

Pacific Journal of Mathematics

**ON SPECTRA OF SIMPLE RANDOM WALKS ON
ONE-RELATOR GROUPS. WITH AN APPENDIX BY PAUL
JOLISSAIN**

PIERRE-ALAIN CHERIX, ALAIN J. VALETTE AND PAUL JOLISSAINT

ON SPECTRA OF SIMPLE RANDOM WALKS ON ONE-RELATOR GROUPS

PIERRE-ALAIN CHERIX AND ALAIN VALETTE
 WITH AN APPENDIX BY PAUL JOLISSAINT

For a one relator group $\Gamma = \langle X : r \rangle$, we study the spectra of the transition operators h_X and h_S associated with the simple random walks on the directed Cayley graph and ordinary Cayley graph of Γ respectively. We show that, generically (in the sense of Gromov), the spectral radius of h_X is $(\#X)^{-1/2}$ (which implies that the semi-group generated by X is free). We give upper bounds on the spectral radii of h_X and h_S . Finally, for Γ the fundamental group of a closed Riemann surface of genus $g \geq 2$ in its standard presentation, we show that the spectrum of h_S is an interval $[-r, r]$, with $r \leq g^{-1}(2g - 1)^{1/2}$. Techniques are operator-theoretic.

1. Introduction.

Let Γ be a finitely generated group. Fix a finite, not necessarily symmetric generating subset X , and let $S = X \cup X^{-1}$ be the symmetrization of X . With X and S are classically associated the usual Cayley graph $G(\Gamma, S)$, but also the Cayley digraph (or directed graph) $G(\Gamma, X)$, where the set of vertices is Γ and, for any $\gamma \in \Gamma$ and $s \in X$, an oriented edge is drawn from γ to γs . We denote by $\#E$ the number of elements in the set E .

We consider the normalized adjacency operators, or transition operators, h_X and h_S ; these are operators of norm at most 1 on $l^2(\Gamma)$, defined by:

$$(h_X \xi)(x) = \frac{1}{\#X} \sum_{s \in X} \xi(xs)$$

$$(h_S \xi)(x) = \frac{1}{\#S} \sum_{s \in S} \xi(xs) \quad (\xi \in l^2(\Gamma), x \in \Gamma).$$

Consider the nearest neighbour simple random walk on $G(\Gamma, X)$ obtained by assigning probability $1/(\#X)$ to each neighbour of a given vertex $\gamma \in \Gamma$ (where a neighbour of γ is the extremity of an oriented edge with origin γ); then, for any $x, y \in \Gamma$, the probability $p^{(n)}(x, y)$ of a transition in n steps from x to y is given by $\langle h_X^n \delta_x | \delta_y \rangle$ (where $(\delta_x)_{x \in \Gamma}$ is the canonical basis of $l^2(\Gamma)$); the

analogous probabilistic interpretation of h_S is classical. We denote by $\text{Sp}(T)$ and $r(T)$ the spectrum and spectral radius of a bounded operator T on a Hilbert space. That the spectra of h_X and h_S capture important information about the pairs (Γ, X) or (Γ, S) follows from the following results of Day and Kesten (see [Day], [Ke1], [Ke2]).

Theorem 1.1.

(a) *The following are equivalent:*

- (i) $r(h_X) = 1$;
- (ii) $1 \in \text{Sp}(h_X)$;
- (iii) Γ is amenable.

(b) *Assume $\#X \geq 2$; then $\frac{\sqrt{2(\#X) - 1}}{\#X} \leq r(h_S)$, with equality if and only if Γ is isomorphic to the free group $\mathbb{F}(X)$ on X ; in this case*

$$\text{Sp}(h_S) = \left[-\frac{\sqrt{2(\#X) - 1}}{\#X}, \frac{\sqrt{2(\#X) - 1}}{\#X} \right].$$

In passing, we recall that, in the symmetric case, $r(h_S)$ is the inverse of the radius of convergence of the Green kernel

$$G(x, y; z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)z^n \quad (x, y \in \Gamma).$$

The qualitative study of the spectra of h_X and h_S was pursued in [HRV1] and [HRV2], where the following result was proved, with the exception of assertions concerning a symmetric X , that were obtained by Cartwright [Car] and Kesten [Ke1] respectively.

Theorem 1.2.

(a) *Let \mathbb{T} be the group of complex numbers of modulus 1; fix $z \in \mathbb{T}$. If there exists a character $\chi : \Gamma \rightarrow \mathbb{T}$ such that $\chi(x) = z$ for any $x \in X$, then $\text{Sp}(h_X)$ is invariant under multiplication by z . The converse is true if either Γ is amenable or X is symmetric (i.e. $X = X^{-1}$).*

(b) *Assume $\#X \geq 2$. Set $\sigma(X) = \limsup_{k \rightarrow \infty} \|h_X^k\|_2^{1/k}$, where h_X is now viewed as the normalized characteristic function of X and h_X^k denotes the k^{th} convolution power of h_X . Then*

$$\frac{1}{\sqrt{\#X}} \leq \sigma(X) \leq r(h_X)$$

with $\frac{1}{\sqrt{\#X}} = \sigma(X)$ if and only if X generates a free semi-group, and $\sigma(X) = r(h_X)$ if either X is symmetric or Γ is hyperbolic in the sense of Gromov (but not in general).

