Pacific Journal of Mathematics

GLOBAL ANALYTIC HYPOELLIPTICITY OF $\square_b$ ON CIRCULAR DOMAINS

SO-CHIN CHEN

Volume 175 No. 1 September 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 1, 1996

GLOBAL ANALYTIC HYPOELLIPTICITY OF $\Box_b$ ON CIRCULAR DOMAINS

So-Chin Chen

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary. In this paper we show that $\Box_b$ is globally analytic hypoelliptic if D is either circular satisfying $\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j}(z) \neq 0$ near the boundary bD, where $r(z)$ is a defining function for D, or Reinhardt.

I. Introduction.

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary, and let $\mathbb{C}^n$ be equipped with the standard Euclidean metric. We consider the real analytic regularity problem of the $\Box_b$- equation on the boundary. Namely, given any $f \in C^{\omega}_{p,q}(bD)$, $0 \leq p \leq n-1$ and $1 \leq q \leq n-1$, let $u = N_b f \in L^2_{p,q}(bD)$ be the solution to the following equation,

$$\Box_b u = \left(\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b\right) N_b f = f. \tag{1.1}$$

Then we ask: is $u = N_b f \in C^{\omega}_{p,q}(bD)$? For the definitions of these notations the reader is referred to Section II.

The existence of the solution $u = N_b f$ is an immediate consequence of the closedness of the range of $\Box_b$ which was proved by M.C.Shaw [**17**] and Boas and M.C.Shaw [**1**], and independently by Kohn [**15**]. Since $u = N_b f$ is the canonical solution to the equation (1.1), it is unique. It also follows from Proposition 2.7. Next the real analyticity of the boundary bD implies that $u = N_b f$ is smooth, i.e., $u \in C^{\infty}_{p,q}(bD)$. For instance see Kohn [**14**][**16**]. Therefore, the main concern here is about the real analytic regularity of the solution u. The only result we know so far is that the answer is affirmative when D is of strict pseudoconvexity which is due to Tartakoff [**18**][**19**][**20**] and Treves [**21**] for $n \geq 3$ and to Geller [**13**] for $n = 2$.

The purpose of this article is to prove the following main results which presumably yield the first positive result to this problem on weakly pseudoconvex domains.

Theorem 1.2. *Let D be a smoothly bounded pseudoconvex domain with real analytic boundary bD in $\mathbb{C}^n$, $n \geq 2$. Suppose that D is circular and that $\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j}(z) \neq 0$ near bD, where $r(z)$ is the defining function for D. Then for any $f \in C^{\omega}_{p,q}(bD)$, $0 \leq p \leq n-1$ and $1 \leq q \leq n-1$, the solution $u = N_b f$ to the $\Box_b$-equation is also in $C^{\omega}_{p,q}(bD)$.*

Here a domain D is called circular if $z \in D$ implies

$$e^{i\theta} \cdot z = \left(e^{i\theta} z_1, \ldots, e^{i\theta} z_n\right) \in D$$

for any $\theta \in \mathbb{R}$. D is called Reinhardt if $z \in D$ implies $\left(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n\right) \in D$ for any $\theta_1, \ldots, \theta_n \in \mathbb{R}$, and D is called complete Reinhardt if $z \in D$ implies $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D$ for any $\lambda_i \in \mathbb{C}$ with $|\lambda_i| \leq 1$, $i = 1, \ldots, n$. Then we also prove

Theorem 1.3. *Let D be a smoothly bounded Reinhardt pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary. Then the same assertion as in the Theorem 1.2 holds.*

Hence, in particular, $\Box_b$ is globally analytically hypoelliptic on any complete Reinhardt domains with real analytic boundary whuch provides a large class of examples. Next we have the following immediate corollary.

