mu)$ two unitarizable highest weight representations with respect to the positive system Δ^+. Then the corresponding unitary representations $(\pi_\lambda, \mathcal{H}_\lambda)$, $(\pi_\mu, \mathcal{H}_\mu)$ of the simply connected group G with $\mathbf{L}(G) = \mathfrak{g}$ are equivalent if and only if $\lambda = \mu$.*

Proof. It is clear that if the corresponding unitary representations π_λ and π_μ are unitarily equivalent, then the same carries over to an isomorphism of the $\mathfrak{g}_{\mathbb{C}}$-modules $\mathcal{H}_\lambda^K$ and $\mathcal{H}_\mu^K$. Therefore $\lambda = \mu$.

If, conversely, $\lambda = \mu$, then we put $\mathcal{H} := \mathcal{H}_\lambda \oplus \mathcal{H}_\mu$ which we endow with the direct sum representation of G. Let $v := (v_\lambda, v_\mu)$. Then $\mathcal{H}^K = \mathcal{H}_\lambda^K \oplus \mathcal{H}_\mu^K$ and $v \in \mathcal{H}^K$ is a primitive element and an analytic vector for G (cf. [**Ne94e**, Cor. III.5]). Let V denote the $\mathfrak{g}_{\mathbb{C}}$-module generated by v. Then V is a highest weight module and if $\mathcal{K}$ is its closure, then the fact that V consists of analytic vectors implies that $\mathcal{K}$ is a G-invariant subspace ([**Wa72**, 4.4.5.6]). Moreover, $\mathcal{K}^K = V$ by construction, and therefore the representation of G on $\mathcal{K}$ is irreducible ([**Ne94e**, Th. III.6]).

Let $p_1 : \mathcal{K} \to \mathcal{H}_\lambda$ denote the projection. Then p_1 is an intertwining operator for G. The same holds for $p_1 p_1^* \in B(\mathcal{H}_\lambda)$. Applying Schur's Lemma to the representation on $\mathcal{K}$ shows that $p_1^* p_1 = c\mathbf{1}_{\mathcal{K}}$ with $c > 0$. Therefore, $A := \frac{1}{\sqrt{c}} p_1$ satisfies $A^*A = \mathbf{1}_{\mathcal{K}}$. Hence $A : \mathcal{K} \to \mathcal{H}_\lambda$ is isometric. Therefore its image is closed and consequently A is surjective. This proves that the representations on $\mathcal{K}$ and $\mathcal{H}_\lambda$ are equivalent. A similar argument applies to $\mathcal{H}_\mu$ showing that the representations on $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are also equivalent. □

3.2. Sufficient conditions for unitarizability. The precise classification of unitarizable irreducible highest weight modules is relatively complicated. For some of the applications as there are the Gelfand-Raïkov Theorem for Ol'shanskiĭ semigroups (cf. Sections IV) and the general Non-linear Convexity Theorem (cf. Section D.V in [**Ne93b**]), it suffices to have in some sense sufficiently many highest weight representations. In the following we derive a sufficient condition for unitarizability which is very easy to check.

Theorem 3.8 (Harish-Chandra). *If $\mathfrak{g}$ is a reductive quasihermitean Lie algebra and $\lambda \in i\mathfrak{t}^*$ is integral such that*

$$\lambda + \rho(\Delta^+) \in i \operatorname{int} C_{\min}^\star, \tag{3.2}$$

where $\rho(\Delta^+) = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$, then the highest weight module $L(\lambda)$ is unitarizable.

Proof. This is Theorem 3 in [**HC55**] which is proved in [**HC56**]. □

The irreducible unitary representations corresponding to highest weight modules $L(\lambda)$, where λ satisfies (3.2), are called *relative holomorphic discrete series representations.*

Theorem 3.9. *Let $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ be as before a $\mathfrak{t}$-invariant semidirect decomposition, where $\mathfrak{g}$ has cone potential, $\mathfrak{l}$ is reductive and $\mathfrak{n}$ is the nilradical, Δ^+ a $\mathfrak{k}$-adapted positive system, and $\lambda \in i \operatorname{int} C_{\min}^\star$ dominant integral such that*

$$\lambda + \rho(\Delta^+) \in i \operatorname{int} C_{\min}(\Delta_s^+)^\star,$$

where $\Delta_s^+ = \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{l}_{\mathbb{C}}\}$. Then $L(\lambda)$ is unitarizable.

Proof. This follows by combining Harish Chandra's Theorem (Theorem 3.8) with Theorem 3.3 and using the trivial fact that $\rho(\Delta_r^+) + \rho(\Delta_s^+) = \rho(\Delta^+)$ and $\widetilde{\Delta}_r^+ = \Delta_r^+$ since $\lambda \in i \operatorname{int} C_{\min}^\star$. □

Remark 3.10. Let $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ be as above such that $Z_{\mathfrak{l}}(\mathfrak{n}) = \{0\}$ and $\dim \mathfrak{z}(\mathfrak{n}) \leq 1$. Suppose that Δ^+ is a $\mathfrak{k}$-adapted positive system such that $C_{\min,z}$ is a pointed cone. Then, according to [**Ne93a**, III.20],

$$C_{\min}(\Delta^+) \subseteq C_{\max}(\Delta^+) \subseteq C_{\max}(\Delta_r^+).$$

It follows in particular that $C_{\min}(\Delta_s^+) \subseteq C_{\max}(\Delta_r^+)$. Since $C_{\max}(\Delta_r^+) \cap (\mathfrak{t} \cap \mathfrak{l})$ is pointed ([**Ne93a**, III.11]), we conclude that the set of positive non-compact roots in Δ_s is determined uniquely by the system Δ_r^+. Hence there are only two choices of a $\mathfrak{k}$-adapted positive system of non-compact roots such that $C_{\min}$ is pointed.

To generalize this observation to general Lie algebras, let $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$, $\mathfrak{l}_1 := Z_{\mathfrak{l}}(\mathfrak{n})$ and $\mathfrak{l}_0$ an ideal of $\mathfrak{l}$ such that $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{l}_1$, where $\mathfrak{g}_0 = \mathfrak{n} \rtimes \mathfrak{l}_0$. If k is the number of hermitean simple ideals in $\mathfrak{l}_1$, it follows that there are exactly 2^{k+1} possibilities for $\mathcal{W}_{\mathfrak{k}}$-invariant sets Δ_p^+ of positive non-compact roots. That there are much more possibilities for Δ^+ comes from the fact that the choice of Δ_k^+ is absolutely arbitrary for a given system Δ_p^+.

4. Coadjoint orbits and holomorphic representations.

Since we are interested in a characterization of those highest weights which might occur via holomorphic representations of an Ol'shanskiĭ semigroup $S = \Gamma(\mathfrak{g}, W, D)$, we have to relate the invariant cone W to the coadjoint orbit $\mathcal{O}_\omega$. One major tool to do this is the following theorem from [**HNP93**] (cf. also [**Ne93b**]).

Theorem 4.1 (The Convexity Theorem for coadjoint orbits). *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra, Δ^+ a $\mathfrak{k}$-adapted positive system, and $\omega \in C^\star_{\min}$. Then*

$$p_{\mathfrak{t}^*}(\mathcal{O}_\omega) = \operatorname{conv}(\mathcal{W}_{\mathfrak{k}}.\omega) + \operatorname{cone}(i\Delta^+_\omega),$$

where

$$\Delta^+_\omega = \left\{\alpha \in \Delta^+_p : (\exists \gamma \in \mathcal{W}_{\mathfrak{k}})(\exists X_\alpha \in \mathfrak{g}^\alpha_{\mathbb{C}}) \ \left\langle \omega, \gamma.i\left[\overline{X}_\alpha, X_\alpha\right]\right\rangle > 0\right\}.$$

If $\mathfrak{g}$ has cone potential and $\omega \in \operatorname{int} C^\star_{\min}$, then $\Delta^+_\omega = \Delta^+_p$.

Proof. This follows from Theorem 5.17 and Remark 5.18 in [**HNP93**]. □

For the following we recall the definitions of the cones $C_{\min}$ and $C_{\max}$ from the end of Section I.