- (c) Let Γ be either the free group $\mathbb{F}(X)$, with $\#X \geq 2$, or the surface group $\Gamma_g \cong \langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^g [a_i, b_i] \rangle$ with $X = a_1, b_1, \dots, a_g, b_g$ and $g \geq 2$; then $\text{Sp}(h_X) = \left\{ z \in \mathbb{C} : |z| \leq \frac{1}{\sqrt{\#X}} \right\}$.

In the case of h_S , quantitative results on the spectrum were obtained mainly for virtually abelian groups (using Fourier analysis, as in [KeS]) or, at the other extreme, for virtually free groups or groups for which the Cayley graph is tree-like (using methods from combinatorics on trees, see e.g. [CS1], [CS2], [IoP], [KuS], [Mlo]). In the present paper, we deal with one-relator groups, i.e. groups of the form $\Gamma = \langle X : r \rangle$ where r , the relator, is a cyclically reduced word in $\mathbb{F}(X)$. This class of groups contains the fundamental groups of all compact surfaces (even non-orientable ones), and one-relator groups share a number of interesting properties with surface groups (e.g., it follows from famous results of Lyndon [Lyn] and Stallings [Sta] that a torsion-free one-relator group which is not free must have cohomological dimension 2). To avoid degeneracies, we shall always assume $\#X \geq 2$ and $|r| \geq 3$, i.e. the word length of r in $\mathbb{F}(X)$ is at least 3.

Here is a summary of our results:

- (a) We propose the statistical result that “most” presentations $\Gamma = \langle X : r \rangle$ give $r(h_X) = \frac{1}{\sqrt{\#X}}$ (which implies in particular that the semi-group generated by X in Γ is free). More precisely, we prove that the ratio

$$\frac{\#\{\text{presentation } r \text{ with } r(h_X) = (\#X)^{-1/2} \text{ and } |r| = N\}}{\#\{\text{presentation } r \text{ with } |r| = N\}}$$

tends (exponentially fast) to 1 when N tends to $+\infty$. This is exactly the sense of genericity introduced by Gromov ([Gro], 0.2(A)), and studied further by Champetier [Ch2].

- (b) Let H_r (resp. H_l) be the subgroup of $\mathbb{F}(X)$ generated by all quotients xy^{-1} , with $x, y \in X$ (resp. $x^{-1}y$, with $x, y \in X$). Suppose that r is not in the union $H_r \cup H_l$; then $\|h_X\| = \frac{2\sqrt{\#X-1}}{\#X}$ and $\max\{r(h_X), r(h_S)\} \leq \frac{2\sqrt{\#X-1}}{\#X}$.

- (c) If r is in the exceptional set $H_r \cup H_l$, then $r(h_X) = \frac{1}{\sqrt{\#X}}$ and $\text{Sp}(h_X)$ is a union of concentric circles centered at 0; this is proved using Jolisaint’s result from the Appendix.

(d) Without restriction on r but with $\#X \geq 4$, we have

$$\max\{r(h_X), r(h_S)\} \leq \frac{2\sqrt{\#X - 2} + 1}{\#X} < 1.$$

(e) For the surface group Γ_g with $g \geq 2$, Peter Sarnak asked for the exact value of $r(h_S)$ in terms of the genus g . We prove that $\text{Sp}(h_S)$ is an interval $[-r, r]$ with the non-trivial estimate $r \leq \frac{\sqrt{2g - 1}}{g}$.

We thank M. Bridson, C. Champetier, T. Delzant and C. Pittet for some useful conversations and correspondence. We are grateful to P. Jolissaint for writing up his results as an Appendix to this paper.

2. Rotational symmetries of spectra.

As is well-known, existence of homomorphisms is the easiest thing to check in the case of a finitely presented group. We exemplify this in the case of a one-relator group $\Gamma = \langle X : r \rangle$; denote by Σ the sum of all exponents in r and fix $z \in \mathbb{T}$; then:

- For z a primitive d -th root of 1, there exists a character $\chi : \Gamma \rightarrow \mathbb{T}$ such that $\chi(x) = z$ for any $x \in X$ if and only if $\Sigma \equiv 0 \pmod{d}$;
- for z not a root of 1, there exists a character $\chi : \Gamma \rightarrow \mathbb{T}$ such that $\chi(x) = z$ for any $x \in X$ if and only if $\Sigma = 0$.

From this and Theorem 1.2 above, we immediately deduce:

Proposition 2.1.

- (a) If $\Sigma \equiv 0 \pmod{d}$, then $\text{Sp}(h_X)$ is invariant under multiplication by $\exp(2\pi i/d)$;
- (b) If $\Sigma = 0$, then $\text{Sp}(h_X)$ is a union of concentric circles, centered at 0;
- (c) $\text{Sp}(h_S)$ is symmetric with respect to 0 if and only if Σ is even.

3. Free semi-groups and small cancellation.

Definition 3.1. A word $w \in \mathbb{F}(X)$ is **positive** if it involves only generators with positive exponents. Any non-empty reduced word $r \in \mathbb{F}(X)$ can be written in a unique way as a product without cancellation, either $r = w_1 w_2^{-1} w_3 \cdots w_n^{\pm 1}$ or $r = w_1^{-1} w_2 w_3^{-1} \cdots w_n^{\pm 1}$, where the w_i 's are positive words. We say that r **alternates enough** if $n \geq 4$, i.e. there are at least 3 changes of signs in the exponents of r .

