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ALAN L. T. PATERSON

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We develop higher dimensional amenability for Banach algebras from the viewpoint of Banach homology theory. In particular, we show that such amenability is equivalent to the flatness of a certain bimodule and a resultant splitting module map gives rise to the higher dimensional virtual diagonals of Effros and Kishimoto. The theory is developed for the non-unital case. Examples of n-amenability are given and it is shown (among other results) that a 2-amenable Banach algebra is amenable if and only if there exists an *inner* 2-virtual diagonal.

1. Introduction.

As observed by Effros and Kishimoto ([**3**]), the problem of "higher cohomological dimension" is one of the most intriguing questions in Functional analysis. The question goes back at least as far as [**15**, 10.10, p. 92]. In dimension 1, there is an important and well-developed theory of *amenable* Banach algebras ([**15**, **11**, **17**, **21**]). A Banach algebra A is called amenable if $H^1(A, X^*) = 0$ for every Banach A-module X. The name is so-called since if G is a locally compact group, then $L^1(G)$ is amenable if and only if G is amenable ([**15**, Theorem 2.5]). For a C^*-algebra, amenability coincides with nuclearity ([**7**]).

For higher dimensions, let us define, for any $n \geq 1$, a Banach algebra A to be *n-amenable*[1] if $H^n(A, X^*) = 0$ for every Banach A-module X. It is well-known that $(n-1)$-amenable implies n-amenable.

As Helemskii notices ([**11**, p. 286]), there are Banach algebras A which are n-amenable for some $n > 1$ but not amenable. These include the biprojective Banach algebras with a one-sided identity but not a two-sided identity. Examples of these are given in [**24**], and we will look in detail at one of these which is two-dimensional. For such algebras, Helemskii has shown that $H^3(A, X) = 0$ for any A-bimodule X. Roger Smith in unpublished work has shown that there are matrix algebras B_n where B_n is $(n+1)$-amenable but

[1]This terminology has been used by B. E. Johnson in a different sense, but for lack of a better alternative, we will use it as above.

not n-amenable. It is shown that the algebra T_2 of upper triangular 2×2 complex matrices is 2-amenable but not amenable[2].

Of particular interest are the questions: Does there exist a non-amenable C^*-algebra which is n-amenable for some n? Does there exist a non-amenable group G such that $L^1(G)$ is n-amenable for some n? The present writer has been unable to solve either of these problems though we note that by the preceding paragraph, the answer to the first is positive if A is allowed to be a non-self-adjoint algebra. Under these circumstances, it seemed natural (as in [**3**]) to study the Banach algebra case.

B. E. Johnson showed ([**14**]) that a Banach algebra A is amenable if and only if there exists $M \in (A\hat{\otimes}A)^{**}$ such that for all $a \in A$,

$$aM = Ma, \qquad \pi^{**}(M)a = a. \tag{1}$$

(Here, π is the product map on A and in the right-hand side of the last equality, we conveniently use a rather than $\hat{a} \in A^{**}$.) Obviously, when A contains an identity element e, this last equality can be replaced by: $\pi^{**}(M) = e$.

Such an element M is called a *virtual diagonal* and is very useful in the theory of amenable Banach algebras. Effros and Kishimoto showed that for a Banach algebra A with unit e, n-amenability is characterized by the existence of a higher-dimensional version of virtual diagonal which we will call an *n-virtual diagonal.* To define this, for any $r \geq 1$, let $C_r(A) = A\hat{\otimes}A\hat{\otimes}\cdots\hat{\otimes}A$ (r copies of A). Let $\pi_{n+1} : C_{n+1}(A) \to C_n(A)$ be the generalized product map: If $z = a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \in C_{n+1}(A)$, then

$$\pi_{n+1}(z) = \sum_{r=1}^{n} (-1)^{r+1} a_1 \otimes a_2 \otimes \cdots \otimes a_{r-1} \otimes a_r a_{r+1} \otimes \cdots \otimes a_{n+1}. \tag{2}$$

Then an n-virtual diagonal is a cocycle $D : C_{n-1}(A) \to C_{n+1}(A)^{**}$ such that for $a_1, \ldots, a_{n-1} \in A$,

$$\pi_{n+1}^{**}(D(a_1 \otimes \cdots \otimes a_{n-1})) = \pi_{n+1}(e \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes e).$$

(Effros and Kishimoto also obtained a fixed-point characterization of n-amenability – this will not be considered in the present paper.)

The main focus of this paper is to understand better the significance of the n-virtual diagonal conditions. For example, a natural question that arises is: is there an n-virtual diagonal characterization of n-amenability in the

[2]Roger Smith and the author have recently shown that this result is true for the algebra T_n of upper triangular $n \times n$ complex matrices. Y. V. Selivanov has informed the author that he has also proved this result using homological techniques.

non-unital case. This is not a problem for ordinary amenability since if A is amenable, it has a bounded approximate identity (bai) and it is essentially for that reason that the virtual diagonal definition above does not involve a unit. The situation is very different for higher dimensional amenability: There are (2-dimensional!) 2-amenable Banach algebras without a bai (§3). There is a natural way to obtain a version of n-virtual diagonal intrinsically in terms of A – in §4, such a map will be called an *intrinsic* n-virtual diagonal. It seems likely that in the presence of a bai, n-amenability is equivalent to the existence of an intrinsic n-virtual diagonal but we have only been able to prove this in one direction (Theorem 4.1).

To deal with the question of the preceding paragraph as well as to clarify the significance of n-virtual diagonals, we will use the approach to Banach cohomology as a relative homology theory (in the sense of Eilenberg and Moore [**4**]). This approach has been extensively developed in the work of A. Ya. Helemskii and others and an invaluable source for the theory is the book [**11**] by Professor Helemskii.[3] Non-unital Banach algebras are not a problem in Helemskii's approach to amenability in terms of Banach cohomology since it is developed in terms of the unitization of an algebra. (As we shall see, the approach readily suggests how to approach n-virtual diagonals in the non-unital case.)

In §2, we sketch briefly the elements of this theory that we will need in the sequel. All of this section (and much more) is contained explicitly or implicitly in [**11**] though sometimes we have presented the material slightly differently. It is hoped that the sketch will be helpful to readers who may be unfamiliar with the theory. It is suggested that readers who know the theory start with §3 and refer to §2 whenever necessary.

The theory develops homology for left Banach A-modules, as in ordinary homology, and uses Banach space versions of the fundamental kinds of module such as free, projective, injective and flat. In Banach homology, the (projective) resolutions used are those that split in the Banach category. The requirement for splitting usually means that we need to use in resolutions A_+ – the Banach algebra A with identity adjoined – rather than A. (Of course, if A has a unit to start with, then we can stay with A.)

As in ordinary homology, one can develop derived functors – in particular Ext – in the theory using projective resolutions or injective coresolutions. To cope with two-sided modules the enveloping algebra $A^{\mathrm{e}} = A_+ \hat{\otimes} A_+^{\mathrm{op}}$ is used, and for any Banach A-module X, we have $H^n(A, X) = \mathrm{Ext}^n_{A^{\mathrm{e}}}(A_+, X)$.

Amenability is formulated in terms of the flatness of A_+ (over A^{e}) or

[3]The author is grateful to Garth Dales and Niels Grønbæk for bringing Professor Helemskii's book to his attention.

equivalently the injectivity of A_+^*. An indication of the relevance of Banach homology to n-amenability is that the natural projective resolution for A_+ (which of course can be used to compute cohomology) is given by the $C_n(A_+)$'s with the π_n-maps as given above. (When A is unital we can replace $C_n(A_+)$ by $C_n(A)$ and the resulting π_{n+1} is exactly the π_{n+1} that occurs in the Effros-Kishimoto n-virtual diagonal. This indicates that n-virtual diagonals should fit naturally into Banach homology theory.)

Helemskii also considers another projective resolution for A_+ "closer" to A in which the $C_n(A_+)$ are replaced by

$$D_n(A) = A_+ \hat{\otimes} A \cdots \hat{\otimes} A \hat{\otimes} A_+$$

($(n-2)$ copies of A), the π_n's having the same formula as before. This gives us a clue for defining n-virtual diagonals in the non-unital case: Such a diagonal D is defined in the same way as in the unital case except that the range of D is in $D_{n+1}(A)^{**}$ rather than $C_{n+1}(A)^{**}$.

In view of Helemskii's flatness characterization of amenability, it is natural to look for a module whose flatness will be equivalent to n-amenability for A. We will show in §3 (using straightforward homology arguments) that *A is n-amenable if and only if $K_n = \ker \pi_n$ is flat.*

Since an A-bimodule is flat if and only if its dual is injective, we would expect splitting properties at the dual level for n-amenable algebras. Indeed the following theorem ([**11**, p. 256]) holds: A is amenable if and only if A has a bai and the dual π^* of the product map $\pi : A\hat{\otimes}A \to A$ is a coretraction. An alternative formulation ([**2**]) states that when A has a bai, then A is amenable if and only if the short exact sequence of A-bimodules

$$0 \to A^* \xrightarrow{\pi^*} (A\hat{\otimes}A)^* \to K^* \to 0 \tag{3}$$

where $K = \ker \pi$ splits as an A-bimodule sequence.