Corollary 1.4. *Let D be a smoothly bounded pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$, $n \geq 2$. Suppose that either D is Reinhardt or D is circular with $\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j}(z) \neq 0$ near bD, where $r(z)$ is the defining function for D. Then we have*

(i) *The Szegö projection S defined on bD preserves the real analyticity globally, and*

(ii) *The canonical solution w to the $\overline{\partial}_b$-equation, i.e., $\overline{\partial}_b w = \alpha$, is in $C^{\omega}_{p,q-1}(bD)$ if the given α is in $C^{\omega}_{p,q}(bD)$ and satisfies $\overline{\partial}_b \alpha = 0$.*

Here the Szegö projection S is defined to be the orthogonal projection from $L^2(bD)$ onto the closed subspace, denoted by $H^2(bD)$, of square-integrable CR- functions defined on the boundary, and by canonical solution w we mean the solution with minimum L^2- norm. We remark that statement (i) has been proved by the author before in [**5**] via a more direct argument, and a special case of (ii), i.e., $n = 2$, is verified by Derridj and Tartakoff in [**11**].

Now if we combine the above theorems and the main result, i.e., the Theorem B, obtained by the author in Chen [**6**], then we can conclude the following theorem.

Theorem 1.5. *Let $D \subseteq \mathbb{C}^n$, $n \geq 3$, be a smoothly bounded pseudoconvex domain with real analytic boundary. Then the Szegö projection S associated with D preserves the real analyticity globally whenever D is defined by*

(i) $D = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \middle| |f(z_1)|^2 + H\left(|z_2|^2, \ldots, |z_n|^2\right) < 1\right\}$, *where $f(z_1)$ is holomorphic in z_1 and $H(x_2, \ldots, x_n)$ is a polynomial with positive coefficient and $H(0, \ldots, 0) = 0$, or*

(ii) $D = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \middle| |f(z_1)|^2 + |g(z)|^2 + \sum_{j=3}^{n} h_j\left(|z_j|^2\right) < 1\right\}$, *where $f(z_1)$ and $g(z_2)$ are holomorphic in one variable z_1 or z_2 respectively, and $h_j(x)$ is a polynomial with positive coefficients satisfying $h_j(0) = 0$, $h_j'(0) > 0$ for $3 \leq j \leq n$.*

The real analytic regularity of the Bergman projection P, which is defined to be the orthogonal projection from $L^2(D)$ onto the closed subspace $H^2(D)$ of square- integrable holomorphic functions defined on D, on the domains (i) and (ii) defined in Theorem 1.5 has been established in Chen [**6**].

We should point out that in general the analytic pseudolocality of the Szegö projection S is false. Counterexamples have been discovered by Christ and Geller [**7**]. However, so far there is no counterexample to the globally real analytic regularity of S. Meanwhile, a number of positive results of the local analytic hypoellipticity for $\Box_b$ have been established on some model pseudoconvex hypersurface by Derridj and Tartakoff. For instance, see [**8**][**9**][**10**].

Finally the author would like to thank Professor Mei-chi Shaw for helpful discussion during the preparation of this paper.

II. Proofs of the Theorems 1.2 and 1.3.

Let D be a smoothly bounded pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$, $n \geq 2$, and let $\mathbb{C}^n$ be equipped with the standard Euclidean metric. Since we assume that the domain D is circular, we can choose a real analytic defining function $r(z)$ for D such that $r(z) = r\left(e^{i\theta} \cdot z\right)$ and that $|\nabla r(z)| = 1$ for $z \in bD$. Let $z_0 \in bD$ be a boundary point. We may assume that $\frac{\partial r}{\partial z_n}(z_o) \neq 0$. Hence a local basis for $T^{1,0}(bD)$ near z_0 can be chosen to be

$$L_j = \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_j} - \frac{\partial r}{\partial z_j}\frac{\partial}{\partial z_n} \cdot \text{for } 1 \leq j \leq n-1.$$

Put $X(z) = \sum_{j=1}^{n} \frac{\partial r}{\partial \bar{z}_j}\frac{\partial}{\partial z_j} - \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}$. We see that

$$L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}$$

and $X(z)$ form a local basis for the complexified tangent space $\mathbb{C}T(bD)$, and $X(z)$ is perpendicular to $T^{1,0}(bD) \oplus T^{0,1}(bD)$. Let $w_1, \dots, w_{n-1}$ be $(1,0)$-form dual to $L_1, \dots, L_{n-1}$ respectively. Put $\eta = 2\left(\partial r - \overline{\partial} r\right)$. Then it is not hard to see that $w_1, \dots, w_{n-1}, \overline{w}_1, \dots, \overline{w}_{n-1}$ and η form a local basis for the complexified cotangent space $\mathbb{C}T^*(bD)$, and η is dual to $X(z)$ and perpendicular to $T^{*^{1,0}}(bD) \oplus T^{*^{0,1}}(bD)$.