Lemma 3.2. Each of the following statements implies the next one:

- (i) $r(h_X) = \frac{1}{\sqrt{\#X}}$;

- (ii) X generates a free semi-group;
- (iii) the relator r alternates enough.

Proof. i) \Rightarrow ii) follows immediately from Theorem 1.2. To show ii) \Rightarrow iii), we assume that r does not alternate enough and prove that X does not generate a free semi-group. There are 3 cases to consider.

- (a) r has no change of signs in its exponents, i.e. r or r^{-1} is a positive word; then we have a positive word that represents the identity in Γ ;
- (b) r has exactly one change of sign, say $r = w_1 w_2^{-1}$, with w_1, w_2 positive, distinct words; then w_1 and w_2 represent the same element in the semi-group generated by X in Γ ;
- (c) r has exactly two changes of sign, i.e. r or r^{-1} is of the form $w_1 w_2^{-1} w_3$, with w_1, w_2, w_3 positive words; then by cyclically permuting we get $w_3 w_1 w_2^{-1}$, i.e. we are back to the preceding case.

□

Example 3.3. We give an example showing that the converse implication ii) \Rightarrow i) does not hold in general. It seems that this example was known to Y. Guivarc'h (private communication). Consider the one-relator group $\Gamma = \langle y, z : yzy^{-1}z^{-1}yz^{-1} \rangle$. We claim that, for $X = \{y, z\}$, we have $r(h_X) = 1$ and X generates a free semi-group. To see it, set $x = zy^{-1}$; in the generators x, y , the group Γ has the famous presentation

$$\Gamma = \langle x, y : yxy^{-1}x^{-2} \rangle$$

(Γ is the first Baumslag-Solitar group). Γ is solvable, hence amenable, thus $r(h_X) = 1$. Let H be the subgroup of Γ generated by x . The relation

$$(*) \quad yxy^{-1} = x^2$$

exhibits Γ as an HNN-extension of H with respect to the monomorphism $\Theta : H \rightarrow H$ such that $x^k \rightarrow x^{2k}$. Therefore, Γ acts on a tree T , whose construction we now recall (see [Ser], I.1.4, I.5.1). The homogeneous space Γ/H will be both the set of vertices and the set of edges of T : We define the extremity of the edge γH as the vertex γH , and the origin of γH as the vertex $\gamma y^{-1} H$; it follows from relation (*) that this is well-defined. The resulting tree T is the homogeneous tree of degree 3 with, at each vertex, one incoming edge and two outgoing edges. We call descendant of order n of the vertex H any of the 2^n vertices at distance n from H that can be reached from H by a positively oriented path. To prove that y and z generate a free semi-group in Γ , it suffices to prove the following

Claim. Any descendant of order n of H can be written as wH , where w is a positive word of length n in y and z .

Note that this writing is necessarily unique, since there are 2^n positive words of length n in y and z . We prove the claim by induction over n , the case $n = 0$ being obvious. So, let γH be a descendant of order $n + 1$ of H ; then $\gamma y^{-1}H$ is a descendant of order n of H , so by the induction assumption we have $\gamma y^{-1}H = wH$ for some positive word w of length n in y and z . Thus $w = \gamma y^{-1}x^k$ for some $k \in \mathbb{Z}$. If k is even, we have $w = \gamma x^{k/2}y^{-1}$, i.e. $\gamma H = wyH$; if k is odd, we have $w = \gamma x^{(k+1)/2}y^{-1}x^{-1}$, i.e. $\gamma H = wzH$. Both wy and wz are positive words of length $n + 1$ in y and z .

Example 3.4. In Lemma 3.2, the converse implication iii) \Rightarrow ii) does not hold either. To see it, let $n \geq 1$ be an integer, and consider the group $\Gamma = \langle a, b : a(ab^{-1})^{n+1} \rangle$. The relator r alternates enough. Set $r' = (ab^{-1})^n a^2 b^{-1}$, a cyclic permutation of r ; then $r^{-1}ar'a^{-1} = bab^{-1}a^{-1}$, so that X does not generate a free semi-group, since $ab = ba$. (Actually, this also shows that Γ is a quotient of \mathbb{Z}^2 ; since the vector $(n + 2, -n - 1)$ is primitive in \mathbb{Z}^2 , one checks easily that Γ is isomorphic to \mathbb{Z} .)

This example typically displays absence of small cancellation, whose definition we recall now (see e.g. [LyS] for an extensive study).

Definition 3.5. Let $\Gamma = \langle X : r \rangle$ be a one-relator group. Denote by R the set of words obtained by cyclic permutations of r and r^{-1} . A **piece** is a prefix u which is common to two distinct elements of R (by prefix we mean any not empty initial part of a word; a word is a prefix of itself). Fix $\lambda \in]0, 1[$. We say that r satisfies **the small cancellation condition** $C'(\lambda)$ if, for any piece u , one has:

$$|u| < \lambda|r|.$$

Definition 3.6. A one-relator group $\Gamma = \langle X : r \rangle$ satisfies a **Dehn's algorithm** if, for any reduced word $w \in \mathbb{F}(X)$ that represents 1 in Γ , there exists a prefix u of some word in R such that u is a subword of w and $|u| > \frac{1}{2}|r|$.