In the n-amenable context, we can no longer expect such a formulation to involve a bai. However we will show that n-amenability for A is equivalent to the splitting of the sequence

$$0 \to K_n^* \to D_{n+1}(A)^* \to K_{n+1}^* \to 0. \tag{4}$$

More precisely, there is an A^e module map $\rho : D_{n+1}(A)^* \to K_n^*$ which is a left inverse for π_{n+1}^*.

The main theme of the paper is that an n-virtual diagonal is essentially a formulation of this splitting at the dual level. In the amenable case, this is clear in [**11**, p. 257] and [**2**]. In that case the relation between ρ and a virtual diagonal is easy to express. In higher dimensions, there are technical difficulties which are addressed in the proof of Theorem 3.2.

The paper concludes with discussing the question of when n-amenable implies $(n-1)$-amenable. The author hopes that this will be helpful in investigating the question of whether or not n-amenability is equivalent to amenability for a C^*-algebra or a group algebra. What we would like to show is that an n-amenable Banach algebra is $(n-1)$-amenable if and only if there exists an n-virtual diagonal *which is a coboundary.* We have only been able to show this when $n = 2, 3$ (Corollary 4.1)[4].

The author is grateful to Glenn Hopkins for advice on homology theory and to Professor Helemskii for invaluable help with Banach homology. He is particularly grateful to Roger Smith for permission to include the example of a 2-virtual diagonal for the group algebra of a discrete amenable group and for showing him an unpublished result exhibiting n-amenable algebras which are not $(n-1)$-amenable. Finally the author is grateful to Ed Effros for helpful discussions on higher dimensional cohomology.

2. Some Banach homological algebra.

In this section, we sketch some of the background and results in Banach homology theory that we will need in the next section. The reader is referred to Helemskii's book [**11**] for details. The papers [**8**, **9**, **10**] are also helpful.

Let A be a Banach algebra. A Banach space X which is a left A-module is called a *left Banach A-module* if the product map $\pi : A\hat{\otimes}X \to X$ is continuous (or equivalently, if there exists $M \geq 0$ such that $\|ax\| \leq M\|a\|\|x\|$ for all $a \in A$ and all $x \in X$). For two such modules X and Y, a *morphism* from X into Y is an element $T \in B(X,Y)$ which is a module map, i.e. is such that $T(ax) = aT(x)$ for all $x \in X$. The space of such morphisms ${}_AB(X,Y)$[5] is a closed subspace of $B(X,Y)$ and so is a Banach space. The resultant category of left Banach A-modules is denoted by ${}_A\mathcal{M}$.

We note that ${}_AB(X,Y)$ above is itself in ${}_A\mathcal{M}$ where for $T \in {}_AB(X,Y)$ and $a \in A$, $aT(x) = aT(x)$.

Similarly, we define the categories of *right Banach A-modules* and *(two-sided) Banach A-modules.* These are denoted respectively by $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$. In the case where $X, Y \in \mathcal{M}_A$, the space of morphisms $T : X \to Y$ is denoted by $B_A(X,Y)$. A two-sided Banach A-module is usually called an *A-bimodule.* The dual of a left Banach A-module is a right Banach A-module under the action: $fa(x) = f(ax)$ for all $f \in X^*, a \in A$ and $x \in X$.

In the right Banach module case, the dual is a left Banach module under the action: $af(x) = f(xa)$. Of course, the dual of a Banach A-bimodule is also a Banach A-bimodule.

[4]The general case has recently been settled by Roger Smith and the present author.
[5]Helemskii [**11**, p. 46] uses the notation ${}_Ah(X,Y)$ where we use ${}_AB(X,Y)$.

Our primary interest is in Banach A-bimodules and it is very convenient to reduce their study to that of left modules by using the enveloping algebra of A. (This parallels the use of such an algebra in homological algebra (cf. [**1**]).)

The Banach algebra A_+ is the algebra A with identity adjoined. So the elements of A_+ are formally sums of the form $a + \lambda 1$ with $a \in A$ and $\lambda \in C$, and $\|a + \lambda 1\| = \|a\| + |\lambda|$. The algebra A_+ plays a fundamental role in Banach algebra homology.

The enveloping algebra of A is defined as follows. Let A^{op} be the Banach algebra A with the multiplication reversed. Any $X \in {}_A\mathcal{M}$ is a right A^{op}-module X^{op} in the obvious way: $x.a = ax$ for $x \in X, a \in A^{\mathrm{op}}$.

We define

$$A^{\mathrm{e}} = A_+ \hat{\otimes} A_+^{\mathrm{op}}. \tag{5}$$

A Banach A-bimodule X then becomes a *left* Banach A^{e}-module by setting $(a \otimes b)x = axb$. The converse obviously holds. (Our main interest is in A-bimodules, but it is very convenient in the development of the theory to treat them as *left* modules, not over A but over A^{e}.)

A (chain) *complex* $\mathcal{K}$ in ${}_A\mathcal{M}$ is a sequence

$$\cdots \overset{\Phi_{m-1}}{\leftarrow} X_m \overset{\Phi_m}{\leftarrow} X_{m+1} \overset{\Phi_{m+1}}{\leftarrow} \cdots \tag{6}$$

where $\Phi_m \in {}_A B(X_{m+1}, X_m)$ is such that $\mathrm{Im}\Phi_m \subset \ker \Phi_{m-1}$. The complex is said to be *exact* if $\ker \Phi_{m+1} = \mathrm{Im}\Phi_m$ for all m. An application of the Hahn-Banach theorem ([**11**, p. 50]) shows that the complex (6) is exact if and only if the dual (cochain) complex in $\mathcal{M}_A$

$$\cdots \to X_m^* \overset{\Phi_m^*}{\to} X_{m+1}^* \overset{\Phi_{m+1}^*}{\to} \cdots$$

is exact.

Recall that a closed subspace W of a Banach space X is *complemented* in X if there exists a closed subspace Y of X such that $W + Y = X$ and $W \cap Y = \{0\}$. Of course, associated with such a Y is the natural (continuous) projection P from X onto W. Conversely, the existence of a continous projection P from X onto W is equivalent to W being complemented in X.

The complex (6) is called *admissible* if it is exact and the kernel of each Φ_m is complemented in X_{m+1}. If X and Y are Banach spaces and $T \in B(X, Y)$, then we say that T is *admissible* if $\ker T$ is complemented in X and $\mathrm{Im}T$ is closed and complemented in Y. (We are interested in the splitting of complexes in ${}_A\mathcal{M}$ – see below – and admissibility ensures that there is no purely Banach space obstruction to this.)

The notions of *retraction* and *coretraction* play an important role in the theory. Let $X, Y \in {}_A\mathcal{M}$, $T \in_A B(X,Y)$ and I_X be the identity map on X. The map T is called a retraction if there exists a morphism $\rho \in_A B(Y,X)$ such that

$$T \circ \rho = I_Y.$$

(So ρ is a right inverse for T.) Of course, every retraction is surjective. Similarly, T is a coretraction if there exists a morphism $\tau \in_A B(Y,X)$ such that $\tau \circ T = I_X$ (so that τ is a left inverse for T). Of course, the notions of retraction and coretraction can be defined in very general categories – in particular, in the category of Banach spaces.

A short exact sequence

$$0 \leftarrow X \overset{\phi}{\leftarrow} Y \overset{\psi}{\leftarrow} Z \leftarrow 0 \tag{7}$$

in ${}_A\mathcal{M}$ is said to *split* if ϕ is a retraction. In this case, Y is the direct sum of the closed submodules $\ker\phi$ and $\rho(X)$ where ρ is a right inverse for ϕ. The resulting projection $y \to y - \rho \circ \phi(y)$ of Y onto $\ker\phi = \operatorname{Im}\psi \cong Z$ gives that ψ is a coretraction. The latter property for ψ is equivalent to the splitting of (7).

As in the usual homology theory, a fundamental role is played by three kinds of modules: *Projective*, *injective* and *flat modules.* We will briefly discuss these in turn.