Now for any $\theta \in \mathbb{R}$, define

$$\begin{aligned} \Lambda_\theta : \overline{D} &\to \overline{D} \\ z &\mapsto e^{i\theta} \cdot z = \left(e^{i\theta} z_1, \dots, e^{i\theta} z_n\right). \end{aligned}$$

Put $\zeta = e^{i\theta} \cdot z$, then we obtain by direct computation $\frac{\partial r}{\partial z_k}(z) = e^{i\theta} \frac{\partial r}{\partial \zeta_k}(\zeta)$, $\Lambda_{\theta *}\left(\frac{\partial}{\partial z_k}\right) = e^{i\theta} \frac{\partial}{\partial \zeta_k}$ and $\Lambda_\theta^*(d\zeta_k) = e^{i\theta} dz_k$ for $1 \le k \le n$. It follows that we have

(2.1) $\Lambda_{\theta *}(X(z)) = X(\zeta)$,

(2.2) $\Lambda_{\theta *}(L_j(z)) = e^{i2\theta} L_j(\zeta)$, $\Lambda_{\theta *}\left(\overline{L}_j(z)\right) = e^{-i2\theta}\overline{L}_j(\zeta)$, for $1 \le j \le n-1$,

(2.3) $\Lambda_\theta^*\left(\overline{\partial} r(\zeta)\right) = \overline{\partial} r(z)$, $\Lambda_\theta^*(\partial r(\zeta)) = \partial r(z)$.

This implies that $\Lambda_\theta^* w_i$ is again a $(1,0)$-form in $\mathbb{C}T^*(bD)$.

Next we recall the definition of $\overline{\partial}_b$ briefly here. let $f \in C^\infty_{p,q}(bD)$, where $C^\infty_{p,q}(bD)$ denotes the space of tangential (p,q)-forms defined on the boundary with smooth coefficients. Namely, any f in $C^\infty_{p,q}(bD)$ can be expressed in the form

$$f = \sum_{\substack{|I|=p \\ |J|=q}}{}' f_{IJ} w_I \wedge \overline{w}_J,$$

where $I = (i_1, \dots, i_p)$ and $J = (j_1 \dots, j_q)$ are strictly increasing multiindices of length p and q respectively, and $w_I = w_{i_1} \wedge \cdots \wedge w_{i_p}$ and $\overline{w}_J = \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q}$, and the prime indicates that the summation is carried over only the strictly increasing multiindices. Then consider f as a (p,q)-form in some open neighbourhood U of the boundary, and apply $\overline{\partial}$ to f. We get

$$\overline{\partial} f = F + r(z)G + \overline{\partial} r \wedge H,$$

where F is a $(p, q+1)$-form involving only the w_i's and $\overline{w}_j$'s, and G is a $(p, q+1)$-form, and H is a (p,q)-form. Then the tangential Cauchy-Riemann operator $\overline{\partial}_b$ is defined to be

$$\overline{\partial}_b f = \pi_{p,q+1}\left(\overline{\partial} f\right) = F\Big|_{bD},$$

where $\pi_{p,q+1}$ maps $\overline{\partial} f$ to the restriction of F on the boundary. For the details the reader is referred to Folland and Kohn [**12**].