It is known that groups satisfying the small cancellation property $C'(\lambda)$, with $\lambda \leq \frac{1}{6}$, also satisfy a Dehn's algorithm (see [LyS], Theorem 4.4 of Chapter V; [Str], Theorem 25). On the other hand, by a result of Gromov, groups with a Dehn's algorithm are hyperbolic ([Gro], Theorem 2.3.D; see also [Str], Theorem 36 for a direct proof that $C'(1/6)$ -groups are hyperbolic).

After these standard definitions, here is another one of our own.

Let $\Gamma = \langle X : r \rangle$ be a one relator-group presentation. For any $r' \in R$, express r' as a reduced product in $\mathbb{F}(X)$ either $r' = w_1 w_2^{-1} w_3 \cdots w_n^{\pm 1}$ or $r' = w_1^{-1} w_2 w_3^{-1} \cdots w_n^{\pm 1}$, with the w_i 's positive words.

Definition 3.7. The presentation $\Gamma = \langle X : r \rangle$ is **balanced** if one has $|w_i| \leq \frac{|r|}{4}$ for $i = 1, \dots, n$.

Clearly, a balanced presentation alternates enough.

We are now in position to present a class of one-relator presentations for which the three conditions of Lemma 3.2 are equivalent.

Lemma 3.8. *Suppose that the presentation $\Gamma = \langle X : r \rangle$ is balanced and satisfies a Dehn's algorithm (this latter assumption being verified if the presentation satisfies condition $C'(1/6)$). Then $r(h_X) = \frac{1}{\sqrt{\#X}}$.*

Proof. We first show that X generates a free semi-group in Γ . Let N be the normal subgroup generated by r in $\mathbb{F}(X)$ (N is the set of consequences of the relation r). Fix $w \in N$, a reduced word. Thanks to the Dehn's algorithm, we find a subword u of w which is also a prefix of some $r' \in R$, with $|u| > \frac{|r|}{2}$. Because the presentation is balanced, we see that u , and a fortiori w , must contain at least 2 changes of sign in their exponents. Now, let v_1, v_2 be distinct positive words. Since $v_1 v_2^{-1}$ has exactly one change of sign in its exponents, we see that $v_1 v_2^{-1}$ does not belong to N , i.e. v_1 is distinct from v_2 in Γ . This shows that the semi-group generated by X in Γ is free¹. Since Γ is hyperbolic, Theorem 1.2 applies to give $r(h_X) = \frac{1}{\sqrt{\#X}}$. \square

Remarks.

- (1) It is stated in Theorem 3 of [New] (for a proof, see [LyS], Theorem 5.5 of Chapter IV) that a one-relator group with torsion satisfies a Dehn's algorithm, and hence is hyperbolic.
- (2) Lemma 3.8 and its proof naturally raise the question: Which one-relator groups are hyperbolic? It is conjectured that a one-relator group is hyperbolic if and only if every non-identity element has a cyclic centralizer (Conjecture 2 in [Juh]).

The following definition is basically due to Gromov ([Gro], 0.2(A)) and was made precise by Champetier ([Ch2]; this paper also contains many impressive results on "generic" properties).

Definition 3.9. Let $\#X \geq 2$ be fixed. For any integer $N \geq 1$, denote by $C(N)$ the number of cyclically reduced words of length N in $\mathbb{F}(X)$. Let (P) be a property of one-relator presentations. We say that (P) is **asymptotically almost sure** if the ratio

$$\frac{\#\{\text{presentation } \langle X : r \rangle \text{ with property } (P) \text{ and } |r| = N\}}{C(N)}$$

¹On p. 100 of [HRV2], it was stated without proof that, in the surface group Γ_g ($g \geq 2$) with the standard presentation, X generates a free semi-group. This provides a proof of this statement.

tends to 1 for N tending to $+\infty$.

Lemma 3.10. *Fix $\lambda \in]0, 1[$. Condition $C'(\lambda)$ is asymptotically almost sure.*

Proof. See [Ch2], Lemma 4.4. Note that the proof reveals in this case that, for $N \rightarrow \infty$, the convergence of the above ratio to 1 is exponentially fast. \square

Lemma 3.11. *A one-relator presentation is asymptotically almost surely balanced.*

Proof. Set $\#X = k$ for simplicity. First, notice that $C(N)$ is not smaller than the number of reduced words of length N in $\mathbb{F}(X)$ whose last letter is not the inverse of the first one, i.e.

$$(1) \quad C(N) \geq 2k(2k - 1)^{N-2}(2k - 2).$$

Now, we estimate the number $B(N)$ of “bad” presentations, i.e those presentations $\langle X : r \rangle$ such that there exists $r' \in R$ beginning with a positive subword of length larger than $N/4$. Since there are at most $2N$ elements in R , we have

$$B(N) \leq 2N \sum_{l=[N/4]+1}^N C(N, l)$$

where $C(N, l)$ is the number of cyclically reduced words of length N beginning with a positive subword of length exactly l . Thus we certainly have:

$$(2) \quad B(N) \leq 2N \sum_{l=[N/4]+1}^N k^l(2k - 1)^{N-l}.$$

Dividing (2) by (1), we estimate the proportion of non-balanced presentations:

$$\begin{aligned} \frac{B(N)}{C(N)} &\leq \frac{N(2k - 1)^2}{2k(k - 1)} \sum_{l=[N/4]+1}^N k^l(2k - 1)^{-l} \\ &= \frac{N(2k - 1)^2}{2k(k - 1)} \frac{k^{[n/4]+1}(2k - 1)^{-[n/4]-1} - k^{N+1}(2k - 1)^{-N-1}}{1 - k(2k - 1)^{-1}}. \end{aligned}$$