Proposition 4.2. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an irreducible representation of G with highest weight $\lambda = i\omega \in iC^\star_{\min}$. Then the following assertions hold:*

(i) *For $X \in \mathfrak{g}$ we have*

$$\sup \operatorname{Spec}\left(id\pi_\lambda(X)\right) = -\inf\langle X, \mathcal{O}_\omega\rangle.$$

(ii) *$B(\mathcal{O}_\omega) := \{X \in \mathfrak{g} : \inf\langle X, \mathcal{O}_\omega\rangle > -\infty\}$ is the set of all elements for which the selfadjoint operator $i \cdot d\pi_\lambda(X)$ is bounded from above.*

(iii) *$\mathcal{O}^\star_\omega = \{X \in \mathfrak{g} : i \cdot d\pi_\lambda(X) \leq 0\}$.*

(iv) *$B(\mathcal{O}_\omega) \cap \mathfrak{t} = (i\Delta^+_\omega)^\star$.*

(v) *$\mathcal{O}^\star_\omega \cap \mathfrak{t} = \{X \in (i\Delta^+_\omega)^\star : (\forall \gamma \in \mathcal{W}_{\mathfrak{k}})\langle \omega, \gamma.X\rangle \geq 0\}$.*

Proof. Let $\Psi : \mathbb{P}(\mathcal{H}^\infty_\lambda) \to \mathfrak{g}^*$ denote the moment map from Section I and I_λ its image. Then $I_\lambda \subseteq \operatorname{conv}(\mathcal{O}_\omega)$ ([**Ne94g**, Th. II.6]) and therefore

$$\operatorname{conv}(\mathcal{O}_\omega) = \operatorname{conv} I_\lambda.$$

For $X \in \mathfrak{g}$ it follows in particular that

$$\sup \operatorname{Spec}\left(i \cdot d\pi_\lambda(X)\right) = \sup\langle -X, I_\lambda\rangle = \sup\langle -X, \mathcal{O}_\omega\rangle = -\inf\langle X, \mathcal{O}_\omega\rangle.$$

This proves (i), and (ii) and (iii) are direct consequences.

Now we apply Theorem 4.1 to see that

$$p_{\mathfrak{t}^*}(\mathcal{O}_\omega) = \operatorname{conv}(\mathcal{W}_{\mathfrak{k}}.\omega) + \operatorname{cone}\left(i\Delta^+_\omega\right).$$

Therefore

$$B(\mathcal{O}_\omega) \cap \mathfrak{t} = \operatorname{cone}\left(i\Delta^+_\omega\right)^\star = \left(i\Delta^+_\omega\right)^\star$$

which is (iv), and (v) follows immediately. $\square$

The following result generalizes Theorem III.8 in [**Ne94e**] because it shows that the assumption that $\mathfrak{g}$ is a (CA) Lie algebra is not necessary for the conclusion. Note that the assumption that $\mathfrak{g}$ contains a compactly embedded Cartan algebra is superfluous if the representation π is assumed to have discrete kernel (cf. [**Ne94d**, III.7]).

Theorem 4.3. *Let $S = \Gamma(\mathfrak{g}, W, D)$ be an Ol'shanskiĭ semigroup, $(\pi, \mathcal{H})$ an irreducible holomorphic representation, and suppose that $\mathfrak{g}$ contains a compactly embedded Cartan algebra and is quasihermitean. Then the following assertions hold:*

(i) *$\mathcal{H}^K$ is an irreducible highest weight module of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to a $\mathfrak{k}$-adapted positive system Δ^+.*

(ii) *For every $s \in \operatorname{int}(S)$ the operator $\pi(s)$ is a trace class operator, i.e., $\pi(\operatorname{int} S) \subseteq B_1(\mathcal{H})$.*

Proof. Since a highest weight module of a quotient algebra $\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_{\mathbb{C}}$, where $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal, can also be considered as a highest weight module of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$, it suffices to assume that the representation π has discrete kernel (cf. Proposition 1.18). Then $\mathfrak{g}$ is an admissible Lie algebra by [**Ne94d**, III.7] and $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$. It follows in particular that there exists a $\mathfrak{k}$-adapted positive system such that $C_{\min,\mathfrak{z}} \subseteq W$ and an element $\alpha \in \operatorname{int} C_{\min}^{\star}$.

In view of Proposition III.15 in [**Ne94d**], the representation π of $G = \mathrm{U}(S)_0$ is also irreducible. Therefore Theorem 3.2 applies and shows that the space $\mathcal{H}^K$ of K-finite vectors is dense in $\mathcal{H}$.

We choose a regular element $X \in W \cap \mathfrak{t}$. Then the fact that $(\pi, \mathcal{H})$ is a holomorphic representation of S entails that the operator $i \cdot d\pi(X)$ is bounded from above ([**Ne94d**, III.1]). Now Corollary III.7 in [**Ne94e**] yields that $\mathcal{H}^K$ is an irreducible highest weight module with respect to the positive system

$$\Delta^+ := \{\alpha \in \Delta : i\alpha(X) > 0\}.$$

(ii) This is the same argument as in [**Ne94e**, Th. III.8(ii)]. $\square$

We recall from [**Ne94d**] that for a simply connected Ol'shanskiĭ semigroup $S = \Gamma(\mathfrak{g}, W)$ the group $H(S)$ of units is always connected and simply connected. Therefore Corollary 1.10 in [**Ne94d**] yields that the mapping $\widetilde{G} \to S$ is injective for the simply connected group $\widetilde{G}$ with $\mathbf{L}(G) = \mathfrak{g}$ whenever $H(W) = W \cap (-W)$ is compact. Hence we can identify for any G with $\mathbf{L}(G) = \mathfrak{g}$ the fundamental group $\pi_1(G)$ with a discrete central subgroup of S so that $\Gamma(G, W) := \Gamma(\mathfrak{g}, W, \pi_1(G))$ is a well defined Ol'shanskiĭ semigroup with $U(\Gamma(G, W))_0 \cong G$.

For the following proposition we recall from [**Ne94d**, Th. III.8] that the compactness of $H(W)$ is necessary for the existence of a holomorphic representation of a semigroup $\Gamma(G, W)$ with discrete kernel. Therefore the compactness of $H(W)$ is a rather natural assumption. The following result is a first characterization of the extendable highest weight representations. In the following we will see how to obtain a characterization which can be checked directly with the data given by ω, the roots, and the Weyl group $\mathcal{W}_{\mathfrak{k}}$.

Proposition 4.4. *Let G be an admissible Lie group, $(\pi_\lambda, \mathcal{H}_\lambda)$ an irreducible unitary representation of G with highest weight $\lambda = i\omega$, and $\Gamma(G, W)$ an Ol'shanskiĭ semigroup with $H(W)$ compact. We define $s_\omega : \mathfrak{g} \to \mathbb{R} \cup \{\infty\}$ by $s_\omega(X) := -\inf\langle X, \mathcal{O}_\omega\rangle$. Then the following are equivalent:*

(1) *π_λ extends to a holomorphic representation of $\Gamma(G, W)$ on $\mathcal{H}_\lambda$.*

(2) *s_ω is bounded on a 0-neighborhood of W.*

Proof. We have seen in Proposition 4.2(i) that

$$s_\omega(X) = \sup \operatorname{Spec}\left(id\pi_\lambda(X)\right)$$

holds for each element $X \in \mathfrak{g}$.

(1) $\Rightarrow$ (2): If π_λ extends to a holomorphic representation of $\Gamma(G, W)$ on $\mathcal{H}_\lambda$, then Lemma III.12 in [**Ne94d**] implies that there exists a norm $\|\cdot\|$ on $\mathfrak{g}$ such that $s_\omega(X) \leq \|X\|$ holds for all $X \in W$ and therefore (2) follows.

(2) $\Rightarrow$ (1): If, conversely, (2) is satisfied, then we find a norm $\|\cdot\|$ on $\mathfrak{g}$ and $m > 0$ such that $X \in W$ and $\|X\| \leq 1$ imply that $s_\omega(X) \leq m$. Renormalizing the norm, we may assume that $m = 1$. Now $s_\omega(X) \leq \|X\|$ for $X \in W$ and the existence of a holomorphic extension follows from [**Ne94d**, Th. III.14]. □

Corollary 4.5. *Let G be an admissible Lie group, $(\pi_\lambda, \mathcal{H}_\lambda)$ an irreducible highest weight representation of G and $\Gamma(G, W)$ an Ol'shanskiĭ semigroup with $H(W)$ compact. If π_λ extends to a holomorphic representation of $\Gamma(G, W)$ on $\mathcal{H}_\lambda$, then $W \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star$.*

Proof. In view of Proposition 4.2, $B(\mathcal{O}_\omega) = \{X \in \mathfrak{g} : s_\omega(X) < \infty\}$, so that Proposition 4.4 shows that if π_λ extends to $\Gamma(G, W)$, then

$$W \cap \mathfrak{t} \subseteq B(\mathcal{O}_\omega) \cap \mathfrak{t} = (i\Delta_\omega^+)^\star$$

(Proposition 4.2(iv)). □

We are aiming at a converse of Corollary 4.5. So we have to show that $W \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star$ implies the condition of Proposition 4.4. This will be shown in several steps. We start with the case of pointed invariant cones.