Let $P \in {}_A\mathcal{M}$. The module P is called *projective* if, whenever $X, Y \in {}_A\mathcal{M}$, $T \in_A B(X,Y)$ is both surjective and admissible and $S \in_A B(P,Y)$, then there exists $R \in_A B(P,X)$ such that $S = T \circ R$. (This is the familiar definition of projective ([**13**, p. 24]) with the extra requirement of admissibility.) It is straightforward to show that P is projective if and only if the functor ${}_AB(P,.)$ is exact, i.e. given an admissible complex (6), the Banach space complex

$$\cdots \overset{\Phi_{m-1*}}{\longleftarrow} {}_A B(P, X_m) \overset{\Phi_{m*}}{\longleftarrow} {}_A B(P, X_{m+1}) \leftarrow \cdots \tag{8}$$

under the natural maps Φ_{m_*} is exact.

The simplest examples of projective modules are (as in the usual theory) the free ones, where a *free* A-module is one of the form $A_+ \hat{\otimes} E$ where E is a Banach space. The latter module is given the natural left multiplication of $A \subset A_+$. Two comments are in order here.

Firstly, in the ordinary theory, an A-module Z is free if it is ([**23**, p. 57]) a sum of copies of A. This can be expressed in our context by saying that Z is of the form $A \otimes C^N$ for some cardinal N. Because we are working with Banach spaces it is reasonable to replace the C^N by a Banach space E and close up in the projective tensor product norm.

Secondly, the reason why we have to replace A by A_+ in defining Z is essentially that A_+ is freely generated as an A_+ module by 1. To illustrate the significance of this, we briefly sketch the proof that $P = A_+\hat{\otimes}E$ is projective ([**11**, p. 138]). Let $T \in_A B(X,Y)$ be an admissible epimorphism and $S \in_A B(P,Y)$. Then there exists $\rho \in B(Y,X)$ such that $T \circ \rho = I_Y$. The required morphism $R : P \to X$ is simply defined by setting $R((a+\lambda 1) \otimes e) = (a + \lambda 1)\rho(S(1 \otimes e))$, where $a \in A$, $\lambda \in C$, $e \in E$ and 1 acts on X as the identity.

As in the usual homology theory, a module is projective if and only if it is direct summand (in ${}_A\mathcal{M}$) of a free module.

We now discuss the functor Ext_A which plays a fundamental role in the theory.

A *projective resolution* for $X \in {}_A\mathcal{M}$ is an admissible complex of the form

$$0 \leftarrow X \stackrel{\epsilon}{\leftarrow} P_0 \stackrel{\Phi_0}{\leftarrow} P_1 \stackrel{\Phi_1}{\leftarrow} P_2 \leftarrow \cdots \tag{9}$$

with every P_m projective. (As we shall see below, there always is such a resolution.) For $Y \in {}_A\mathcal{M}$, we "miss out" (as in the usual homology theory [**13**, p. 131]) the "X" term to define the ${}_A\mathcal{M}$-complex of Banach spaces

$$0 \to {}_AB(P_0, Y) \stackrel{\Phi_0^*}{\to} {}_AB(P_1, Y) \stackrel{\Phi_1^*}{\to} {}_AB(P_2, Y) \to \cdots \tag{10}$$

with the natural bounded linear maps Φ_m^*.

The Ext-groups $\mathrm{Ext}_A^m(X,Y)$ are then the homology groups of the complex (10): Explicitly, $\mathrm{Ext}_A^m(X,Y) = \ker \Phi_m^*/\mathrm{Im}\Phi_{m-1}^*$ for $m \geq 0$. It is easy to show that $\mathrm{Ext}_A^0(X,Y) = {}_AB(X,Y)$. The group $\mathrm{Ext}_A^1(X,Y)$ is often called $\mathrm{Ext}_A(X,Y)$. As in usual homology, the Ext-groups don't depend on the choice of projective resolution ([**11**, p. 151]).

There always is a projective resolution for any $X \in {}_A\mathcal{M}$. Indeed, the standard one – a free one – is given by ([**11**, p. 145]):

$$0 \leftarrow X \stackrel{\partial_1}{\leftarrow} A_+\hat{\otimes}X \stackrel{\partial_2}{\leftarrow} A_+\hat{\otimes}A\hat{\otimes}X \stackrel{\partial_3}{\leftarrow} A_+\hat{\otimes}A\hat{\otimes}A\hat{\otimes}X \leftarrow \cdots \tag{11}$$

where the morphisms ∂_{n+1} $(n \geq 0)$ are defined:

$$\begin{aligned} &\partial_{n+1}(\alpha_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} \otimes x) = \\ &\alpha_1 a_2 \otimes a_3 \otimes \cdots a_{n+1} \otimes x - \alpha_1 \otimes a_2 a_3 \otimes a_4 \otimes \cdots a_{n+1} \otimes x + \\ &\cdots + (-1)^n \alpha_1 \otimes a_2 \otimes a_3 \otimes \cdots \otimes a_{n+1} x. \end{aligned} \tag{12}$$

Of course, ∂_1 is just the product map.

We note that in each of the modules in (11), A's appear between the initial A_+ and the final X. We could also have obtained a projective resolution using A_+'s everywhere (apart from the X at the end). In relating Ext to the H^n-groups below, it turns out that (11) is easier to deal with. This illustrates the usefulness of being able to choose whatever projective resolution is most convenient in the calculation of Ext.

In the case where A is unital, we can replace A_+ by A in the above sequence to get a projective resolution. In this case, when $X = A$, the map ∂_{n+1} coincides with the map π_{n+2} of Effros and Kishimoto in [**3**].

It is straightforward to show that (11) is admissible. (Compare the discussion below in the case when $X = A_+$.)

Of course, as for any cofunctor that is *derived* in the appropriate categorical sense, Ext_A has the *long exact sequence* property: If

$$0 \leftarrow X'' \leftarrow X \leftarrow X' \leftarrow 0 \tag{13}$$

is an admissible sequence in ${}_A\mathcal{M}$ then for any $Y \in {}_A\mathcal{M}$, there is a long exact sequence ([**11**, p. 153])

$$\begin{aligned}&0 \to \mathrm{Ext}^0_A(X'', Y) \to \mathrm{Ext}^0_A(X, Y) \to \ldots \to \mathrm{Ext}^n_A(X'', Y) \to \mathrm{Ext}^n_A(X, Y)\\ &\to \mathrm{Ext}^n_A(X', Y) \to \mathrm{Ext}^{n+1}_A(X'', Y) \to \ldots .\end{aligned} \tag{14}$$

When $X, Y \in {}_A\mathcal{M}_A$, the following equality (cf [**11**, p. 156]) will be useful later:

$$\mathrm{Ext}_{A^e}(X, Y^*) = \mathrm{Ext}_{A^e}(Y, X^*). \tag{15}$$

Of particular importance is the case where $X = A_+$. Then $\mathrm{Ext}^n_{A^e}(A_+, Y)$ is the *n-dimensional cohomology group of A with coefficients in $Y \in {}_A\mathcal{M}_A$* and is denoted by $H^n(A, Y)$.

Historically (for example, in [**16**]) the groups $H^n(A, Y)$ were realized as the cohomology groups of the *standard cohomology complex*

$$0 \to C^0(A, Y) \xrightarrow{\delta^0} C^1(A, Y) \xrightarrow{\delta^1} C^2(A, Y) \to \ldots \tag{16}$$

where $C^n(A, Y)$ is the Banach space of bounded linear maps

$$T : A \hat{\otimes} A \hat{\otimes} \cdots \hat{\otimes} A \to Y \ (n \text{ copies of } A)$$

(or equivalently, the space of bounded n-linear maps from $A \times \ldots \times A$ into Y). Such maps are called *n-cochains*. The space $C^0(A, Y)$ is defined to be Y. Further the morphisms

$$\delta^n : C^n(A, Y) \to C^{n+1}(A, Y)$$

are given by:

$$\delta^n f(a_1,\ldots,a_{n+1}) =$$
$$a_1 f(a_2,\ldots,a_{n+1}) + \sum_{k=1}^{n}(-1)^k f(a_1,\ldots,a_{k-1},a_k a_{k+1},a_{k+2},$$
$$\ldots,a_{n+1}) + (-1)^{n+1} f(a_1,\ldots,a_n)a_{n+1}. \tag{17}$$

The elements of $\ker\delta^n$ are called *n-cocyles* and the elements of Im δ^{n-1} are called the *n-coboundaries.* The nth cohomology group of (16) is thus the quotient of the group of n-cocycles by the subgroup of n-coboundaries. As is normal, we will often omit the n in δ^n when it is clear which n is intended.

For any Banach algebra B, let $C_n(B) = B\hat{\otimes}B\hat{\otimes}\cdots\hat{\otimes}B$ (n copies of B) and $D_n(B)$ be the space $B_+\hat{\otimes}B\hat{\otimes}\ldots\hat{\otimes}B\hat{\otimes}B_+$ ($(n-2)$ copies of B).