Since $k \geq 2$, this ratio tends exponentially fast to 0 for $N \rightarrow \infty$. \square

From this, we deduce:

Theorem 3.12. *Let $\#X \geq 2$ be fixed. A presentation $\Gamma = \langle X : r \rangle$ has asymptotically almost surely $r(h_X) = \frac{1}{\sqrt{\#X}}$.*

Proof. We combine Lemma 3.10 (with $\lambda = 1/6$) and Lemma 3.11, and use the fact that the conjunction of two asymptotically almost sure properties is asymptotically almost sure. Thus, asymptotically almost surely, a one-relator presentation is balanced and satisfies $C'(1/6)$, so also satisfies $r(h_X) = \frac{1}{\sqrt{\#X}}$, by Lemma 3.8. \square

Remark. Fix an integer $k \geq 1$. Let $\Gamma_n = \langle X_n : r_n \rangle$ be a sequence of one-relator groups on k generators, with $|r_n|$ tending to infinity for $n \rightarrow \infty$. Set $S_n = X_n \cup X_n^{-1}$. It was proved by Grigorchuk [Gri] (and recently reproved by Champetier [Ch1]) that, if all Γ_n 's satisfy the small cancellation condition $C'(\lambda)$, with $\lambda < 1/6$, then

$$\lim_{n \rightarrow \infty} r(h_{S_n}) = \frac{\sqrt{2(\#X) - 1}}{\#X}.$$

This corresponds to the intuitive idea that, as $|r_n|$ becomes larger, the Cayley graph of Γ_n looks more and more like a tree.

4. Estimates on norms and spectral radii.

First, we recall that, for any group Γ , the **right regular representation** is the representation ρ of Γ on $l^2(\Gamma)$ defined by:

$$(\rho(g)\xi)(h) = \xi(hg) \quad (\xi \in l^2(\Gamma), g, h \in \Gamma).$$

If X is a finite generating subset of Γ and $S = X \cup X^{-1}$, our transition operators h_X, h_S may be expressed simply in terms of ρ as:

$$h_X = \frac{1}{\#X} \sum_{s \in X} \rho(s);$$

$$h_S = \frac{1}{\#S} \sum_{s \in S} \rho(s).$$

We shall need the following result of Akemann-Ostrand [AkO] (see also [Woe]).

Lemma 4.1. *Let x_1, x_2, \dots, x_n be elements of Γ that generate a free subgroup on n generators. Then*

$$\left\| \sum_{i=1}^n \rho(x_i) \right\| = 2\sqrt{n-1};$$

$$\left\| 1 + \sum_{i=1}^n \rho(x_i) \right\| = 2\sqrt{n}.$$

With this we may estimate the norm of h_X .

Proposition 4.2. *Let $\Gamma = \langle X : r \rangle$ be a one-relator group, with $\#X \geq 4$, $|r| > 2$ and r cyclically reduced. Then*

$$\max\{r(h_X), r(h_S)\} \leq \|h_X\| \leq \frac{2\sqrt{\#X-2} + 1}{\#X} < 1.$$

Proof. The inequality $r(h_X) \leq \|h_X\|$ holds for any bounded operator. Now, since $|r| > 2$ and r is cyclically reduced, the intersection $X \cap X^{-1}$ is empty, so

$$r(h_S) = \|h_S\| = \left\| \frac{h_X + h_X^*}{2} \right\| \leq \|h_X\|,$$

where the first equality holds for any bounded self adjoint operator. This proves the first inequality in the statement. To prove the second, set $X = \{x_1, \dots, x_k\}$; without loss of generality, we may assume that x_k appears in the relator r . Then

$$\|h_X\| \leq \frac{1}{k} \left(\left\| \sum_{i=1}^{k-1} \rho(x_i) \right\| + 1 \right).$$

Now, by Magnus' Freiheitssatz (see [LyS], Proposition 5.1 of Chapter II), the subgroup of Γ generated by x_1, \dots, x_{k-1} is free on $k-1$ generators, so Lemma 4.1 applies. Finally, the expression is $1/k(2\sqrt{k-2} + 1)$ less than 1 provided $k \geq 4$. \square

Remark. It was observed in Proposition 4 (iv) of [HRV2] that, for $\#X = 2$, one always has $\|h_X\| = 1$.