Now the above argument shows Λ_θ^* maps the tangential component to the tangential component and maps the normal component to the normal component. Therefore, if $f \in C_{p,q}^{\infty}(bD)$ with $1 \le q \le n-2$ we obtain

$$\begin{aligned}
\overline{\partial}\left(\Lambda_\theta^* f(\zeta)\right) &= \pi_{p,q+1} \circ \overline{\partial} \circ \Lambda_\theta^* f \\
&= \pi_{p,q+1} \circ \Lambda_\theta^* \circ \overline{\partial} f \\
&= \pi_{p,q+1} \circ \Lambda_\theta^* \left(F + r(\zeta)\, G + \overline{\partial} r \wedge H\right) \\
&= \pi_{p,q+1} \circ \left(\Lambda_\theta^* F + r(z)\Lambda_\theta^* G + \overline{\partial} r \wedge \Lambda_\theta^* H\right) \\
&= \Lambda_\theta^* F\Big|_{bD} \\
&= \Lambda_\theta^* \circ \pi_{p,q+1}\left(F + r(\zeta)\, G + \overline{\partial} r \wedge H\right) \\
&= \Lambda_\theta^* \circ \pi_{p,q+1} \circ \overline{\partial} f \\
&= \Lambda_\theta^*\left(\overline{\partial}_b f\right).
\end{aligned}$$

Hence we have proved the following lemma.

Lemma 2.4. $\overline{\partial}_b \Lambda_\theta^* f = \Lambda_\theta^* \overline{\partial}_b f$ *for any* $f \in C_{p,q}^{\infty}(bD)$ *with* $1 \le q \le n-1$.

In general, $\overline{\partial}_b \circ h^* \neq h^* \circ \overline{\partial}_b$ if h is just smooth CR- mapping. Denote by $L_{p,q}^2(bD)$ the space of tangential (p,q)-forms with square-integrable coefficients. Then we have

Lemma 2.5. *For any* u *in* $L_{p,q}^2(bD)$, *we have* $(\Lambda_\theta^* u, v) = (u, \Lambda_{-\theta}^* v)$ *for any* $\theta \in \mathbb{R}$.

Proof. Put $\zeta = e^{i\theta} \cdot z$, and express u and v in terms of the Euclidean coordinates, we get

$$u(\zeta) = \sum_{\substack{|I|=p \\ |J|=q}}{}' u_{IJ}(\zeta)\, d\zeta_I \wedge d\overline{\zeta}_J \text{ and } v(z) = \sum_{\substack{|I|=p \\ |J|=q}}{}' v_{IJ}(z) dz_I \wedge d\overline{z}_J .$$

Let $d\sigma$ be the surface element defined on bD. We see that $d\sigma$ is invariant under rotation, i.e., $\Lambda_\theta^* d\sigma_\zeta = d\sigma_z$. For instance, see Chen [**5**]. Hence if we set $z = e^{-\theta} \cdot \zeta$, we obtain

$$
\begin{aligned}
(\Lambda_\theta^* u, v) &= \left(\sum\nolimits' u_{IJ}\left(e^{i\theta}\cdot z\right) e^{i(p-q)\theta} dz_I \wedge d\bar{z}_J, \sum\nolimits' v_{IJ}(z) dz_I \wedge d\bar{z}_J\right) \\
&= \sum\nolimits' \int_{bD} u_{IJ}\left(e^{i\theta}\cdot z\right) e^{i(p-q)\theta}\overline{v_{IJ}(z)} d\sigma_z \\
&= \sum\nolimits' \int_{bD} u_{IJ}(\zeta)\cdot \overline{e^{-i(p-q)\theta} v_{IJ}\left(e^{-i\theta}\cdot\zeta\right)} d\sigma_\zeta \\
&= \left(u, \Lambda_{-\theta}^* v\right) .
\end{aligned}
$$

This completes the proof of the lemma. $\square$

Lemma 2.6. *$\bar\partial_b^* \Lambda_\theta^* \alpha = \Lambda_\theta^* \bar\partial_b^* \alpha$ for any $\alpha \in C_{p,q}^\infty(bD)$ with $1 \le q \le n-1$, where $\bar\partial_b^*$ is the L^2-adjoint of $\bar\partial_b$.*