To see the connection between the cohomology groups of (16) and the groups $H^n(A,Y) = \mathrm{Ext}^n_{A^e}(A_+,Y)$, we consider the resolution (11) with A_+ in place of X:

$$0\leftarrow A^+ \xleftarrow{\partial_1} D_2(A) \xleftarrow{\partial_2} D_3(A) \leftarrow \cdots . \tag{18}$$

This resolution is projective for the algebra A^e ([**11**]). (One can also prove this using the map G of (24).)

Now $T \in {}_{A^e}B(D_n,Y)$ is, via the module map property, determined by its values as a bounded linear map on $1\hat{\otimes}C_{n-2}(A)\hat{\otimes}1$ and when the latter is identified with $C_{n-2}(A)$, the resultant sequence reduces to that of (16) thus identifying the respective cohomology groups. (In this connection, see [**11**, p. 155].)

In preparation for the rest of the paper, we look more closely at the sequence (18).

Firstly in order to relate the notation to that of [**3**], we will set $\partial_n = \pi_{n+1}$ for $n \geq 1$. Then π_2 is the product map π and π_3 is given by:

$$\pi_3(\alpha\otimes b\otimes\gamma) = \alpha b\otimes\gamma - \alpha\otimes b\gamma = \alpha(b\otimes 1 - 1\otimes b)\gamma. \tag{19}$$

The map $\pi_n : D_n \to D_{n-1}$ for general n is given by:

$$\pi_n(\alpha_1\otimes a_2\cdots\otimes a_{n-1}\otimes\alpha_n) =$$
$$\alpha_1 a_2\otimes a_3\otimes\cdots\alpha_n$$
$$-\alpha_1\otimes a_2 a_3\otimes\cdots\alpha_n + \cdots + (-1)^n\alpha_1\otimes a_2\otimes\cdots a_{n-1}\alpha_n. \tag{20}$$

For the following details, cf. [**11**, p. 145]. It is obvious that for $1 < r < n$, we have

$$\pi_n(\alpha_1\otimes a_2\otimes\cdots\otimes a_{n-1}\otimes\alpha_n) =$$
$$\pi_r(\alpha_1\otimes\cdots\otimes a_r)\otimes a_{r+1}\otimes\cdots\otimes\alpha_n +$$
$$(-1)^{r+1}\alpha_1\otimes\cdots\otimes a_{r-1}\otimes\pi_{n-r+1}(a_r\otimes\cdots\otimes\alpha_n)$$

which we will abbreviate to:

$$\pi_n = \pi_r \otimes I + (-1)^{r+1} I \otimes \pi_{n-r+1}. \tag{21}$$

Throughout the paper, K_n, C_n and D_n will stand for $\ker \pi_n$, $C_n(A)$ and $D_n(A)$ respectively. We now discuss a simple complementation result.

Let $E_n = A\otimes \cdots \otimes A \otimes A_+$ ($(n-1)$ copies of A). The map T, where $Tx = 1\otimes x$ is clearly a norm continuous linear map from E_n into D_n. Let $Tx = \sum_{i=1}^N \alpha_1^i \otimes a_2^i \otimes \ldots \otimes a_{n-1}^i \otimes \alpha_n^i$ with $\alpha_1^i, \alpha_n^i \in A_+$ and $a_j^i \in A$. Let $\alpha_j^i = b_j^i + \lambda_j^i 1$ with $b_j^i \in A$ and $\lambda_j^i \in C$. Then $x = \sum_{i=1}^N \lambda_1^i 1 \otimes a_2^i \otimes \ldots \otimes a_{n-1}^i \otimes \alpha_n^i \in E_n$, and

$$\begin{aligned}\sum_{i=1}^N \|\alpha_1^i\| \ldots \|\alpha_n^i\| &\geq \|1\| \sum_{i=1}^N \mid \lambda_1^i \mid \|a_2^i\| \ldots \|\alpha_n^i\| \\ &\geq \|1\| \|x\|.\end{aligned}$$

It follows that $\|Tx\| \geq \|1\| \|x\|$. It also follows that T extends to a bicontinuous linear map from $C_{n-1}\hat{\otimes} A_+$ onto a closed subspace of D_n which we shall conveniently call $1\hat{\otimes} C_{n-2} \hat{\otimes} A_+$.

A similar simple argument shows that D_n is the Banach space direct sum:

$$D_n = (C_{n-1}\hat{\otimes} A_+) \oplus (1 \hat{\otimes} C_{n-2} \hat{\otimes} A_+) \tag{22}$$

with associated natural projection maps. We also note here – and this is useful in connection with Definition 3.1 – that the map $w \to 1 \otimes w \otimes 1$ from $A\otimes \cdots \otimes A$ ($(n-1)$ copies of A) into D_{n+1} extends to a Banach space isomorphism from C_{n-1} onto a closed subspace of D_{n+1}. The image of $w \in C_{n-1}$ under this map will be denoted by $1 \otimes w \otimes 1$.

Let $Q : A_+ \to A$ be the linear map which is the identity on A and is zero on the unit 1. Let $Q' = Q \otimes I : D_n \to D_n$ and for $u \in D_n$ let

$$u' = Q'(u). \tag{23}$$

So Q is the natural projection from D_n onto $C_{n-1}\hat{\otimes} A_+$: Note that π_n maps $C_{n-1}\hat{\otimes} A_+$ into $C_{n-2}\hat{\otimes} A_+$.

Define $G : D_n \to D_n$ by:

$$G(u) = u' - 1 \otimes \pi_n(u').$$

Then

$$\pi_n(G(u)) = \pi_n(u') - \pi_n(u') + 1 \otimes \pi_{n-1}(\pi_n(u')) = 0$$

so that $G(u) \in K_n$. Let $k \in K_n$. By (22), we can write $k = k' + 1 \otimes v$ where $v \in C_{n-2}\hat{\otimes}A_+$. Then $0 = \pi_n(k) = \pi_n(k') + v - 1 \otimes \pi_{n-1}(v)$ giving (after applying the D_{n-1} version of Q') that

$$v + \pi_n(k') = 0, \qquad k = G(k). \tag{24}$$

So G is a Banach space retraction onto K_n.

A module $J \in {}_A\mathcal{M}$ is called *injective* ([**11**, p. 136]) if whenever $X, Y \in {}_A\mathcal{M}$, $\rho \in {}_AB(X, Y)$ is an admissible monomorphism and $\phi \in {}_AB(X, J)$, then there exists $\psi \in {}_AB(Y, J)$ such that $\psi \circ \rho = \phi$. Equivalently, J is injective if and only if, whenever

$$0 \to X_1 \to X_2 \to X_3 \to \ldots \tag{25}$$

is an admissible sequence, then the associated Banach space complex

$$0 \leftarrow {}_AB(X_1, J) \to {}_AB(X_2, J) \to {}_AB(X_3, J) \to \ldots \tag{26}$$

is exact, i.e. ${}_AB(., J)$ is an exact functor. It is easily proved from the definition that if E is injective and $J \in {}_A\mathcal{M}$ is a direct module summand of E (i.e. if there is a retraction from E onto J) then J is injective.

Injectivity in the categories $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ are defined in the obvious ways.

Any space of the form $B(A_+, E)$, where E is a Banach space is in ${}_A\mathcal{M}$, where the module action is given by: $(aT)(\alpha) = T(\alpha a)$. These are the *cofree* modules ([**9**, p. 212]), and are easily shown to be injective.

As in the usual homology theory, the Ext-functor can also be defined using *injective coresolutions* ([**11**, p. 141]). An admissible complex in ${}_A\mathcal{M}$

$$0 \to Y \xrightarrow{\eta} J_0 \xrightarrow{\phi_0} J_1 \to \ldots \tag{27}$$

is called an injective coresolution for Y if all the J_i's are injective. (An injective coresolution for Y always exists using cofree modules in a natural way.) Then the cohomology groups of the associated sequence

$$0 \to {}_AB(X, J_0) \xrightarrow{\phi_{0*}} {}_AB(X, J_1) \xrightarrow{\phi_{1*}} {}_AB(X, J_2) \to \ldots \tag{28}$$

coincide with the $\mathrm{Ext}^n_A(X, Y)$. So Ext can be calculated either by a projective resolution in the first variable or an injective coresolution in the second variable.

For the next result see [**11**, p. 154]. (The corresponding result in ordinary homology appears in [**23**, p. 199].)

Proposition 2.1. *Let $P, J \in {}_A\mathcal{M}$. Then P is projective if and only if $\mathrm{Ext}_A(P, X) = 0$ for all $X \in {}_A\mathcal{M}$. The module J is injective if and only if $\mathrm{Ext}_A(X, J) = 0$ for all $X \in {}_A\mathcal{M}$.*

It would be nice if $X \in {}_A\mathcal{M}$ were projective if and only if X^* were injective. This is not true. However, in the usual homology theory, there is a version of such a result with flat in place of projective. Indeed ([**23**, p. 87]) if R is a ring and B is a right R-module, then B is flat if and only if the "character module" $\mathrm{Hom}_Z(B, Q/Z)$ is injective as a left R-module. We would expect in the Functional analytic context that the character module would be replaced by the Banach space dual. We now discuss the appropriate notion of a flat module in ${}_A\mathcal{M}$.