Lemma 4.6. *Let $\Delta^+ \subseteq \Delta$ be a $\mathfrak{k}$-adapted positive system, $\omega \in C_{\min}^\star$, and $W \subseteq \mathfrak{g}$ a pointed generating invariant cone. Then the following assertions hold:*

(i) *If $W \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star$, then there exists a $\mathfrak{k}$-adapted positive system $\widetilde{\Delta}^+$ such that $W \cap \mathfrak{t} \subseteq C_{\max}\left(\widetilde{\Delta}^+\right)$ and $\omega \in C_{\min}\left(\widetilde{\Delta}^+\right)^\star$.*

(ii) *If $W \cap \mathfrak{t} \subseteq C_{\max}$, then s_ω is bounded on a 0-neighborhood in W.*

Proof. (i) Let $C := W \cap \mathfrak{t}$. Since W is pointed and generating, there exists a $\mathfrak{k}$-adapted positive system $\widetilde{\Delta}^+ \subseteq \Delta$ such that

$$C_{\min}\left(\widetilde{\Delta}^+\right) \subseteq C \subseteq C_{\max}\left(\widetilde{\Delta}^+\right)$$

([**Ne93a**, III.15]). In view of (i), we have $C \subseteq (i\Delta_\omega^+)^\star$. Thus, for each $\alpha \in \Delta_\omega^+$, there exists an open subset of $C_{\max}\left(\widetilde{\Delta}^+\right)$ on which $i\alpha$ is non-negative and therefore $\alpha \in \widetilde{\Delta}_p^+$. We conclude that $\Delta_\omega^+ \subseteq \widetilde{\Delta}_p^+$.

For $\alpha \in \widetilde{\Delta}_p^+$ let $X_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$. If $\alpha \in \Delta_p^+$, then $\omega\left(i\left[\overline{X}_\alpha, X_\alpha\right]\right) \geq 0$ follows from $\omega \in C_{\min}^\star$. If $\alpha \notin \Delta_p^+$, then $-\alpha \in \Delta_p^+$ and since $\Delta_\omega^+ \subseteq \widetilde{\Delta}_p^+$, it follows that $-\alpha \notin \Delta_\omega^+$. We conclude that $\omega\left(i\left[\overline{X}_\alpha, X_\alpha\right]\right) = 0$ so that we eventually see that $\omega \in C_{\min}\left(\widetilde{\Delta}^+\right)^\star$.

(ii) Let $C := W \cap \mathfrak{t}$. Then $C^\star = W^\star \cap \mathfrak{t}^*$ ([**Ne93a**, Th. I.10]). We choose a $\mathcal{W}_\mathfrak{k}$-invariant element $\alpha_0 \in \operatorname{int} C^\star$ (cf. [**Ne94e**, Lemma II.4]). Then $\mathfrak{t}^* = \mathbb{R}^+\alpha_0 - C^\star$. Hence there exists $t > 0$ with $-\omega \in t\alpha_0 - C^\star$. We set $\alpha := t\alpha_0$. Then $\alpha + \omega \in C^\star$. We note that $\alpha \in \operatorname{int} C_{\min}^\star$ because $C \subseteq C_{\max}$ implies that $C_{\min} \subseteq C$ and hence that $C^\star \subseteq C_{\min}^\star$ ([**Ne93a**, Prop. III.15]).

Let $X \in C$. Then the Convexity Theorem (Theorem IV.1) implies that

$$\begin{aligned}
&\langle X, \mathcal{O}_\alpha + \mathcal{O}_\omega\rangle \\
&\subseteq \langle X, \operatorname{conv}(\mathcal{W}_\mathfrak{k}.\alpha) + \operatorname{cone}(i\Delta_p^+) + \operatorname{conv}(\mathcal{W}_\mathfrak{k}.\omega) + \operatorname{cone}(i\Delta_p^+)\rangle \\
&= \langle X, \alpha + \operatorname{conv}(\mathcal{W}_\mathfrak{k}.\omega) + \operatorname{cone}(i\Delta_p^+)\rangle \\
&\subseteq \langle X, \alpha + \operatorname{conv}(\mathcal{W}_\mathfrak{k}.\omega)\rangle + \mathbb{R}^+ \\
&= \langle X, \operatorname{conv}\left(\mathcal{W}_\mathfrak{k}.(\omega + \alpha)\right)\rangle + \mathbb{R}^+ \subseteq \mathbb{R}^+
\end{aligned}$$

because $\mathcal{W}_\mathfrak{k}.(\omega + \alpha) \subseteq C^\star$.

Next we recall that $W = \overline{\operatorname{Inn}_\mathfrak{g}.C}$ ([**Ne94c**, Prop. II.3]) and use the invariance of the set $\mathcal{O}_\alpha + \mathcal{O}_\omega$ to see that

$$\langle Y, \mathcal{O}_\alpha + \mathcal{O}_\omega\rangle \subseteq \mathbb{R}^+$$

holds for all $Y \in W$. It follows in particular that $\alpha + \mathcal{O}_\omega \subseteq W^\star$.

Finally we apply Lemma IV.4.2 in [**HHL89**] to obtain a norm $\|\cdot\|$ on $\mathfrak{g}$ such that $\|X\| = \alpha(X)$ for all $X \in W$. Now the condition

$$\langle \nu, X\rangle \geq -\|X\| = -\alpha(X) \qquad \forall X \in W, \nu \in \mathcal{O}_\omega$$

is satisfied because $\alpha + \mathcal{O}_\omega \subseteq W^\star$. Therefore $s_\omega(X) \leq \|X\|$ holds for $X \in W$ and the assertion follows. □

Proposition 4.7. *Let $\Delta^+ \subseteq \Delta$ be a $\mathfrak{k}$-adapted positive system, $\omega \in C^\star_{\min}$, and $W \subseteq \mathfrak{g}$ a pointed generating invariant cone. Then the following are equivalent:*

(1) *s_ω is bounded on a 0-neighborhood in W.*

(2) *$W \subseteq B(\mathcal{O}_\omega)$.*

(3) *$W \cap \mathfrak{t} \subseteq (i\Delta^+_\omega)^\star$.*

Proof. (1) $\Rightarrow$ (2): Is trivial since s_ω is positively homogeneous, hence finite on W if it is bounded on a 0-neighborhood.
(2) $\Rightarrow$ (3): This is immediate from Proposition 4.2(iv).
(3) $\Rightarrow$ (1): First we use Lemma 4.6(i) to see that after replacing Δ^+ by $\widetilde{\Delta}^+$, we may w.l.o.g. assume that $W \cap \mathfrak{t} \subseteq C_{\max}$. Then Lemma 4.6(ii) applies. □

Next we have to prepare the case where $H(W)$ is central.

Lemma 4.8 (Factorization Lemma for invariant cones). *Let $\mathfrak{g}$ be a Lie algebra with cone potential and $W \subseteq \mathfrak{g}$ a generating invariant cone such that $H(W)$ is central. Then there exists a pointed generating invariant cone $W_1 \subseteq W$ such that*

$$H(W) \cap W_1 = \{0\} \quad \text{and} \quad W = W_1 + H(W),$$

and the mapping

$$\beta : W_1 \times H(W) \to W, \quad (X, Y) \mapsto X + Y$$

is proper.

Proof. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra and $\alpha : \mathfrak{g} \to \mathfrak{g}/H(W)$ the quotient homomorphism. Then there exists a $\mathfrak{k}$-adapted positive system Δ^+ of roots in $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) = \Delta(\mathfrak{g}_\mathbb{C}/H(W)_\mathbb{C}, \mathfrak{t}_\mathbb{C}/H(W)_\mathbb{C})$ such that

$$C_{\min} \subseteq C \subseteq C_{\max}$$

([**Ne93a**, III.15]). Moreover, there exists a $\mathcal{W}_\mathfrak{k}$-invariant vector space complement $\mathfrak{t}'$ for $H(W)$ in $\mathfrak{t}$. Set $C' := C \cap \mathfrak{t}'$. Then C' is invariant under $\mathcal{W}_\mathfrak{k}$ and pointed.

We claim that $C_{\min} \cap H(W) = \{0\}$. To see this, we note that the quotient Lie algebra $\mathfrak{g}/H(W)$ contains the pointed generating invariant cone $W/H(W)$. Hence it has cone potential (cf. [**Ne93a**, Prop. III.15]) and therefore $0 \neq X \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$, $\alpha \in \Delta_p^+$ implies that $0 \neq i\left[\overline{X}, X\right] \in C_{\min} \setminus H(W)$. Let $\alpha \in \operatorname{algint} W^\star$. Then it follows that $\alpha\left(i\left[\overline{X}, X\right]\right) > 0$ whenever $0 \neq X$. Thus $\alpha \in \operatorname{int} C_{\min}^\star$ (cf. [**Ne94f**, IV.21]) and this shows that $C_{\min} \cap H(W) = \{0\}$.