Proof. Let β be any tangential $(p, q-1)$-form, i.e., $\beta \in C_{p,q-1}^\infty(bD)$. We have

$$
\begin{aligned}
\left(\bar\partial_b^* \Lambda_\alpha^*, \beta\right) &= \left(\Lambda_\theta^*\alpha, \bar\partial_b\beta\right) \\
&= \left(\alpha, \Lambda_{-\theta}^*\bar\partial_b\beta\right) \\
&= \left(\alpha, \bar\partial_b\Lambda_{-\theta}^*\beta\right) \\
&= \left(\bar\partial_b^*\alpha, \Lambda_{-\theta}^*\beta\right) \\
&= \left(\Lambda_\theta^*\bar\partial_b^*\alpha, \beta\right) .
\end{aligned}
$$

This proves the lemma. $\square$

Now denote by $H_{p,q} = \left\{u \in L_{p,q}^2(bD) \,\middle|\, \Box_b\, u = 0\right\}$. We have the following fact.

Proposition 2.7. (i) *$H_{p,q} = 0$ for $1 \le q \le n-2$, and*
(ii) $H_{p,n-1} = \left\{u \in L_{p,n-1}^2(bD) \,\middle|\, u \in \operatorname{Dom}\left(\bar\partial_b^*\right) \text{ and } \bar\partial_b^* u = 0\right\}$.

In general, $H_{p,n-1} \neq 0$. Now let $f \in C_{p,q}^\infty(bD)$, $f \perp H_{p,q}$, for $1 \le q \le n-1$ be given, and let $u = N_b f \in C_{p,q}^\infty(bD)$ be the canonical solution to the $\Box_b$-equation,

$$\Box_b u = \Box_b N_b f = f,$$

where N_b is the so-called boundary Neumann operator. Let T be the vector field generated by the rotation, namely, T is defined by

$$
\begin{aligned}
T(z) &= \frac{1}{2}\pi_{z^*}\left(\left.\frac{\partial}{\partial\theta}\right|_{\theta=0}\right) \\
&= \frac{i}{2}\left(\sum_{j=1}^n z_j\frac{\partial}{\partial z_j} - \sum_{j=1}^n \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right),
\end{aligned}
$$

where π_z is the mapping defined for any $z \in bD$ by

$$\begin{aligned} \pi_z : & S^1 \to \overline{D} \\ & e^{i\theta} \mapsto e^{i\theta} \cdot z = \left(e^{i\theta} z_1, \ldots, e^{i\theta} z_n\right). \end{aligned}$$

By our hypotheses stated in the Theorem 1.2, $T(z)$ is tangential and pointing in the bad direction for any $z \in bD$.

From now on, we will assume that f has real analytic coefficients, namely, $f \in C^{\omega}_{p,q}(bD)$ with $1 \leq q \leq n-1$, and that $f \perp H_{p,n-1}$ if $q = n-1$. Write f as

$$f = \sum\nolimits'_{I,J} f_{IJ}(z) \omega_I \wedge \bar{\omega}_J.$$

Define Tf by

$$Tf = \sum\nolimits'_{I,J} Tf_{IJ}(z) \omega_I \wedge \bar{\omega}_J.$$

It is not hard to see that Tf is still a tangential (p,q)-form, i.e., $Tf \in C^{\omega}_{p,q}(bD)$. Then we have the following key lemma.

Lemma 2.8. *$T^k u = T^k N_b f = N_b T^k f$ for any $k \in \mathbb{N}$.*

Proof. Since, in general, $H_{p,n-1} \neq 0$, we need to check that if $u \perp H_{p,n-1}$, then $\Lambda^*_\theta u \perp H_{p,n-1}$. So, let $w \in H_{p,n-1}$. By Lemma 2.6 we have $\Lambda^*_\theta w \in H_{p,n-1}$. It follows that

$$(\Lambda^*_\theta u, w) = (u, \Lambda^*_{-\theta} w) = 0.$$

Hence $\Lambda^*_\theta u \perp H_{p,n-1}$. This proves our assertion.