A module $Y \in {}_A\mathcal{M}$ is called *flat* ([**11**, p. 239]) if for any admissible complex

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

in $\mathcal{M}_A$, the associated complex

$$0 \to X_1 \hat{\otimes}_A Y \to X_2 \hat{\otimes}_A Y \to X_3 \hat{\otimes}_A Y \to 0 \tag{29}$$

is exact. Here, for $X \in \mathcal{M}_A$, $X \hat{\otimes}_A Y$ is the projective tensor product of X and Y over A (so that $xa \otimes y$ is identified with $x \otimes ay$ for $x \in X$, $y \in Y$ and $a \in A$).

As in ordinary homology ([**23**, p. 85]) every projective module in ${}_A\mathcal{M}$ is flat.

It is straightforward to show that $(X \hat{\otimes}_A Y)^*$ is canonically identified with $B_A(X, Y^*)$. Using the dual of (29) yields the following beautiful theorem (due to M. V. Sheinberg): *the left module Y is flat if and only if the right module Y^* is injective.* In particular, for bimodules, replacing A by A^{e} gives:

Theorem 2.1. *Let $Y \in {}_A\mathcal{M}_A$. Then Y is flat if and only if Y^* is injective.*

This result ennables one to investigate flatness in terms of dual injectivity.

3. Cohomology and n-amenability.

The Banach algebra A is called *n-amenable* if the cohomology groups $H^n(A, X^*) = 0$ for every Banach A-bimodule X. As discussed in the preceding section, the $H^n(A, X^*)$ can be regarded either as the $\mathrm{Ext}^n_{A^{\mathrm{e}}}(A_+, X^*)$ or with the cohomology groups of the standard Banach space complex (16). Both points of view will be used in this paper. We stress that since we are dealing throughout with bimodules, the Banach algebra whose homology we are discussing is A^{e}. (Thus when we refer to projective, flat etc. these will be with reference to A^{e}, not A.)

Amenability for A is the same as 1-amenability (in Johnson's definition [**15**, p. 60]) and implies n-amenability for all n. (This follows from the straightforward equality ([**15**, p. 9]): For any Banach A-module X, $H^{n+1}(A, X^*) = H^n(A, B(A, X^*))$ where $B(A, X^*) = (A\hat{\otimes}X)^*$ is a dual Banach A-bimodule in a natural way. More generally, if $m < n$, then m-amenability implies n-amenability. For an Ext-approach to this, see [**11**, p. 254].)

We note that n-amenability for A is equivalent to n-amenability for A_+. Indeed, in general, for any Banach algebra A and any Banach A-bimodule Z, we have

$$H^n(A, Z) = H(A_+, Z)$$

where the identity of A_+ acts as the unit on both left and right sides of Z. To see this, for such a bimodule, we have ([**11**, p. 155]) that $H^n(A_+, X) = \mathrm{Ext}_{A^{\mathrm{e}}}(A_+, X) = H^n(A, X)$. A result of Johnson ([**15**, p. 14]) gives that if B is a unital Banach algebra, Y is a Banach B-bimodule, and $X = eYe$ where e is the identity of B, then $H^n(B, Y^*) = H^n(B, X^*)$. It follows that A is n-amenable if and only A_+ is n-amenable.

In ([**11**, p. 253]), Helemskii defines A to be amenable if the module A_+ is flat in ${}_A\mathcal{M}_A$. We will show that n-amenability is equivalent to the flatness of the kernel bimodule K_n defined earlier. (This will give Helemskii's definition in the case $n = 1$ where $K_1 = A_+$.)

Once we have proved this, we will dualize and obtain another characterization of n-amenability parallel to the splitting of (3) given earlier. We then show how this splitting naturally gives rise to n-virtual diagonals. We conclude the section by looking at some examples of n-amenable algebras and n-virtual diagonals.

We will use the notations introduced in the previous two sections.

Theorem 3.1. *The Banach algebra A is n-amenable if and only if the bimodule $K_n \in {}_A\mathcal{M}_A$ is flat.*

Proof. By Theorem 2.1, the flatness of K_n is equivalent to the injectivity of $(K_n)^*$. By Proposition 2.1, the injectivity of K_n^* is equivalent to:

$$\mathrm{Ext}_{A^{\mathrm{e}}}(X, K_n^*) = 0 \tag{30}$$

for all $X \in {}_A\mathcal{M}_A$. Now by (15), the equality (30) is equivalent to:

$$\mathrm{Ext}_{A^{\mathrm{e}}}(K_n, X^*) = 0 \tag{31}$$

for all $X \in {}_A\mathcal{M}_A$. It remains to show that for all such X,

$$\operatorname{Ext}_{A^e}(K_n, X^*) = \operatorname{Ext}^n_{A^e}(A_+, X^*) = H^n(A, X^*). \tag{32}$$

(In fact this is true with any Banach A-bimodule in place of X^*.) The rest of the argument is (with slight modification) similar to a proof of the corresponding result [**23**, Corollary 6.19] in ordinary homology and essentially appears (in the context of projective homological dimension) on [**11**, p. 162]. (Indeed, as Professor Helemskii has pointed out, the present theorem is the "flat" analogue of the "projective" [**11**, Theorem III.5.4].) We will be content with a brief summary.

From the admissibility of (18), we see that for each $n \geq 2$, the short exact sequence

$$0 \to K_n \to D_n \to K_{n-1} \to 0 \tag{33}$$

of bimodules is admissible. Applying the long exact sequence of cohomology (14) to (33) and using the projectivity of D_n and Proposition 2.1, we obtain (with $\operatorname{Ext} = \operatorname{Ext}_{A^e}$) that $\operatorname{Ext}^r(K_n, X^*) = \operatorname{Ext}^{r+1}(K_{n-1}, X^*)$. So $\operatorname{Ext}^1(K_n, X^*) = \operatorname{Ext}^2(K_{n-1}, X^*) = \cdots = \operatorname{Ext}^n(K_1, X^*)$. Since $K_1 = A_+$, we obtain the required equality (32). □

We now define the notion of an n-virtual diagonal for A.

Definition 3.1. An *n-virtual diagonal* for A is an $(n-1)$-cocycle $D : C_{n-1} \to D^{**}_{n+1}$ such that for all $w \in C_{n-1}$,

$$\pi^{**}_{n+1}(D(w)) = \pi_{n+1}(1 \otimes w \otimes 1). \tag{34}$$

In the case where $n = 1$, D above is interpreted as being a virtual diagonal for A_+ as in (1).

When A has a unit, an n-virtual diagonal with values in C^{**}_{n+1} will also be called an n-virtual diagonal, the context determining which kind of n-virtual diagonal is intended.

Dualizing the admissible short exact sequence (33) with n replaced by $(n+1)$ gives the admissible short exact sequence

$$0 \to K^*_n \to D^*_{n+1} \to K^*_{n+1} \to 0. \tag{35}$$

We note that in the next theorem, for the case where A is unital, we can replace D_{n+1} by C_{n+1} and in that case, the equivalence of (a) and (c) is due to Effros and Kishimoto ([**3**]). Their method of proof uses the approach of the standard complex (16) while in our approach to the following theorem, use of the standard complex is combined with homological techniques. (Some use of the standard cohomology complex is essential since n-virtual diagonals are *cocycles*.)

Theorem 3.2. *The following conditions are equivalent for a Banach algebra A:*

(a) *A is n-amenable.*

(b) *The sequence* (35) *splits in terms of A-bimodule morphisms.*

(c) *There exists an n-virtual diagonal for A.*

Proof. (a)$\Leftrightarrow$ (b). Suppose that A is n-amenable. By Theorem 3.1, the bimodule K_n is flat. Hence (Theorem 2.1) K_n^* is injective. Now π_{n+1} is surjective and (35) is admissible. So π_{n+1}^* is an admissible monomorphism, and it follows from the definition of *injective* that π_{n+1}^* is a Banach space coretraction. So (35) splits and (b) follows.

Conversely, suppose that (b) holds and let $\rho : D_{n+1}^* \rightarrow K_n^*$ be a morphism such that $I_{K_n^*} = \rho \circ (\pi_{n+1})^*$. Then K_n^* is a direct module summand of D_{n+1}^*. The latter is injective since it is the dual of a projective module and projective implies flat. So K_n^* is injective and hence by Theorem 3.1, A is n-amenable.