We define $C_1 := C' + C_{\min}$. To see that C_1 is closed and pointed, in view of [**Ne93a**, I.7], we only have to note that

$$C' \cap -C_{\min} \subseteq H(W) \cap -C_{\min} = \{0\}.$$

It is clear that C_1 is invariant under the Weyl group $\mathcal{W}_{\mathfrak{k}}$. Now we use [**Ne93a**, III.33] to see that there exists a pointed generating invariant cone $W_1 \subseteq \mathfrak{g}$ such that $W_1 \cap \mathfrak{t} = C_1$.

Since

$$\begin{aligned} W_1 \cap H(W) &= C_1 \cap H(W) = (C' + C_{\min}) \cap H(W) \\ &= (C' \cap H(W)) + (C_{\min} \cap H(W)) = \{0\} \end{aligned}$$

follows from $C_1 \subseteq C$ and $H(W) = C \cap (-C)$, we conclude that $W_1 \cap H(W) = \{0\}$. Hence $W_1 + H(W)$ is closed by [**Ne93a**, I.7]. On the other hand it is an invariant cone contained in W which contains $C = C_1 + H(W)$, so it has to contain W by [**Ne93a**, III.34]. It follows that $W = W_1 + H(W)$.

To see that β is a proper mapping, we pick $\alpha \in \operatorname{algint} W^\star$ as above. Then $\alpha(\beta(X, Y)) = \alpha(X)$ and since $H(W) \cap W_1 = \{0\}$, the functional α is contained in $\operatorname{int} W_1^\star$. It follows that the inverse image of a compact subset $K \subseteq W$ under β is contained in $\alpha^{-1}([0, m]) \times H(W)$, where $m = \max \alpha(K)$. Then the projection of $\beta^{-1}(K)$ onto $H(W)$ is also compact and therefore β is proper. $\square$

The following proposition clarifies the assumptions made on the Lie algebra $\mathfrak{g}$ in the remainder of this section.

Proposition 4.9. *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Then the following are equivalent:*

(1) *$\mathfrak{g}$ is admissible.*

(2) *$\mathfrak{g} \oplus \mathbb{R}$ contains a pointed generating invariant cone.*

(3) *$\mathfrak{g}$ contains a generating invariant cone W such that $H(W)$ is a compact Lie algebra whose center is central in $\mathfrak{g}$.*

(4) *$\mathfrak{g}$ has cone potential and there exists a generating invariant cone W with $H(W)$ compact.*

Proof. (1) $\Leftrightarrow$ (2): This is the definition.
(1) $\Rightarrow$ (3): If $\mathfrak{g}$ is admissible, then there are two cases (cf. [**Ne93a**, III.39]). Either $\mathfrak{g}$ is compact semisimple, then we put $W := \mathfrak{g}$, or $\mathfrak{g}$ contains a pointed generating invariant cone W.
(3) $\Rightarrow$ (1): Suppose that $H(W) = W \cap (-W)$ is compact. Then $H(W)' = [H(W), H(W)]$ is a compact ideal and therefore $\mathfrak{g} = \mathfrak{g}_0 \oplus H(W)'$ is a direct sum. Moreover $W = W_0 \oplus H(W)'$, where $W_0 := W \cap \mathfrak{g}_0$. Then the edge of W_0 is central in $\mathfrak{g}_0$ and with Lemma IV.8 we find a pointed generating invariant cone $W_1 \subseteq \mathfrak{g}_0$. Therefore $\mathfrak{g}_0$ is admissible and consequently $\mathfrak{g}$ is admissible ([**Ne93a**, III.37]).
(1) $\Rightarrow$ (4): Since every admissible Lie algebra has cone potential (cf. [**Ne93a**, III.15]), this follows from the fact that (1) implies (3).
(4) $\Rightarrow$ (3): The center of $H(W)$ is an abelian ideal of $\mathfrak{g}$, hence central since $\mathfrak{g}$ has cone potential (cf. [**HiNe93**, Lemma 7.14]). □

Lemma 4.10. *Let W_j, $j = 1, 2$ be closed convex cones in the vector space V_j and $f : V_1 \to V_2$ a linear mapping which induces a proper map of $f_0 : W_1 \to W_2$ which is surjective. Then f_0 is open in* 0.

Proof. Let U_1 be an open 0-neighborhood in W_1. We claim that $f(U_1)$ is a 0-neighborhood in W_2. So let U_2 be a compact 0-neigborhood in W_2. Then the fact that f is proper implies that $f^{-1}(U_2)$ is a compact subset of W_1. On the other hand $W_1 = \bigcup_{\lambda > 0} \lambda U_1$ is an open covering so that we find $\lambda_0 > 0$ with $f^{-1}(U_2) \subseteq \lambda_0 U_1$. Since f is surjective, we conclude that

$$U_2 = f(f^{-1}(U_2)) \subseteq \lambda_0 f(U_1),$$

hence that $\lambda_0^{-1} U_2 \subseteq f(U_1)$. This proves the assertion. □

The following theorem is a version of the *Holomorphic Extension Theorem* ([**Ne94d**, Th. III.14]) for irreducible representations which has the feature that it guarantees the existence of some holomorphic extensions in all cases. The third condition has the remarkable feature that it can be checked rather easily.

Theorem 4.11 (Characterization of the extendable highest weight representations). *Let $\mathfrak{g}$ be an admissible Lie algebra, G a corresponding connected Lie group, $\omega \in C_{\min}^\star$ with respect to the $\mathfrak{k}$-adapted positive system Δ^+, and $(\pi_\lambda, \mathcal{H}_\lambda)$ an irreducible unitary representation of G with highest weight $\lambda = i\omega$. Further let $W \subseteq \mathfrak{g}$ be a generating invariant cone with $H(W)$ compact. Then the following are equivalent:*
(1) *π_λ extends to a holomorphic representation of $\Gamma(G, W)$.*
(2) *s_ω is bounded on a 0-neighborhood in W.*

(3) $W \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star$.

Proof. (1) $\Leftrightarrow$ (2): This is Proposition 4.4.
(2) $\Rightarrow$ (3): This is Corollary 4.5.
(3) $\Rightarrow$ (2): First we note that the commutator algebra $\mathfrak{h}_c$ of $H(W)$ is a compact semisimple ideal of $\mathfrak{g}$, so that we have a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{h}_c$ and accordingly $W = W_1 \oplus \mathfrak{h}_c$ with $W_1 \cap \mathfrak{t}_1 \subseteq (i\Delta_{\omega_1}^+)^\star$, where $\Delta_\omega^+ = \Delta_{\omega_1}^+$ and $\mathfrak{t}_1 = \mathfrak{t} \cap \mathfrak{g}_1$. We also have a corresponding decomposition $\omega = (\omega_1, \omega_c)$ with $\omega_1 \in \mathfrak{g}_1^* \cong \mathfrak{h}_c^\perp$ and $\omega_c \in \mathfrak{h}_c^* \cong \mathfrak{g}_1^\perp$. Then $\mathcal{O}_\omega = \mathcal{O}_{\omega_1} \times \mathcal{O}_{\omega_c}$ and therefore

$$s_\omega(X_1 + X_c) = s_{\omega_1}(X_1) + s_{\omega_c}(X_c)$$

for $X_1 \in \mathfrak{g}_1$ and $X_c \in \mathfrak{h}_c$. Since $\mathcal{O}_{\omega_c}$ is compact, the function s_{ω_c} is trivially locally bounded on $\mathfrak{h}_c$. Hence s_ω is bounded in a 0-neighborhood of W if and only if s_{ω_1} is bounded in a 0-neighborhood of W_1.

This provides a reduction to the case where $\mathfrak{h} := H(W)$ is abelian, hence central (**[HiNe93**, Lemma 7.14]). From now on we assume that $\mathfrak{h}$ is central. This permits us to apply the Factorization Lemma (Lemma 4.8) to the cone W. We find a pointed generating cone $W_1 \subseteq W$ such that the addition mapping $\beta : W_1 \times \mathfrak{h} \to W$ is proper, surjective, and by Lemma IV.10 open in 0.

For $Z \in \mathfrak{h}$ we have $\langle Z, \mathcal{O}_\omega \rangle = \{\omega(Z)\}$ so that

$$s_\omega(X + Z) = s_\omega(X) + \omega(Z)$$

for $X \in W_1$ and $Z \in \mathfrak{h}$. Since β is open in 0, it therefore remains to show that s_ω is bounded on a 0-neighborhood in the pointed generating cone W_1.