Example 4.3. For $\#X = 3$, Proposition 4.2 just gives the obvious bound $\|h_X\| \leq 1$. The following example shows that we cannot expect better in this case. Indeed, consider the group $\Gamma = \langle a, b, c : [ac^{-1}, bc^{-1}] \rangle$. Then, factoring out $\rho(c)$ on the right, we get:

$$\|h_X\| = \frac{1}{3} \|\rho(ac^{-1}) + \rho(bc^{-1}) + 1\|.$$

Now, ac^{-1} and bc^{-1} commute and actually they generate a subgroup H isomorphic to \mathbb{Z}^2 . Under Fourier transform $C_r^*(H)$ is isometrically isomorphic to $C(\mathbb{T}^2)$, the C*-algebra of continuous functions on the 2-torus

$\mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$, the norm on $C(\mathbb{T}^2)$ being the sup-norm. Thus

$$\|\rho(ac^{-1}) + \rho(bc^{-1}) + 1\| = \sup_{(z_1, z_2) \in \mathbb{T}^2} |z_1 + z_2 + 1| = 3.$$

We are now going to improve the bound of Proposition 4.2, and drop the assumption $\#X \geq 4$. For that, we shall have to exclude some presentations (like the one in Example 4.3). First we fix some notations. Let $\Gamma = \langle X : r \rangle$ be a one-relator group; fix $y \in X$. We define a new generating subset X_y as

$$X_y = \{xy^{-1} : x \in X, x \neq y\} \cup \{y\};$$

note that $\#X_y = \#X$. Now, let r_y be the word r written in the alphabet X_y ; more precisely, if we set $x' = xy^{-1}$ for $x \in X$, $x \neq y$, the word r_y is obtained from r by the change of variables (Nielsen transformation)

$$T_y : \begin{cases} x \rightarrow x'y & (x \neq y) \\ y \rightarrow y \end{cases}.$$

Then we define r'_y as the word obtained from r_y by cyclically reducing.

Lemma 4.4. *If r is a cyclically reduced word in $\mathbb{F}(X)$, then y is the only element that may disappear in r when T_y is applied. More precisely, if we denote by $(\alpha_1, \dots, \alpha_l)$ the ordered set of elements in $X \cup X^{-1} - \{y, y^{-1}\}$ appearing in r (i.e. $r = y^{\nu_1} \alpha_1 y^{\nu_2} \alpha_2 y^{\nu_3} \dots y^{\nu_l} \alpha_l y^{\nu_{l+1}}$ where $\nu_i \in \mathbb{Z}$), then the ordered set in r_y and hence in r'_y is $(\alpha'_1, \dots, \alpha'_l)$.*

Proof. We view T_y as an isomorphism from $\mathbb{F}(X)$ to $\mathbb{F}(X_y)$. Note that T_y^{-1} is defined on the generators of $\mathbb{F}(X_y)$ by $T_y^{-1}(x') = xy^{-1}$ for $x' \in X_y - \{y\}$ and $T_y^{-1}(y) = y$. Then for $r = y^{\nu_1} \alpha_1 y^{\nu_2} \alpha_2 y^{\nu_3} \dots y^{\nu_l} \alpha_l y^{\nu_{l+1}}$:

$$\begin{aligned} r_y &= T_y(r) = T_y(y^{\nu_1} \alpha_1 y^{\nu_2} \alpha_2 y^{\nu_3} \dots y^{\nu_l} \alpha_l y^{\nu_{l+1}}) \\ &= T_y(y^{\nu_1}) T_y(\alpha_1) T_y(y^{\nu_2}) T_y(\alpha_2) \dots T_y(y^{\nu_l}) T_y(\alpha_l) T_y(y^{\nu_{l+1}}) \\ &= y^{\mu_1} \alpha'_1 y^{\mu_2} \alpha'_2 y^{\mu_3} \dots y^{\mu_l} \alpha'_l y^{\mu_{l+1}}. \end{aligned}$$

Suppose that α'_i and α'_{i+1} cancel out in r_y . Then applying T_y^{-1} , we see that r cannot contain α_i and α_{i+1} , a contradiction. So the ordered set of r_y is exactly $(\alpha'_1, \dots, \alpha'_l)$.

As r is cyclically reduced, by the same argument, we conclude that we cannot cancel α'_1 and α'_l by cyclic permutation of r_y . So $(\alpha'_1, \dots, \alpha'_l)$ is also the ordered set of r'_y . That concludes the proof of 4.4. \square

Recall from the introduction that we defined a subgroup H_r (resp. H_l) of $\mathbb{F}(X)$ as the subgroup generated by all right (left) quotients xy^{-1} , with $x, y \in X$

(resp. all left quotients $x^{-1}y$, with $x, y \in X$); in passing, notice that H_r and H_l are free on $(\#X) - 1$ generators.

Lemma 4.5. *For r cyclically reduced in $\mathbb{F}(X)$, the following are equivalent:*

- (i) $r \in H_r \cup H_l$;
- (ii) For any $y \in X$, the letter y does not appear in r'_y ;
- (iii) There exists $y \in X$ such that y does not appear in r'_y .

Proof. (i) \Rightarrow (ii) Suppose that r is in H_r ; write $r = \prod_{i=1}^n a_i b_i^{-1}$ with $a_i, b_i \in X$.

We can have three kinds of factors $a_i b_i^{-1}$:

- (1) $a, b \in X - \{y\}$: then $T_y(ab^{-1}) = a'y y^{-1}(b')^{-1} = a'(b')^{-1}$
- (2) $T_y(ay^{-1}) = a'y y^{-1} = a'$
- (3) $T_y(ya^{-1}) = y(a'y)^{-1} = (a')^{-1}$.

Thus r_y does not contain y and so does r'_y .