(b) $\Leftrightarrow$ *(c)*. Assume that (b) holds and let ρ be a morphism as above. Then $\rho^* : K_n^{**} \rightarrow D_{n+1}^{**}$ is also a morphism. Regard $K_n \subset K_n^{**}$ and note that for $w \in C_{n-1}$, we have $1 \otimes w \otimes 1 \in D_{n+1}$ and $\pi_{n+1}(1 \otimes w \otimes 1) \in K_n$. Define $D : C_{n-1} \rightarrow D_{n+1}^{**}$ by:

$$D(w) = \rho^*(\pi_{n+1}(1 \otimes w \otimes 1)).$$

We claim that D is an n-virtual diagonal.

Firstly we show that D is an $(n-1)$-cocycle. Let $w_i \in A$, $w = w_1 \otimes \cdots \otimes w_{n-1} \in C_{n-1}$ and $a \in A$. Then using the facts that ρ^* and π_{n+1} are morphisms and that $\pi_{n+1}\pi_{n+2} = 0$,

$$\begin{aligned}
&\delta D(a \otimes w) = \\
&\quad aD(w) - D(aw_1, \dots , w_{n-1}) + \cdots + (-1)^{n-1} D(a, w_1, \dots , w_{n-2}w_{n-1}) \\
&\quad +(-1)^n D(a, w_1, \dots , w_{n-2})w_{n-1} \\
&= \rho^* \pi_{n+1}(a \otimes w \otimes 1 - 1 \otimes aw_1 \otimes \cdots \otimes w_{n-1} \otimes 1 + \cdots \\
&\quad +(-1)^{n-1} 1 \otimes a \otimes w_1 \otimes \cdots \otimes w_{n-2}w_{n-1} \otimes 1 \\
&\quad + (-1)^n 1 \otimes a \otimes w_1 \otimes \cdots \otimes w_{n-1}) \\
&= \rho^* \pi_{n+1}(\pi_{n+2}(1 \otimes a \otimes w \otimes 1)) \\
&= 0.
\end{aligned}$$

Next,

$$\pi_{n+1}^{**}(D(w)) = (\rho\pi_{n+1}^{*})^{*}\pi_{n+1}(1 \otimes w \otimes 1) = \pi_{n+1}(1 \otimes w \otimes 1).$$

So D is an n-virtual diagonal.

Now suppose that (c) holds and let D be an n-virtual diagonal for A. Then the multilinear map $\alpha \otimes w \otimes \beta \to \alpha(Dw)\beta$ extends to a bounded linear map $\tilde{D} : D_{n+1} \to D_{n+1}^{**}$. Define $\sigma : K_n \to D_{n+1}^{**}$ by: $\sigma(k) = \tilde{D}(1 \otimes k')$ where we are using the notation of (23). We claim that σ is a morphism.

Indeed, let $k \in K_n$ and $\alpha \in A_+$. Since the dash operation only affects the left component, we have $(k\alpha)' = k'\alpha$, and since $\tilde{D}$ is a right morphism, we have $\sigma(k\alpha) = \tilde{D}(1 \otimes k'\alpha) = \sigma(k)\alpha$. We now show that we also have $\sigma(\alpha k) = \alpha\sigma(k)$.

Let $a \in A$ and $v = b_1 \otimes \cdots \otimes b_{n-1} \otimes 1$ where $b_i \in A$. Since D is a cocycle,

$$\begin{aligned}
&\tilde{D}(a \otimes v) = \\
&\quad aD(b_1 \otimes \cdots \otimes b_{n-1})1 \\
&\quad = \delta D(a \otimes b_1 \otimes \cdots \otimes b_{n-1}) + D(ab_1 \otimes \cdots \otimes b_{n-1}) \\
&\quad -D(a \otimes \pi_{n-1}(b_1 \otimes \cdots \otimes b_{n-1})) + (-1)^{n+1} D(a \otimes b_1 \otimes \cdots \otimes b_{n-2})b_{n-1} \\
&\quad = D(ab_1 \otimes b_2 \otimes \cdots \otimes b_{n-1}) - \tilde{D}(1 \otimes a \otimes \pi_{n-1}(b_1 \otimes \cdots \otimes b_{n-1}) \otimes 1) + \\
&\quad (-1)^{n+1}\tilde{D}(1 \otimes a \otimes b_1 \otimes \cdots b_{n-2} \otimes b_{n-1}) \\
&\quad = D(ab_1 \otimes b_2 \otimes \cdots \otimes b_{n-1}) - \tilde{D}(1 \otimes a \otimes \pi_n(b_1 \otimes \cdots \otimes b_{n-1} \otimes 1)).
\end{aligned}$$

So

$$\tilde{D}(a \otimes v) = \tilde{D}(1 \otimes av) - \tilde{D}(1 \otimes a \otimes \pi_n(v)). \tag{36}$$

Since $\tilde{D}$ is a right morphism, (36) holds when the 1 at the end of v is replaced by an arbitrary element of A_+.

Let $g \in D_n$. By (22), we can write $g = g' + 1 \otimes v$ where $v \in C_{n-2}\hat{\otimes}A_+$. Let $a \in A$. Then using (36),

$$\begin{aligned}
\tilde{D}(1 \otimes (ag)') &= \tilde{D}(1 \otimes (ag' + a \otimes v)) \\
&= \tilde{D}(1 \otimes ag') + \tilde{D}(1 \otimes a \otimes v) \\
&= [\tilde{D}(a \otimes g') + \tilde{D}(1 \otimes a \otimes \pi_n(g'))] + \tilde{D}(1 \otimes a \otimes v) \\
&= a\tilde{D}(1 \otimes g') + \tilde{D}(1 \otimes a \otimes (\pi_n(g') + v)). \qquad (37)
\end{aligned}$$

In particular, if $g = k \in K_n$, then by (24), we have $\pi_n(k') + v = 0$, and so

$$\sigma(ak) = \tilde{D}(1 \otimes (ak)') = a\tilde{D}(1 \otimes k') = a\sigma(k).$$

Hence σ is an A_+-bimodule morphism.

We next show that $\pi^{**}_{n+2}\sigma = I$, the identity map on K_n. Let $h = b_1 \otimes \cdots \otimes b_{n-1} \otimes \beta_n$ where $b_i \in A$ and $\beta_n \in A_+$. Then using the second virtual diagonal condition,

$$\begin{aligned}\pi^{**}_{n+1}(\tilde{D}(1 \otimes h)) &= \pi^{**}_{n+1}(D(b_1 \otimes \cdots \otimes b_{n-1})\beta_n) \\ &= \pi^{**}_{n+1}(D(b_1 \otimes \cdots \otimes b_{n-1}))\beta_n \\ &= \pi_{n+1}(1 \otimes b_1 \otimes \ldots \otimes b_{n-1} \otimes 1)\beta_n \\ &= \pi_{n+1}(1 \otimes h).\end{aligned}$$

Clearly this is also true for any $h \in C_{n-1}\hat{\otimes}A_+$, in particular for $h = k'$ for $k \in K_n$. For such a k, we then have, using (24), $\pi^{**}_{n+1}\sigma(k) = \pi^{**}_{n+1}(\tilde{D}(1 \otimes k')) = \pi_{n+1}(1 \otimes k') = k' - \pi_n(k') = G(k) = k$. So $\pi^{**}_{n+1}\sigma = I$.

Finally, let $i : D^*_{n+1} \to D^{***}_{n+1}$ be the natural injection map and $\rho = \sigma^* \circ i$. Then ρ is a morphism since both i and σ^* are. Further, $\sigma^* \circ \pi^{***}_{n+1}$ is the identity map on $K^*_n \subset K^{***}_n$. Now π^{***}_{n+1} restricted to K^*_n coincides with π^*_{n+1}, and so has range in D^*_{n+1}. Hence $\rho \circ \pi^*_{n+1}$ is the identity on K^*_n and (b) now follows. □

Examples. (1) Let G be a discrete amenable group. By Johnson's theorem, the Banach algebra $\ell^1(G)$ is amenable. So $\ell^1(G)$ is 2-amenable. Since $\ell^1(G)$ is unital, it has a 2-virtual diagonal D with values in $C_3(\ell^1(G))^{**}$.

The following 2-virtual diagonal on $\ell^1(G)$ is due to Roger Smith.