Now by combining Lemma 2.4 and 2.6, we obtain

$$\begin{aligned} \Box_b \Lambda^*_\theta N_b f &= \Lambda^*_\theta \Box_b N_b f \\ &= \Lambda^*_\theta f \\ &= \Box_b N_b \Lambda^*_\theta f. \end{aligned}$$

Therefore, by Proposition 2.7 and our assertion we conclude that

$$\Lambda^*_\theta N_b f = N_b \Lambda^*_\theta f \quad \text{for any } \theta \in \mathbb{R}. \tag{2.9}$$

So now one can argue as we did in Chen [**2**] to get $TN_b f = N_b Tf$. Inductively we have $T^k N_b f = N_b T^k f$. This completes the proof of the lemma. $\square$

Lemma 2.8 enables us to estimate the derivatives of the solution $u = N_b f$ in the bad direction as follows,

$$\left\|T^k u\right\| = \left\|T^k N_b f\right\| = \left\|N_b T^k f\right\| \leq C_0 \left\|f\right\|_k \leq C C^k k!\ ,$$

for some constant $C > 0$ and any $k \in \mathbb{N}$, where $\| \ \|_k$ is the Sobolev k-norm.

Therefore, what we need to estimate is the mixed derivatives of u, namely, the differentiations involving L_i's, $\overline{L}_i$'s and T. For dealing with the $\overline{\partial}$-Neumann problem we can avail ourselves of the so-called basic estimate to achieve this goal. However, for the $\overline{\partial}_b$- Neumann problem, in general, the energy norm Q_b does not control the barred terms. But if we add the differentiation in T-direction to the right hand side, then we do have the following estimate,

$$\|u\| + \sum_{j=1}^{n-1} \|L_j u\| + \sum_{j=1}^{n-1} \left\|\overline{L}_j u\right\| \leq C \left(\left\|\overline{\partial}_b u\right\| + \left\|\overline{\partial}_b^* u\right\| + \|Tu\| \right), \tag{2.10}$$

for any $u \in C^\omega_{p,q}(bD)$ with support in some open neighbourhood of z_0. The estimate (2.10) is essentially proved in [**12**]. Since we know how to control the T-derivatives of the solution $u = N_b f$, then a standard argument can be used to obtain the estimates of all the other mixed derivatives. For the details the reader is referred to Chen [**2**][**3**][**4**]. This completes the proof of Theorem 1.2.

A similar argument can be applied to prove the Theorem 1.3. Let D be a smoothly bounded Reinhardt pseudoconvex domain with real analytic boundary in $\mathbb{C}$, $n \geq 2$. Let $z_0 \in bD$ be a boundary point. First one can choose a direction, say z_n, such that $\left(z_n \frac{\partial r}{\partial z_n}\right)(z_0) \neq 0$, where $r(z)$ is the defining function for D. Next we simply consider the rotation in z_n-direction, namely, for each $\theta \in \mathbb{R}$, define

$$\begin{aligned} \Lambda_\theta : &\overline{D} \to \overline{D} \\ &z \mapsto e^{i\theta} \cdot z = \left(z_1, \dots, z_{n-1}, e^{i\theta} z_n\right). \end{aligned}$$

Then by following the proof we present here for circular domains we can show without difficulty that $\Box_b$ is globally analytically hypoelliptic on any smoothly bounded Reinhardt pseudoconvex domain with real analytic boundary . Details can be found in Chen [**3**]. This also completes the proof of the Theorem 1.3.

Finaly we make a concluding remark that the method we present here can be used to obtain the Sobolev H^s-regularity for $\Box_b$ on any smoothly bounded pseudoconvex domain which is either Reinhard or circular with $\displaystyle\sum_{j=1}^{n} z_j \frac{\partial r}{\partial z_j} \neq 0$ near bD, where $r(z)$ is a smooth defining function for D. For instance, see Chen [**4**].

References

[1] H. Boas and M.C. Shaw, *Sobolev estimates for the Lewy operator on weakly pseudoconvex boundaries*, Math. Ann., **274** (1986), 221-231.

[2] S.C. Chen, *Global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem on circular domains*, Invent. Math., **92** (1988), 173-185.

[3] ———, *Global real analyticity of solutions to the $\bar{\partial}$-Neumann problem on Reinhardt domains*, Indiana Univ. Math. J., **37(2)** (1988), 421-430.