In view of (3), we have $W_1 \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star$ so that Proposition 4.7 applies and the proof is complete. $\square$

Corollary 4.12 (Classification of the irreducible holomorphic representations). *Let $\mathfrak{g}$ be an admissible Lie algebra, G a corresponding connected Lie group, $W \subseteq \mathfrak{g}$ a generating invariant cone with $H(W)$ compact, and $S := \Gamma(G, W)$. Then the holomorphic irreducible representations of S restrict exactly to those representations of G which are highest weight representations with respect to a $\mathfrak{k}$-adapted positive system Δ^+ and highest weight $\lambda = i\omega \in iC_{\min}^\star$ satisfying*

$$W \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star.$$

Two holomorphic representations with highest weight λ and λ' are equivalent if and only if $\lambda = \lambda'$.

Proof. If $(\pi, \mathcal{H})$ is an irreducible representation of S, then it follows from Theorem 4.3 that the restriction to G is a highest weight representation with respect to a $\mathfrak{k}$-adapted positive system of roots and from Proposition 1.18 that $\omega \in C_{\min}^\star$. Hence Theorem 4.11 implies that $W \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star$.

If, conversely, $(\pi_\lambda, \mathcal{H}_\lambda)$ is a unitary representation of G with highest weight $\lambda = i\omega$ with respect to Δ^+ such that $W \cap \mathfrak{t} \subseteq (i\Delta_\omega^+)^\star$, then π_λ is irreducible (cf. [**Ne94e**, Th. III.6]) and, according to Theorem 4.11, it extends to a holomorphic representation of S on $\mathcal{H}_\lambda$.

The last assertion follows from Proposition 3.7 for the group G and to pass to S we have to apply Proposition III.15 in [**Ne94d**]. □

The following result shows that the class of highest weight representations of a Lie group G coincides with the class of those representations which have a holomorphic extension to some Ol'shanskiĭ's semigroup containing G as $\mathrm{U}(S)_0$.

Corollary 4.13. *Let G be admissible and $(\pi, \mathcal{H})$ a unitary highest weight representation. Then there exists a generating invariant cone $W \subseteq \mathfrak{g}$ such that π extends to a holomorphic representation of $\Gamma(G, W)$.*

Proof. We may w.l.o.g. assume that $\ker \pi$ is discrete because it suffices to prove the assertion for the quotient group $G/(\ker \pi)_0$. So we assume that $\ker \pi$ is discrete. If $\lambda = i\omega$ is the highest weight, then Theorem 1.17 shows that $\omega \in \operatorname{int} C_{\min}^\star$. In view of Theorem 4.11 and $\Delta_\omega^+ = \Delta_p^+$, it therefore suffices to find a generating invariant cone $W \subseteq \mathfrak{g}$ such that $H(W)$ is compact and $W \cap \mathfrak{t} \subseteq C_{\max}$.

If $\mathfrak{g}$ is compact, then $C_{\max} = \mathfrak{t}$ and we set $W := \mathfrak{g}$. If $\mathfrak{g}$ is non-compact, then $\operatorname{int} C_{\min}^\star \cap Z(\mathfrak{k})^\perp = \emptyset$ because otherwise $0 \in \operatorname{int} C_{\min}^\star$ which in turn implies that $C_{\min} = \{0\}$. But this is impossible if $\mathfrak{g}$ is not compact. Let $W := \mathcal{O}_\omega^\star$. Then, since $\ker \pi$ is discrete, Proposition 4.2 yields that

$$W \cap \mathfrak{t} = \{X \in C_{\max} : \langle \mathcal{W}_{\mathfrak{k}}.\omega, X \rangle \subseteq \mathbb{R}^+\}$$

and therefore $W \cap \mathfrak{t}$ is generating because $\mathcal{W}_{\mathfrak{k}}.\omega$ is a compact subset of the interior of the non-trivial cone $C_{\min}^\star$. This shows in particular that W is generating. Moreover the fact that $\mathfrak{g}^\omega$ is strictly reduced (Theorem 1.17) means that W is pointed and therefore in particular $H(W) = \{0\}$ is compact. The inclusion $W \cap \mathfrak{t} \subseteq C_{\max}$ is a consequence of Proposition 4.2. □

So far we have a classification of all irreducible holomorphic representations of a given admissible Ol'shanskiĭ's semigroup S via highest weight representations and the classification of the highest weight representations from Section III. Our next objective is the Gelfand-Raïkov Theorem for admissible Ol'shanskiĭ semigroups.

5. The Gelfand-Raĭkov Theorem for Ol'shanskiĭ semigroups.

In this section we deal with the problem to find sufficiently many irreducible holomorphic representations for an Ol'shanskiĭ's semigroup $S = \Gamma(\mathfrak{g}, W, D)$. We assume that $H(W)$ is a compact Lie algebra and that $\mathfrak{g}$ is admissible, i.e., $\mathfrak{g} \oplus \mathbb{R}$ contains pointed generating invariant cones. We recall from **[Ne94d]** that these conditions are necessary for S to have a holomorphic representation with discrete kernel (**[Ne94d**, Lemma III.7; Th. III.8]).

Lemma 5.1. *Let S be an Ol'shanskiĭ semigroups . Then the following are equivalent:*

(1) *There exists a holomorphic representation of S with discrete kernel.*

(2) *For every $X \in \mathfrak{g}$ there exists a holomorphic representation $(\pi, \mathcal{H})$ of S such that $d\pi(X) \neq 0$.*

Proof. (1) $\Rightarrow$ (2): If π has discrete kernel, then $d\pi$ is injective, hence satisfies (2).
(2) $\Rightarrow$ (1): Using (2), we find inductively a finite set $\{\pi_1, \ldots, \pi_n\}$ of representations such that $\bigcap_{i=1}^n \ker d\pi_i = \{0\}$. Hence $\pi := \pi_1 \times \ldots \times \pi_n$ is a representation with discrete kernel. $\square$

Lemma 5.2. *If $S = \Gamma(\mathfrak{g}, W, D)$ is an Ol'shanskiĭ's semigroup with $H(W)$ compact, then $S = \mathrm{U}(S)_0 \operatorname{Exp}(iW)$.*

Proof. We may w.l.o.g. assume that $D = \{\mathbf{1}\}$. If $\mathfrak{h}$ is a compact semisimple Lie algebra and $H_{\mathbb{C}}$ the corresponding simply connected complex Lie group, then $H_{\mathbb{C}} = H \exp(i\mathfrak{h})$ follows from the theorem on the Cartan decomposition of semisimple Lie groups (cf. **[HiNe91**, III.6.7]).

Since all compact semisimple ideals of $\mathfrak{g}$ which are contained in $\mathfrak{h} := H(W)$ split, we may assume that $\mathfrak{h}$ is abelian, hence central (**[HiNe93**, Lemma 7.14]). Now we use Theorem I.5(v) in **[Ne94d]** to see that

$$S = H_{\mathbb{C}} U(S)_0 \operatorname{Exp}(iW) = U(S)_0 H \operatorname{Exp}(i\mathfrak{h}) \operatorname{Exp}(iW) = U(S)_0 \operatorname{Exp}(iW)$$

since $H \subseteq U(S)_0$ and $i\mathfrak{h}$ is central and contained in iW. $\square$

The following proposition shows that the kernel congruence of a holomorphic representation $(\pi, \mathcal{H})$ of an Ol'shanskiĭ's semigroup S with discrete kernel is simply the congruence defined by the subgroup $\ker \pi := \{s \in H(S) : \pi(s) = \mathbf{1}\}$ of the group $H(S)$ of units of the semigroup S.

Proposition 5.3. *Let $(\pi, \mathcal{H})$ be a holomorphic representation of the Ol'shanskiĭ semigroup S with discrete kernel. Then $\pi(s) = \pi(s')$ holds if and *only if there exists $d \in \ker \pi$ such that $s' = sd$.*

Proof. If there exists $d \in \ker \pi$ with $s' = sd$, then $\pi(s') = \pi(s)$ holds trivially. According to Lemma V.2, there exist $X, X' \in iW$ and $g, g' \in \mathrm{U}(S)_0$ such that $s = g\,\mathrm{Exp}(X)$ and $s' = g'\,\mathrm{Exp}(X')$. Suppose that $\pi(s) = \pi(s')$. Then

$$\pi\left(\mathrm{Exp}(2X)\right) = \pi(s^*s) = \pi(s)^*\pi(s) = \pi(s'^*s') = \pi\left(\mathrm{Exp}(2X')\right).$$

Note that $d\pi(X)$ and $d\pi(X')$ are self-adjoint operators on $\mathcal{H}$ which satisfy

$$\pi(\mathrm{Exp}\,2X) = e^{2d\pi(X)} = e^{2d\pi(X')}$$

([**Ne94d**, Th. III.1]). Using functional calculus for unbounded self-adjoint operators, we conclude that

$$d\pi(X) = \frac{1}{2}\log\left(\pi(\mathrm{Exp}\,2X)\right) = d\pi(X').$$

Hence

$$\pi(\mathrm{Exp}\,X) = e^{d\pi(X)} = e^{d\pi(X')} = \pi(\mathrm{Exp}\,X')$$

and therefore $\pi(g)\pi(\mathrm{Exp}\,X) = \pi(g')\pi(\mathrm{Exp}\,X)$.