Let m be right invariant mean for the discrete amenable group G with identity e. (So m is a state on $\ell^\infty(G)$ and $m(x\phi) = m(\phi)$ for all $x \in G$ and all $\phi \in \ell^\infty(G)$, where $x\phi(t) = \phi(tx)$ for all $t \in G$.) Any 2-virtual diagonal on $\ell^1(G)$ is a derivation into the module $C_3(\ell^1(G))^{**} = \ell^1(G \times G \times G)^{**} = \ell^\infty(G \times G \times G)^*$ and we only need to specify its values on the group elements. A 2-virtual diagonal D for $\ell^1(G)$ is then given by:

$$D(a) = \int (x^{-1} \otimes xa \otimes e - x^{-1} \otimes x \otimes a)\, dm(x) \tag{38}$$

where $a \in G$. The above equation can be interpreted: for $f \in \ell^\infty(G \times G \times G)$,

$$D(a)(f) = m(x \to [f(x^{-1}, xa, e) - f(x^{-1}, x, a)]).$$

The map D is actually an inner derivation. Indeed, since $x^{-1} \otimes xa \otimes e = a(xa)^{-1} \otimes xa \otimes e$ and m is right invariant, a change of variable in the first term of the right-hand side of (38) gives that for all a, $D(a) = az - za$ where

$$z = \int (x^{-1} \otimes x \otimes e)\, dm(x).$$

It is easy to check that the second 2-virtual diagonal condition holds for D so that D is an inner 2-virtual diagonal.

This is of interest since we have here an amenable Banach algebra with a coboundary 2-virtual diagonal. This was the main motivation for Corollary 4.1, which relates $(n-1)$-amenability to the existence of a coboundary n-virtual diagonal in much greater generality.

It seems likely that a similar formula can be given for a 2-virtual diagonal on an amenable unital C^*-algebra using the right invariant mean that always exists on its unitary group ([**18**, **19**]). (More precisely, the mean exists on the space of bounded left uniformly continuous complex valued functions on the unitary group regarded as a topological group in the relative weak topology.) Similar issues arise for von Neumann algebras. In fact, ([**7**, **20**]) a von Neumann algebra is amenable if and only if there exists a right invariant mean (in a suitable sense) on its isometry semigroup.[6]

(2) The Banach algebra A in this example is two-dimensional but is of interest since it is a non-unital (and hence non-amenable) 2-amenable finite dimensional Banach algebra. Here the 2-virtual diagonal takes its values in $D_3^{**} = D_3 = A_+ \otimes A \otimes A_+$, and it does not seem possible to formulate it in terms of A only.

Let A be the algebra of 2×2-complex matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

and

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The multiplication in A is determined by the products: $e^2 = e$, $ef = f$ and $fe = 0 = f^2$. We note that e is a left unit for A.

A theorem of Helemskii ([**11**, p. 215]) can be used to show that $H^3(A, X) = 0$ for all Banach A-modules X, so that in particular, A is 3-amenable. Indeed, A satisfies: $A^2 = A$ and is biprojective in the sense ([**11**, p. 188]) that the product map $\pi : A \hat{\otimes} A \to A$ is a retraction in ${}_A\mathcal{M}_A$. Indeed, a right inverse morphism ρ for π is given by:

$$\rho(e) = e \otimes e \qquad \rho(f) = e \otimes f.$$

Helemskii's theorem then applies.

[6]The author is grateful to Professor Helemskii for pointing out that the characterization of amenability for a von Neumann algebra given in [**20**, Corollary 1] is homological in character and was proved earlier by him in [**12**].

In fact $H^2(A,X) = 0$ for all Banach A-modules X. I am grateful to Professor Helemskii for pointing out that this is a consequence of a theorem in an unpublished paper by Y. V. Selivanov. (See [**11**, p. 286] for some details.)

However, we can also show 2-amenability for A by exhibiting a 2-virtual diagonal. To check that a linear map $D : A \to A_+ \otimes A \otimes A_+$ is a 2-virtual diagonal, we need only check the derivation conditions $De = eDe + (De)e$, $Df = eDf + (De)f$, $fDe + (Df)e = 0 = fDf + (Df)f$, and the conditions:

$$\pi_3(De) = e \otimes 1 - 1 \otimes e, \quad \pi_3(Df) = f \otimes 1 - 1 \otimes f.$$

The reader can verify that the map D below satisfies these conditions:

$$D(e) = e \otimes e \otimes 1 - 2e \otimes e \otimes e + 1 \otimes e \otimes e$$
$$D(f) = -e \otimes f \otimes e + e \otimes f \otimes 1 - f \otimes e \otimes e + 1 \otimes e \otimes f.$$

We note that if A is finite-dimensional, then n-amenability is equivalent to the formally stronger condition that $H^n(A,X) = 0$ for every Banach A-module X. Indeed, if $D : C_n \to X$ is a cocycle, then its range is finite dimensional and so a Banach dual module. So D is a coboundary if A is n-amenable. (The condition $H^n(A,X) = 0$ for every Banach A-module is important for the study of the *homological bidimension* dbA of A – see [**11**, p. 164].)

We conclude this section by briefly discussing other examples of n-amenability for finite dimensional algebras.

Let T_2 be the algebra of upper triangular 2×2 complex matrices. The algebra T_2 is 2-amenable. Indeed, T_2 is the algebra obtained by adjoining an identity to the algebra of Example 2 above and so is also 2-amenable. Since the algebra of Example 2 is not amenable, it also follows that T_2 is not amenable.

The examples of finite dimensional n-amenable algebras below were shown to the author by Roger Smith. He shows that if A_4 is the join (in the sense of Gilfeather and Smith ([**5**])) of D_2 with D_2, where D_2 is the algebra of 2×2 diagonal complex matrices, then $H^n(A_4, X) = 0$ for any any $n \geq 2$ and any A_4-bimodule X. Forming repeated suspensions of A_4 ([**6**]) yields algebras $B_n \subset M_{2n+2}$ (where M_r is the algebra of $r \times r$ complex matrices) and he shows that $H^m(B_n, X) = 0$ for all $m > n$ while $H^n(B_n, M_{2n+2}) = C$ (where M_{2n+2} has the natural multiplication module action). In particular, B_n is $(n+1)$-amenable but not n-amenable.

4. n-virtual diagonals.

In this section, we formulate a version of n-virtual diagonals that involves A only (without mention of A_+). The natural way that this can be done is to replace D^{**}_{n+1} by C^{**}_{n+1} and to multiply (34) on the left and right by arbitrary elements of A. We shall call the resulting map an *intrinsic* n-virtual diagonal.

The situation is parallel to what happens for the classical virtual diagonals, where the condition $\pi^{**}(M) = 1$ condition in the unital case converts to the condition $a\pi^{**}(M) = a$. The existence of such a virtual diagonal is equivalent to the amenability of A. Both conditions entail a bai and (roughly) we do not lose anything by multiplying by an arbitrary element of A because of Cohen's theorem. The situation is different for higher dimensional virtual diagonals since as Example 2 of §3 shows, n-amenability does not imply the existence of a bai. In fact as we shall see, the above Example 2 does not admit an intrinsic 2-virtual diagonal. We will however show that if A does possess a bai – and both C^*-algebras and group algebras admit such – then the n-amenability of A implies the existence of an intrinsic n virtual diagonal.

Definition 4.1. An *intrinsic n-virtual diagonal* for A is an $(n-1)$-cocycle $D : C_{n-1} \to C^{**}_{n+1}$ such that for all $w \in C_{n-1}$ and $a, b \in A$,

$$a\pi^{**}_{n+1}(D(w))b = \pi_{n+1}(a \otimes w \otimes b). \tag{39}$$

In the case where A is unital, an intrinsic n-virtual diagonal is the same as an n-virtual diagonal with values in C_{n+1} as discussed before Theorem 3.2.

Theorem 4.1. *Let A be an n-amenable Banach algebra with a bai. Then A has an intrinsic n-virtual diagonal.*

Proof. Let $\{e_\delta\}$ be a bai for A and D be an n-virtual diagonal. Define $D_\delta \in B(C_{n-1}, C^{**}_{n+1}) = (C_{n-1}\hat{\otimes}C^*_{n+1})^*$ by: $D_\delta(w) = e_\delta D(a)e_\delta$. By weak* compactness, we can suppose that there exists $T \in B(C_{n-1}, C^{**}_{n+1})$ such that $D_\delta(w) \to T(w)$ weak* for all $w \in C_{n-1}$. We claim that T is an intrinsic n-virtual diagonal.

Indeed for $a \in A$, both $\|(e_\delta a - ae_\delta)D(w)e_\delta\|$ and $\|e_\delta D(w)(ae_\delta - e_\delta a)\|$ converge to 0. So in the weak* topology, we have $e_\delta aD(w)e_\delta \to aT(w)$ and $e_\delta D(w)be_\delta \to (T(w))b$ for all $a, b \in A$. It follows that $(\delta T)(a \otimes w)$ is the weak* limit of $e_\delta(\delta T)(a \otimes w)e_\delta = 0$, so that T is an $(n-1)$ cocycle.

Next, for $a, b \in A$ and any δ, we have $\pi^{**}_{n+1}(ae_\delta D(w)e_\delta b) = \pi_{n+1}(ae_\delta \otimes w \otimes e_\delta b)$. Since π^{**}_{n+1} is weak* continuous and $\{e_\delta\}$ is a bai, it follows that $\pi^{**}_{n+1}(aT(w)b) = \pi_{n+1}(a \otimes w \otimes b)$. So T is an intrinsic n-virtual diagonal. ∎

The present writer suspects that the converse to the preceding theorem is true but has been unable to prove it.