[4] ———, *Global regularity of the $\bar{\partial}$-Neumann problem on circular domains*, Math. Ann., **285** (1989), 1-12.

[5] ———, *Real analytic regularity of the Szegö projection on circular domains*, Pacific J. Math., **148(2)** (1991), 225-235.

[6] ———, *Real analytic regularity of the Bergman and Szegö projections on decoupled domains*, Math. Zeit., **213** (1993), 491-508.

[7] M. Christ and D. Geller, *Counterexamples to analytic hypoellipticity for domains of finite type*, Ann. of Math., **135** (1992), 551-566.

[8] M. Derridj and D.S. Tartakoff, *Local analyticity for $\Box_b$ and the $\bar{\partial}$-Neumann problem at certain weakly pseudoconvex points*, Commun. in P.D.E., **13(12)** (1988), 1521-1600.

[9] ———, *Local analyticity for the $\bar{\partial}$-Neumann problem and $\Box_b$ - some model domains without maximal estimates*, Duke Math. J., **64(2)** (1991), 377-402.

[10] ———, *Microlocal analyticity for $\Box_b$ in block-decoupled pseudoconvex domains*, Preprint.

[11] ——— *Global analyticity for $\Box_b$ on three dimensional pseudoconvex domains*, Preprint.

[12] G.B. Folland and J.J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. Math. Studies, **75** Princeton, Princeton University Press 1972.

[13] D. Geller, *Analytic pseudodifferential operators for the Heisenberg group and local solvability*, Math. Notes, **37** Princeton, Princeton University Press 1990.

[14] J.J. Kohn, *Estimates for $\bar{\partial}_b$ on pseudoconvex CR manifolds*, Proc. Sympos. Pure Math., **43** Amer. Math. Soc., Providence, R.I. (1985).

[15] ———, *The range of the tangential Cauchy-Riemann operator*, Duke Math. J., **53(2)** (1986), 525-545.

[16] ———, *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions*, Acta Math., **142** (1979), 79-122.

[17] M.C. Shaw, *L^2 estimates and existence theorems for the tangential Cauchy-Riemann complex*, Invent. Math., **82** (1985), 133-150.

[18] D.S. Tartakoff, *On the global analyticity of solutions to $\Box_b$ on compact manifolds*, Commun. in P.D.E., **1** (1976), 283-311.

[19] ———, *Local analytic hypoellipticity for $\Box_b$ on non-degenerate Cauchy-Riemann manifolds*, Proc. Natl. Acad. Sci. USA, **75(7)** (1978), 3027-3028.

[20] ———, *On the local real analyticity of solutions to $\Box_b$ and the $\bar{\partial}$-Neumann problem*, Acta Math., **145** (1980), 177-204.

[21] F. Treves, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to the $\bar{\partial}$-Neumann problem*, Commun. in

P.D.E., **3** (1978), 475-642.

Received November 10, 1993 and revised July 20, 1994.

NATIONAL TSING HUA UNIVERSITY
HSINCHU 30043
TAIWAN, R.O.C.
E-mail address: scchen@am.nthu.edu.tw

Added in proof: M.Christ has recently proved the following, M.Christ, The Szegö projection need not preserve global analyticity, Annals of Math., **143** (1996), 301-330.

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by

E. F. Beckenbach (1906-1982) F. Wolf (1904-1989)

Manuscripts must be prepared in accordance with the instructions provided on the inside back cover.
The table of contents and the abstracts of the papers in the current issue, as well as other information about the Pacific Journal of Mathematics, may be found on the Internet at http://www.math.uci.edu/pjm.html.

The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: $245.00 a year (10 issues). Special rate: $123.00 a year to individual members of supporting institutions.
Subscriptions, back issues published within the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at the University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 6143, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,
Berkeley, CA 94720, A NON-PROFIT CORPORATION
This publication was typeset using AMS-LATEX,
the American Mathematical Society's TEX macro system.

PACIFIC JOURNAL OF MATHEMATICS

Volume 175 No. 1 September 1996