If r is in H_l then the element $s = yry^{-1}$ belongs to H_l . So y does not appear in s_y , s_y is cyclically reduced (because r is cyclically reduced and Lemma 4.4) and $s_y = r'_y$.

(ii) \Rightarrow (iii) is obvious.

(iii) \Rightarrow (i) We assume that y does not appear in r'_y and analyse r in several steps. Write $r = a_1 \cdots a_n$, a word on $X \cup X^{-1}$, and look at the ordered subset $(a_{i_1}, \dots, a_{i_n})$ of all letters in r from $X \cup X^{-1} - \{y, y^{-1}\}$. By Lemma 4.4, the corresponding ordered set in r'_y is $(a'_{i_1}, \dots, a'_{i_n})$. We will systematically use the following argument: Suppose that, after applying T_y to each letter of r , we find a word r_y in which y appears and there is no obvious cancellation to remove it: Then it is really impossible to remove y , because it would be necessary first to remove some a'_{i_j} , contradicting Lemma 4.4.

First step: r has no subword of the form ab or $a^{-1}b^{-1}$, with $a, b \in X - \{y\}$. Indeed, if this would be the case, by applying T_y , we would get either $a'yb'$ or $(a')^{-1}y^{-1}(b')^{-1}$ and y would appear in r'_y .

2nd Step: r does not contains y^2 or y^{-2} . To see this, we suppose by contradiction that r contains such a subword, and show that r'_y contains y or y^{-1} . There are three cases to consider.

- (1) r contains $a^{\pm 1}y^n b^{\pm 1}$ with $n \geq 2$, ($a \neq y \neq b$). If the exponents of a and b are positive, then $ay^n b$ becomes $a'y^{n+1}b'$ after T_y . If the exponents of a and b are different, $ay^n b^{-1}$ becomes $a'y y^n y^{-1}(b')^{-1} = a'y^n (b')^{-1}$ and $a^{-1}y^n b$ becomes $(a')^{-1}y^n b'$. Finally if the exponents of a and b are negative, $a^{-1}y^n b^{-1}$ becomes $(a')^{-1}y^{n-1}(b')^{-1}$. The same can be done for n negative with $|n| \geq 2$.
- (2) r begins with $y^n b^{\pm 1}$ ($b \in X - \{y\}$). First assume $n \geq 2$. Since r is cyclically reduced, r ends with some letter $a \in X \cup X^{-1} - \{y^{-1}\}$. Then

$r_y = T_y(r)$ cannot end with y^{-1} , so there will be no cancellation when cyclically reducing r_y to get r'_y . On other hand, if r begins with $y^n b$, then r_y begin with $y^n b'$, and if r begins with $y^n b^{-1}$, then r_y begin with $y^{n-1}(b')^{-1}$. Since $n \geq 2$, y appears in r'_y . The same can be done for $n \leq -2$.

(3) Similar arguments hold if r ends with $b^{\pm 1} y^n$ ($b \in X - \{y\}$, $|n| \geq 2$).

Note that cases 2 and 3 also show that r cannot begin with yb or $y^{-1}b^{-1}$, neither end with by or $b^{-1}y^{-1}$ ($b \in X - \{y\}$).

3rd step: All exponents in r are equal to ± 1 , and exponents alternate in sign, i.e. $r = a_1^{-1}a_2a_3^{-1} \cdots a_n^{\pm 1}$ or $r = a_1a_2^{-1}a_3 \cdots a_n^{\pm 1}$. Indeed, we have to show that r contains no subword of the form ab or $a^{-1}b^{-1}$, for $a, b \in X$. We already know that this holds if either $a, b \in X - \{y\}$ (first step) or $a = b = y$ (second step). It remains to show that r contains no subword of the form ay or yb ($a, b \in X - \{y\}$), or an inverse of these. The remark at the end of the second step already shows that r cannot begin or end with such a subword. If $ayb^{\pm 1}$ appears then we see that after applying T_y , $ayb^{\pm 1}$ becomes either $a'yyb'y$ or $a'yyy^{-1}(b')^{-1} = a'y(b')^{-1}$. Similar arguments holds for yb , $a^{-1}y^{-1}$, $y^{-1}b^{-1}$.

Final step: To see that r is in $H_r \cup H_l$, we have to see that the exponent of a_n is the opposite of the exponent of a_1 . By contradiction suppose that a_1 and a_n have the same exponent $+1$ (resp. -1). Then $r = a_1a_2^{-1}a_3 \cdots a_n$ becomes, after applying T_y , $a'_1(a'_1)^{-1}a'_3 \cdots a'_n y$ so y appear in r'_y (because $a_1 \neq y^{-1}$). The argument for $r = a_1^{-1}a_2a_3^{-1} \cdots a_n^{-1}$ is similar. That ends the proof. \square

Remark. The group $\Gamma = \langle X : r \rangle$ clearly also admits the presentation $\Gamma = \langle X_y : r'_y \rangle$. If $r \in H_r \cup H_l$, Lemma 4.5 reveals that Γ is the free product of $\mathbb{Z} = \langle y \rangle$ with the one-relator group $\Gamma_y = \langle X_y - y : r'_y \rangle$. Now Shenitzer [She] has characterized those presentations $\Gamma = \langle X : r \rangle$ such that Γ is isomorphic to the free product of \mathbb{Z} with another group; the criterion is that at least one generator of X must disappear from r by applying Nielsen transformations. Our Lemma 4.5 however does not seem to be a consequence of the result in [She] because we only consider very special Nielsen transformations, namely the T_y 's.