For $F \in C_r^{**}$ and $x \in A$, we define $F \otimes x \in C_{r+1}^{**}$ as follows: For $g \in C_{r+1}^{*}$,

$$(F \otimes x)(g) = F(w \to g(w \otimes x))$$

where $w \in C_r$. When A is finite dimensional, $F \otimes x \in C_{r+1}$ would literally be $F \otimes x$ in the usual sense. For general A, the notation $F \otimes x$ is heuristically helpful. If $N \in C^{n-1}(A, C_{n+1}^{**})$, we define the $(n-1)$-cochain $N \otimes x$ by: $(N \otimes x)(w) = N(w) \otimes x$.

In (40) below, we interpret the right-hand side to be 0 when $n = 2$.

Proposition 4.1. *Let A have a unit e, $n \geq 2$ and $N \in C^{n-2}(A, C_n^{**})$ be an $(n-1)$-virtual diagonal. Let $a_i \in A$ $(1 \leq i \leq (n-1))$ and $a = a_1 \otimes \cdots \otimes a_{n-1}$. Then*

$$\begin{aligned} &\pi_{n+1}^{**}(\delta(N \otimes e)(a) + (-1)^{n+1} e \otimes a \otimes e) = \\ &(-1)^{n+1} \pi_{n+1}(e \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes e \otimes a_{n-1}). \end{aligned} \tag{40}$$

Proof. Let a be as in the above statement. Then for any $N \in C^{n-2}(A, C_n^{**})$,

$$\begin{aligned} &\delta(N \otimes e)(a) = \\ &a_1 N(a_2 \otimes \cdots \otimes a_{n-1}) \otimes e - N(\pi_{n-1}(a)) \otimes e + \\ &(-1)^{n-1} N(a_1 \otimes \cdots \otimes a_{n-2}) \otimes a_{n-1} \\ &= (\delta N)(a) \otimes e + (-1)^n N(a_1 \otimes \cdots \otimes a_{n-2}) a_{n-1} \otimes e + \\ &(-1)^{n-1} N(a_1 \otimes \cdots \otimes a_{n-2}) \otimes a_{n-1}. \end{aligned} \tag{41}$$

Now suppose that N is an $(n-1)$-virtual diagonal. Then $\delta N = 0$, and using the second virtual diagonal condition, we obtain from (41),

$$\begin{aligned} &\pi_{n+1}^{**}(\delta(N \otimes e)(a)) = \\ &\quad (-1)^n [\pi_n^{**}(N(a_1 \otimes \cdots \otimes a_{n-2})) a_{n-1} \otimes e + (-1)^{n+1} N(a_1 \otimes \cdots \otimes a_{n-2}) a_{n-1}] + \\ &\quad (-1)^{n-1} [\pi_n^{**}(N(a_1 \otimes \cdots \otimes a_{n-2})) \otimes a_{n-1} + (-1)^{n+1} N(a_1 \otimes \cdots \otimes a_{n-2}) a_{n-1}] \\ &= (-1)^n [\pi_n^{**}(N(a_1 \otimes \cdots \otimes a_{n-2})) a_{n-1} \otimes e - \pi_n^{**}(N(a_1 \otimes \cdots \otimes a_{n-2})) \otimes a_{n-1}] \\ &= (-1)^n [\pi_n(e \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes e) a_{n-1} \otimes e - \pi_n(e \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes e) \otimes a_{n-1}] \\ &= (-1)^n [\pi_n(e \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1}) \otimes e - \\ &\quad [\pi_{n-1}(e \otimes a_1 \otimes \cdots \otimes a_{n-2}) \otimes e \otimes a_{n-1} + \\ &\quad (-1)^n e \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1}]] \\ &= (-1)^n [\pi_{n+1}(e \otimes a \otimes e) - (-1)^{n+1} e \otimes a_1 \otimes \cdots \otimes a_{n-1} - \\ &\quad [\pi_{n-1}(e \otimes a_1 \otimes \cdots \otimes a_{n-2}) \otimes e \otimes a_{n-1} + (-1)^n e \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes a_{n-1}]] \\ &= (-1)^n [\pi_{n+1}(e \otimes a \otimes e) - \pi_{n-1}(e \otimes a_1 \otimes \cdots \otimes a_{n-2}) \otimes e \otimes a_{n-1}]. \end{aligned}$$

The equation (40) now follows.

Theorem 4.2. *Let A be a Banach algebra with unit e. Let $n \geq 2$ and assume that there exists a cochain $Z : C_{n-2} \to C_{n+1}^{**}$ such that, with a, a_i as in Proposition* 4.1,

$$\pi_{n+1}^{**}((\delta Z)(a) - e \otimes a_1 \otimes \cdots \otimes a_{n-2} \otimes e \otimes a_{n-1}) = 0. \tag{42}$$

Then A is $(n-1)$-amenable if and only if there exists an n-virtual diagonal for A which is a coboundary.

Proof. Suppose that A is $(n-1)$-amenable and let N be an $(n-1)$-virtual diagonal on A. Let Z be as in (42) and $D' = (-1)^n \delta(N \otimes e) + \delta Z$. Then D' is an n-virtual diagonal which is a coboundary by (40).

Conversely, suppose that D is an n-virtual diagonal for A which is a coboundary. Let $W : C_{n-2} \to C_{n+1}^{**}$ be a cochain such that $D = \delta W$. Define the cochain $V : C_{n-2} \to C_n^{**}$ by:

$$V(a) = \pi_{n+1}^{**}(W(a)) - g(a)$$

where $g(a) = e \otimes a \otimes e$. We claim that V is an $(n-1)$-virtual diagonal for A.

Firstly, let $w_i \in A$ $(1 \leq i \leq (n-1))$ and $w = w_1 \otimes \cdots \otimes w_{n-1}$. Then

$$\begin{aligned}(\delta g)(w) &= w \otimes e - e \otimes \pi_{n-1}(w) \otimes e + (-1)^{n-1} e \otimes w \\ &= \pi_{n+1}(e \otimes w \otimes e).\end{aligned} \tag{43}$$

Using (43), the fact that $D = \delta W$ and the second virtual diagonal condition for D, we have

$$\begin{aligned}(\delta V)(w) &= w_1 \pi_{n+1}^{**}[W(w_2 \otimes \cdots \otimes w_{n-1})] - \pi_{n-1}^{**}[W(\pi_{n-1}(w))] + \\ &\quad (-1)^n \pi_{n+1}^{**}[W(w_1 \otimes \cdots \otimes w_{n-2})] w_{n-1} - (\delta g)(w) \\ &= \pi_{n+1}^{**}(D(w)) - \pi_{n+1}(e \otimes w \otimes e) \\ &= 0.\end{aligned}$$

So V is a cocycle.

Next, for $a = a_1 \otimes \cdots \otimes a_{n-1}$, we have

$$\begin{aligned}\pi_n^{**}(V(a)) &= (\pi_n \pi_{n+1})^{**}(W(a)) - \pi_n(g(a)) \\ &= \pi_n(e \otimes a \otimes e).\end{aligned}$$

So V is an $(n-1)$-virtual diagonal. It follows that A is $(n-1)$-amenable.

□

Corollary 4.1. *A 2-amenable unital Banach algebra is amenable if and only if it admits an inner 2-virtual diagonal. A 3-amenable unital Banach algebra is 2-amenable if and only if it admits a coboundary 3-virtual diagonal.*

Proof. By the preceding theorem, we only have to show that there exists a Z satisfying (42) for the cases $n = 2, 3$. For the case $n = 2$ we have to find $Z \in C_3^{**}$ such that $\pi_3^{**}(a_1 Z - Z a_1) = 0$ – trivially, $Z = 0$ will do. For the case $n = 3$, we have to find a cochain $Z : C_1 \to C_4^{**}$ such that

$$\pi_4^{**}(a_1 Z(a_2) - Z(a_1 a_2) + Z(a_1) a_2) = \pi_4(e \otimes a_1 \otimes e \otimes a_2) = a_1 \otimes e \otimes a_2.$$

It is left to the reader to check that we can take $Z(a_1) = a_1 \otimes e \otimes e \otimes e$. □

It seems very likely that for general $n \geq 2$, if unital A is n-amenable, then A is $(n-1)$-amenable if and only if there exists an n-virtual diagonal for A that is a coboundary. By the above, this is equivalent to the existence of a Z satisfying (42) for any n.

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UNIVERSITY OF MISSISSIPPI
UNIVERSITY, MS 38677
E-mail address: mmap@vm.cc.olemiss.edu

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