To conclude that $\pi(g) = \pi(g')$, we have to show that the operator $\pi(\mathrm{Exp} X)$ has dense range. Let E denote a spectral measure of $d\pi(X)$ and $I \subseteq \mathbb{R}$ a compact interval. Then $E(I)\pi(\mathrm{Exp}\,X) = e^{E(I)d\pi(X)}$ is invertible on the subspace $E(I)\mathcal{H}$. Thus $E(I)\mathcal{H}$ is contained in the range of $\pi(\mathrm{Exp}\,X)$. Now the density of the union of the subspaces $E(I)\mathcal{H}$ shows that $\pi(\mathrm{Exp}\,X)$ has dense range.

Therefore $\pi(g) = \pi(g')$ and we find $d \in \ker \pi$ such that $g' = gd$. So far we have not used that π has discrete kernel. Now this property entails that $d\pi$ is injective which in turn shows that $X = X'$ and consequently $s' = sd$. □

This proposition has some important corollaries.

Corollary 5.4. *A holomorphic representation π of S is injective if and only if it has trivial kernel.*

Corollary 5.5. *The semigroup $S = \Gamma(\mathfrak{g}, W, D)$ has an injective holomorphic representation if and only if $G = \mathrm{U}(S)$ has an injective continuous unitary representation which extends to S.*

5.1. The existence of sufficiently many holomorphic representations. We keep the notation from the preceding subsection. The basic idea to find sufficiently many irreducible representations of an Ol'shanskiĭ's semigroup S is to use highest weight representations where the weights are contained in $i \operatorname{int} C_{\min}^\star$ with respect to a suitable positive system Δ^+. Before we turn to the representations, we make some remarks on the geometry of the situation we are dealing with.

Proposition 5.6. *Let G be a connected Lie group such that $\mathfrak{g}$ contains a compactly embedded Cartan algebra and $(\pi_\lambda, \mathcal{H}_\lambda)$ an irreducible representation of highest weight $\lambda = i\omega$ such that $\mathfrak{g}^\omega$ is reduced. Then*

$$\ker \pi_\lambda = \exp\{X \in \mathfrak{t} : (\forall \alpha \in \Delta \cup \{\lambda\})\ \alpha(X) \in 2\pi i \mathbb{Z}\}.$$

Proof. First we note that the reducedness of $\mathfrak{g}^\omega$ shows that the Lie algebra $\mathfrak{a}$ of $\ker \pi_\lambda$ is central because it is an ideal contained in $\mathfrak{g}^\omega$ (Lemma 1.4(v)).
"$\subseteq$": Let $g \in \ker \pi_\lambda$, $A := \exp \mathfrak{a} = (\ker \pi_\lambda)_0$, and $q : G \to G/A$ the quotient homomorphism. Then $q(\ker \pi_\lambda) \subseteq G/A$ is a discrete normal subgroup. Hence it is central. Therefore it is contained in the exponential image of the compactly embedded Cartan algebra $\mathfrak{t}/\mathfrak{a}$ ([**HiNe91**, III.7.11]). We conclude that $g \in A\exp(\mathfrak{t}) = \exp(\mathfrak{t})$. Let $X \in \mathfrak{t}$ with $g = \exp(X)$. Then the fact that g is central in G/A implies that $\alpha(X) \in 2\pi i \mathbb{Z}$ holds for all roots $\alpha \in \Delta$ ([**HiNe91**, III.7.11]). But if v_λ is a unit highest weight vector, then we also have that

$$v_\lambda = \pi_\lambda(g).v_\lambda = e^{\lambda(X)}.v_\lambda$$

which shows that $\lambda(X) \in 2\pi i \mathbb{Z}$.
"$\supseteq$": If, conversely, $X \in \mathfrak{t}$ with $\lambda(X) \in 2\pi i \mathbb{Z}$ and $\alpha(X) \in 2\pi i \mathbb{Z}$ for all roots α, then all eigenvalues of $d\pi_\alpha(X)$ on the highest weight module $L(\lambda)$ are multiples of $2\pi i \mathbb{Z}$ ([**Ne94e**, Th. II.9]). Thus $\pi_\lambda(\exp X) = e^{d\pi_\lambda(X)} = \mathbf{1}$, where the exponential has to be considered in the sense of Stone's Theorem. □

Corollary 5.7. *Let G be a connected Lie group such that $\mathfrak{g}$ contains a compactly embedded Cartan algebra and $(\pi_\lambda, \mathcal{H}_\lambda)$ an irreducible representation of highest weight $\lambda = i\omega$ of the simply connected covering group $\widetilde{G}$ such that $\mathfrak{g}^\omega$ is reduced. Let further $E := \{X \in \mathfrak{t} : \exp_G X = \mathbf{1}\}$. Then π_λ factors to a representation of G if and only if $\lambda(E) \subseteq 2\pi i \mathbb{Z}$.*

Proof. In view of Proposition 5.6, we only have to recall that the fundamental group $\pi_1(G) \subseteq \widetilde{G}$ is central and therefore contained in $\exp \mathfrak{t}$ (cf. [**HiNe91**, III.7.11]). □

Definition 5.8. Let G be a connected Lie group such that $\mathfrak{g}$ contains a compactly embedded Cartan algebra $\mathfrak{t}$ and $E := \{X \in \mathfrak{t} : \exp_G X = \mathbf{1}\}$. Then an element $\omega \in \mathfrak{t}^*$ is said to be *G-integral* if $\omega(E) \subseteq 2\pi\mathbb{Z}$ holds for $E := \{X \in \mathfrak{t} : \exp_G X = \mathbf{1}\}$. Note that an element $\omega \in \mathfrak{t}^*$ is integral if it is $\widetilde{G}$-integral (cf. Definition 1.11).

The following lemmas are more general than we need it in this subsection. But in the next subsection we will need them in full generality to prove the Gelfand-Raïkov Theorem for contraction representation.

Lemma 5.9. *Let V be a real vector space and C a generating cone in V. Then the following assertions hold:*

(i) *For every $X \in V$ and every compact subset $K \subseteq \operatorname{int} C$ there exists $n \in \mathbb{N}$ such that $nK + X \subseteq \operatorname{int} C$.*

(ii) *For linearly indepedent elements $\alpha_1, \ldots, \alpha_n \in V^*$ and $X \in V$ the set*

$$C' := \{Y \in \operatorname{int} C : (\forall j)\alpha_j(Y) \in \mathbb{Z}, Y + X \in C\}$$

satisfies $C = \overline{\mathbb{R}^+ C'}$ and there exists a finite subset $C'' \subseteq C'$ such that $\{\alpha \in V^* : (\forall Y \in C'')\alpha(Y) \in \mathbb{Z}\}$ coincides with the subgroup generated by $\alpha_1, \ldots, \alpha_n$.

Proof. (i) The relation $nK + X \subseteq \operatorname{int} C$ is equivalent to $K + \frac{1}{n}X \subseteq \operatorname{int} C$ which is clearly satisfied for sufficiently large $n \in \mathbb{N}$.
(ii) Let $c \in \operatorname{int} C$ and B a compact convex neighborhood of 0 in V such that $c + B \subseteq \operatorname{int} C$. Pick elements $v_1, \ldots, v_n \in V$ such that $\alpha_j(v_k) = \delta_{jk}$.

Then, according to (i), there exists $m \in \mathbb{N}$ such that

$$m(c + B) + X \subseteq C \quad \text{and} \quad mB \supseteq \sum_{j=1}^{n} [0,1]v_j.$$

We conclude that $mc + mB$ contains an element $c' \in C'$. Hence $(c + B) \cap \mathbb{R}^+ C' \neq \emptyset$. Since B was arbitrary, it follows that $\mathbb{R}^+ C'$ is dense in C.

It remains to prove the last assertion. Fix $c' \in C'$ and choose B as above such that $c' + B \subseteq \operatorname{int} C$. Let $B' = \{v \in B : (\forall j)\alpha_j(v) = 0\}$, $k := \dim(\operatorname{span} B') + 1$, and $b_1, \ldots, b_k \in B'$ such that these elements generate a dense subgroup of $\operatorname{span} B'$.

Next we use (i) to find an $m' \in \mathbb{N}$ such that

$$m'(c' + B) + X \subseteq C \quad \text{and} \quad m'B \supseteq \sum_{j=1}^{n} [0,1]v_j.$$

Then $m'c' \in C'$ and $m'c' + v_j \in C'$ for all $j = 1, \ldots, n$. We define

$$C'' := \{c'\} \cup \{c' + b_1, \ldots, c' + b_k\} \cup \{m'c' + v_1, \ldots, m'c' + v_n\}.$$

Then $C'' \subseteq C'$ is finite. Let $\alpha \in V^*$ with $\alpha(C'') \subseteq \mathbb{Z}$. Then $\alpha(c') \in \mathbb{Z}$ and therefore $\alpha(b_j) \in \mathbb{Z}$ for $j = 1, \dots, k$. It follows that α takes integral values on the subgroup of V generated by $b_1, \dots, b_k$. Since this subgroup is dense in $\operatorname{span} B'$, it follows that $\alpha(B') = \{0\}$, and therefore $\alpha \in \operatorname{span}\{\alpha_1, \dots, \alpha_n\}$. Since on the other hand $\alpha(m'c') \in \mathbb{Z}$, it also follows that $\alpha(v_j)$ is integral for $j = 1, \dots, n$. Consequently α is an integral linear combination of $\alpha_1, \dots, \alpha_n$. □

In the following we say that an element $\omega \in \mathfrak{t}^*$ is *regular* if $\mathfrak{g}^\omega = \mathfrak{t}$ (cf. [**Ne94f**, Lemma II.4]). For a given positive system we define $\check{\Delta}_k := \{\check{\alpha} : \alpha \in \Delta_k^+\}$ (cf. Definition 1.11). Then $\check{C}_k := \left(\check{\Delta}_k^+\right)^\star \subseteq i\mathfrak{t}^*$ is the cone which cuts the *dominant integral weights* out of the lattice of all integral weights. This is a chamber for the system $\Delta_k^+ \subseteq \mathfrak{t}^*$ of positive compact roots.

Lemma 5.10. *Let Δ^+ be a $\mathfrak{k}$-adapted positive system, C a pointed $\mathcal{W}_\mathfrak{k}$-invariant cone in $\mathfrak{t}$ lying between $C_{\min}$ and $C_{\max}$, and write $\mathcal{P}_C$ for the set of all dominant G-integral regular elements ω in $\operatorname{int} C^\star$ such that the highest weight module $L(\lambda)$ with $\lambda = i\omega$ is unitarizable. Then*

$$\mathcal{P}_C^\star = C + i\check{C}_k^\star, \qquad \bigcap_{\gamma \in \mathcal{W}_\mathfrak{k}} \gamma.\mathcal{P}_C^\star = C,$$

and there exists a finite subset $\mathcal{P}_C' \subseteq \mathcal{P}_C$ with

$$\{Y \in \mathfrak{t} : (\forall \alpha \in \mathcal{P}_C')\ \alpha(Y) \in 2\pi i\mathbb{Z}\} = \exp_G^{-1}(\mathbf{1}) \cap \mathfrak{t}.$$

Proof. Let $C_1 := C^\star \cap (-i\check{C}_k)$. Then C_1 a generating cone in $\mathfrak{t}^*$ because $\operatorname{int} C^\star$ contains elements which are fixed under $\mathcal{W}_k$ and therefore trivially contained in $i\check{C}_k$ ([**Ne94e**, Lemma II.4]).

The G-integrality condition for elements ω in C_1 means that $\omega(X) \in 2\pi i\mathbb{Z}$ holds for all elements X in the discrete subgroup $\exp^{-1}(\mathbf{1}) \cap \mathfrak{t}$ of the vector space $\mathfrak{t}$. So it clearly suffices to have this condition satisfied for a finite set of generators of this subgroup. Note that any integral element ω in $\operatorname{int} C_1$ is automatically regular because it satisfies $i\omega(\check{\alpha}) \in \mathbb{N}$ for all $\alpha \in \Delta_k^+$ since $i\omega \in \check{C}_k$.

The unitarizability of $L(\lambda)$ is guaranteed by the condition $\lambda + \underline{\rho(\Delta^+)} \in C^\star \subseteq C^\star_{\min,s}$ (Theorem 3.7). Therefore Lemma 5.9(ii) implies that $\overline{\mathbb{R}^+\mathcal{P}_C} = C_1$ and that there exists a finite subset $\mathcal{P}_C' \subseteq \mathcal{P}_C$ with

$$\{Y \in \mathfrak{t} : (\forall \omega \in \mathcal{P}_C')\ \omega(Y) \in 2\pi i\mathbb{Z}\} = \exp_G^{-1}(\mathbf{1}) \cap \mathfrak{t}.$$

Hence

$$\mathcal{P}_C^\star = C_1^\star = C - (i\check{C}_k)^\star = C + i\check{C}_k^\star$$

because the cone on the right hand side is closed by [**Ne94b**, II.11]. Moreover

$$\bigcap_{\gamma\in\mathcal{W}_{\mathfrak{k}}} \gamma(\mathcal{P}_C^\star) = \bigcap_{\gamma\in\mathcal{W}_{\mathfrak{k}}} \gamma(C + i\check{C}_k^{\ \star}) = C$$

follows from Lemmas II.5, II.11, and II.12 in [**Ne94b**]. □

Theorem 5.11 (Gelfand-Raïkov-Theorem for Ol'shanskiĭ semigroups). *Let $S = \Gamma(\mathfrak{g}, W, D)$ be an Ol'shanskiĭ semigroups . Then the irreducible holomorphic representations of S separate the points of S if and only if S is admissible, i.e., if $H(W)$ is a compact Lie algebra and $\mathfrak{g}$ is admissible, i.e., if $\mathfrak{g}\oplus\mathbb{R}$ contains pointed generating invariant cones. Every admissible Ol'shanskiĭ's semigroup has an injective holomorphic representation.*

Proof. In view of Lemma 5.1, the necessity of the admissibility of S follows from [**Ne94d**, Lemma III.7; Th. III.8].

To show that it is sufficient, let $\mathfrak{h} := H(W)$, $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{h}$, and $\alpha : \mathfrak{g} \to \mathfrak{g}_1$ denote the quotient morphism. The Lie algebra $\mathfrak{g}_1$ contains the pointed generating invariant cone $W_1 := W/\mathfrak{h}$. Hence there exists a $\mathfrak{k}$-adapted positive system $\Delta_1^+ \subseteq \Delta_1 := \Delta\left((\mathfrak{g}_1)_{\mathbb{C}}, (\mathfrak{t}_1)_{\mathbb{C}}\right)$, where $\mathfrak{t}_1 = \alpha(\mathfrak{t})$, $\mathfrak{t}$ is a compactly embedded Cartan algebra in $\mathfrak{g}$, and

$$C_{\min}(\Delta_1^+) \subseteq W_1 \cap \mathfrak{t}_1 \subseteq C_{\max}(\Delta_1^+)$$

([**Ne93a**, III.15]).

Since $\mathfrak{h}$ is a compact ideal, its commutator algebra splits and $\mathfrak{h}$ is contained in $\mathfrak{k}$. Thus there exists a $\mathfrak{k}$-adapted positive system $\Delta^+ \subseteq \Delta$ such that $(\Delta_1)_p^+$ can be identified with the set Δ_p^+ since the former set consists of functionals vanishing on $\mathfrak{h}\cap\mathfrak{t}$. Then

$$C_{\min}(\Delta^+) \subseteq C := W \cap \mathfrak{t} \subseteq C_{\max}(\Delta^+)$$

because $\mathfrak{h}\cap\mathfrak{t} \subseteq C_{\max}(\Delta^+)$.

We want to apply Lemma V.10 with $C = C_{\min}$. Then $\mathcal{P}_C$ is the set of all dominant G-integral regular elements ω in $\operatorname{int} C_{\min}^\star$ such that the highest weight module $L(\lambda)$ with $\lambda = i\omega$ is unitarizable. Note that $\mathfrak{g}^\omega = \mathfrak{t}$ is reduced for all these functionals. According to Theorem 4.11, every highest weight representation π_λ with $\omega \in \mathcal{P}_C$ extends holomorphically to S. Since the last statement in Lemma 5.10, in view of Proposition 5.6, entails that there exists a finite subset $\mathcal{P}_C' \subseteq \mathcal{P}_C$ such that the common kernel of the finite set $\mathcal{R} := \{\pi_\lambda : \omega \in \mathcal{P}_C'\}$ is trivial, the direct product representation of $\mathcal{R}$ has trivial kernel, i.e., it is an injective holomorphic representation of S. □

5.2. Contraction representations of Ol'shanskiĭ semigroups. We keep the notation from the preceding subsection and write $C(\mathcal{H})$ for the semigroup of all contractions on the Hilbert space $\mathcal{H}$. Since every contraction representation $\pi : S \to C(\mathcal{H})$ of $S = \Gamma(\mathfrak{g}, W, D)$ annihilates the subgroup $\langle \operatorname{Exp} H(W)_{\mathbb{C}} \rangle$ of S (cf. [**HiNe93**, Lemma 9.14]), we may w.l.o.g. assume that the cone W is pointed. Then we find a $\mathfrak{k}$-adapted positive system Δ^+ such that the cone $C := W \cap \mathfrak{t}$ satisfies

$$C_{\min} \subseteq C \subseteq C_{\max}.$$

Note that this implies that $C^\star \subseteq C_{\min}^\star$. We set $G := \mathrm{U}(S)_0$.

Lemma 5.12. *Let $\mathcal{P}_C$ denote the set of all dominant G-integral regular elements ω in* $\operatorname{int} C^\star$ *such that the highest weight module $L(\lambda)$ with $\lambda = i\omega$ is unitarizable. Then for every $\omega \in \mathcal{P}_C$ the unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G extends to a holomorphic contraction representation of S.*

Proof. In view of Theorem 4.11, every highest weight representation π_λ with $\omega \in \mathcal{P}_C$ extends holomorphically to S. It follows from Proposition 4.2 that this representation is contractive because $W \subseteq \mathcal{O}_\omega^\star$ follows from $\omega \in W^\star$. □

Theorem 5.13 (Gelfand-Raïkov-Theorem for Contraction Representation). *Let $S = \Gamma(G, W)$ be an Ol'shanskiĭ's semigroup with $G = \mathrm{U}(S)_0$, W pointed, and $\mathcal{P}_C$ as above. Then*

$$W = \bigcap_{\omega \in \mathcal{P}_C} W_\omega,$$

S has an injective holomorphic contraction representation, and the irreducible holomorphic contraction representations separate the points.

Proof. In view of Lemma 5.12, this follows with the same argument as Theorem 5.11. Here Lemma 5.10 provides a finite set of holomorphic representations which are contractive by Lemma 5.12 and which separate the points. The direct product of these representations is an injective contraction representation. The representation of W as an intersection of the cones W_ω, $\omega \in \mathcal{P}_C$ follows from Lemma 5.10. □

References

[Bou90] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 7 et 8*, Masson, Paris, 1990.

[Bou82] ———, *Groupes et algèbres de Lie, Chapitre 9*, Masson, Paris, 1982.

[BK80] M. Brunet and P. Kramer, *Complex extensions of the representation of the symplectic group associated with the canonical commutation relations*, Reports on Math. Physics, **17** (1980), 205–215.

[Br85] M. Brunet, *The metaplectic semigroup and related topics*, Reports on Math. Physics, **22** (1985), 149–170.

[Dix64] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.

[DoNa88] J. Dorfmeister and K. Nakajima, *The fundamental conjecture for homogeneous Kähler manifolds*, Acta. Math., **161** (1988), 23–70.

[EHW83] T.J. Enright, R. Howe and N. Wallach, *A classification of unitary highest weight modules*, Proc. Representation theory of reductive groups (Park City, UT, 1982), pp. 97–149; Progr. Math., **40** (1983), 97–143.

[Fo89] G.B. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press, Princeton, New Jersey, 1989.

[GS84] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.

[HC55] Harish-Chandra, *Representations of semi-simple Lie groups*, IV, Amer. J. Math., **77** (1955), 743–777.

[HC56] ———, *Representations of semi-simple Lie groups*, VI, Amer. J. Math., **78** (1956), 564–628.

[Hi89] J. Hilgert, *A note on Howe's oscillator semigroup*, Annales de l'institut Fourier, **39** (1989), 663–688.

[HHL89] J. Hilgert, K.H. Hofmann and J.D. Lawson, *Lie Groups, Convex Cones, and Semigroups*, Oxford University Press, 1989.

[HiNe91] J. Hilgert and K.-H. Neeb, *Lie-Gruppen und Lie-Algebren*, Vieweg, Braunschweig, 1991.

[HiNe93] ———, *Lie semigroups and their applications*, Lecture Notes in Math., **1552**, Springer, 1993.

[HNP93] J. Hilgert, K.-H. Neeb and W. Plank, *Symplectic convexity theorems and coadjoint orbits*, Comp. Math., **94** (1994), 129–180.

[HiOl92] J. Hilgert and G. Olafsson, *Analytic continuations of representations, the solvable case*, Jap. Journal of Math., **18** (1992), 213–290.

[How88] R. Howe, *The Oscillator semigroup*, in "The Mathematical Heritage of Hermann Weyl", Proc. Symp. Pure Math., **48**, R. O. Wells ed., AMS Providence, 1988.

[Ja79] J.C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Math., **750**, Springer, 1979.

[Jak83] H.P. Jakobsen, *Hermitean symmetric spaces and their unitary highest weight modules*, J. Funct. Anal., **52** (1983), 385–412.

[La94] J.D. Lawson, *Polar and Ol'shanskiĭ decompositions*, J. für Reine Ang. Math., **448** (1994), 191–219.

[Li90] W. Lisiecki, *Kähler coherent state orbits for representations of semisimple Lie groups*, Ann. Inst. Henri Poincaré, **53(2)** (1990), 245–258.

[Li91] ———, *A classification of coherent state representations of unimodular Lie groups*, Bull. of the AMS, **25(1)** (1991), 37–43.

[Ne92] K.-H. Neeb, *On the fundamental group of a Lie semigroup*, Glasgow Math. J., **34** (1992), 379–394.

[Ne93a] ———, *Invariant subsemigroups of Lie groups*, Memoirs of the AMS, **499** (1993).

[Ne93b] ______, *Holomorphic representation theory and coadjoint orbits of convexity type*, Habilitationsschrift, Technische Hochschule Darmstadt, January, 1993.

[Ne94a] ______, *Contraction semigroups and representations*, Forum Math., **6** (1994), 233–270.

[Ne94b] ______, *A convexity theorem for semisimple symmetric spaces*, Pacific Journal of Math., **162(2)** (1994), 305–349.

[Ne94c] ______, *On closedness and simple connectedness of adjoint and coadjoint orbits*, Manuscripta Math., **82** (1994), 51–65.

[Ne94d] ______, *Holomorphic representation theory* I, Math. Annalen, **301** (1995), 155–181.

[Ne94e] ______, *Holomorphic representation theory* II, Acta Math., **173** (1994), 103–133.

[Ne94f] ______, *Kähler structures and convexity properties of coadjoint orbits*, Forum Math., **7** (1995), 349–384.

[Ne94g] ______, *On the convexity of the moment mapping for a unitary highest weight representation*, J. Funct. Anal., **127(2)** (1995), 301–325.

[Ol82] G.I. Ol'shanskiĭ, *Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series*, Funct. Anal. and Appl., **15** (1982), 275–285.

[Pe86] A. Perelomov, *Generalized coherent states and their applications*, Texts and Monographs in Physics, Springer, 1986.

[Sa71] I. Satake, *Unitary representations of a semi-direct product of Lie groups on $\overline{\partial}$-cohomology spaces*, Math. Ann., **190** (1971), 177–202.

[Sta86] R.J. Stanton, *Analytic Extension of the holomorphic discrete series*, Amer. J. Math., **108** (1986), 1411–1424.

[Wal88] N.R. Wallach, *Real reductive groups* I, Academic Press Inc., Boston, New York, Tokyo, 1988.

[Wal92] ______, *Real reductive groups* II, Academic Press Inc., Boston, New York, Tokyo, 1992.

[Wa72] G. Warner, *Harmonic analysis on semisimple Lie groups* I, Springer, Berlin, Heidelberg, New York, 1972.

[We76] J. Weidmann, *Lineare Operatoren in Hilberträumen*, Teubner, Stuttgart, 1976.

Received December 20, 1993 and revised July 13, 1994.

MATHEMATISCHES INSTITUT
UNIVERSITÄT ERLANGEN
BISMARCKSTR. $1\frac{1}{2}$
D-91054 ERLANGEN
FRG
E-mail address: neeb@mi.uni-erlangen.de

PACIFIC JOURNAL OF MATHEMATICS

Volume 174 No. 2 June 1996