Theorem 4.6. *Let $\Gamma = \langle X : r \rangle$ be a one-relator group, with $\#X \geq 2$, $|r| > 2$, r cyclically reduced and $r \notin H_r \cup H_l$. Then:*

$$\max\{r(h_X), r(h_S)\} \leq \|h_X\| = \frac{2\sqrt{\#X - 1}}{\#X}.$$

Proof. The inequality is proved as in Proposition 4.2. Fix $y \in X$; then

$$\|h_X\| = \frac{1}{\#X} \left\| 1 + \sum_{x \in X - \{y\}} \rho(xy^{-1}) \right\| = \frac{1}{\#X} \left\| 1 + \sum_{x' \in X_y - \{y\}} \rho(x') \right\|.$$

Since r is not in $H_r \cup H_l$, it follows from Lemma 4.5 that y appears in r'_y ; again by Magnus' Freiheitssatz, $X_y - \{y\}$ freely generates a free group on $(\#X) - 1$ generators; Lemma 4.1 then applies to give the result.

Note that Example 4.3 above of a presentation with $r \in H_r$, $\#X = 3$ and $\|h_X\| = 1$ shows that the assumption on r in Theorem 4.6 cannot be dropped. We now discuss somewhat the "exceptional" presentations in $H_r \cup H_l$. We choose to work with H_r (the analogous results for H_l following by interchanging left and right). \square

Proposition 4.7. *Let $\Gamma = \langle X : r \rangle$ be a one-relator presentation, with $r \in H_r$. Then X generates a free semi-group in Γ and $r(h_X) = \frac{1}{\sqrt{\#X}}$. Moreover $\text{Sp}(h_X)$ is a union of concentric circles centered at 0.*

Proof. Fix $y \in X$; as mentioned in the remark following Lemma 4.5, Γ is the free product of $\mathbb{Z} = \langle y \rangle$ with the one-relator group $\Gamma_y = \langle X_y - y : r_y \rangle$. We first prove that X generates a free semi-group. So, let w_1, w_2 be two distinct positive words in $\mathbb{F}(X)$; using the change of variables T_y together with the normal form for elements in a free product, we see that w_1 and w_2 define distinct elements of Γ . It follows from Theorem 1.2 that $\sigma(X) = \frac{1}{\sqrt{\#X}} \leq r(h_X)$. To prove the converse inequality $r(h_X) \leq \sigma(X)$, we appeal to Jolissaint's result from the Appendix: there exists a constant $C > 0$ such that, for any integer $k \geq 0$:

$$\|h_X^k\| \leq C(1+k)^3 \|h_X^k\|_2.$$

The desired inequality follows then straight from the definition of $\sigma(X)$. The final assertion follows from Proposition 2.1 by noticing that the sum of all exponents in r is 0. \square

Remarks.

- (1) In the case of h_S , it would of course be desirable to find an upper bound on $r(h_S)$ that depends on the relator r (for example on the length of r); but we did not succeed in achieving that. Note that such a lower bound for $r(h_S)$ was recently obtained by Paschke [Pas]: one has

$$r(h_S) \geq \min_{s>0} \left\{ \cosh(s) + (\#X - 1) Q \left(\frac{\cosh(|r|s) + 1}{\text{sh}(s) \text{sh}(|r|s)} \right) \right\}$$

where $Q(t) = \frac{\sqrt{t^2 + 1} - 1}{t}$.

- (2) Proposition 4.2 and Theorem 4.6 above show that one-relator groups are, in a certain sense, uniformly non-amenable: if we fix the number of generators, then the spectral radius of h_S is uniformly bounded away from 1. Since Proposition 4.2 and Theorem 4.6 provide upper bounds on $r(h_S)$, it is possible to deduce from them lower bounds on quantities that are known to depend on the width of the spectral gap, i.e. the quantity $\epsilon = 1 - r(h_S)$. One such quantity is the *Kazhdan constant* of the right regular representation ρ with respect to S , defined as

$$\kappa(\rho, S) = \inf_{\xi \in \ell^2(\Gamma), \|\xi\|=1} \max_{s \in S} \|\rho(s)\xi - \xi\|.$$

It is proved in Proposition I(6) of [HRV1] that $\kappa(\rho, \Gamma) \geq \sqrt{2}\epsilon$. Another such quantity is the isoperimetric constant of the Cayley graph $G(\Gamma, S)$, defined as

$$\iota(G(\Gamma, S)) = \inf \left\{ \frac{\#\partial A}{\#A} : A \text{ a finite subset of } \Gamma \right\};$$

here ∂A is the boundary of A , i.e. the set of edges of $G(\Gamma, S)$ with one extremity in A and the other in $\Gamma - A$. It is well-known that

$$\iota(G(\Gamma, S)) \geq |S|\epsilon$$

(see e.g. Theorem 3.3 of [Moh] for a slightly better inequality).

5. Some computations of spectra.

We recall that the reduced C^* -algebra of the group Γ , denoted by $C_r^*(\Gamma)$, is the C^* -algebra generated by $\rho(\Gamma)$. If Γ is torsion free, a tantalizing conjecture of Kaplansky and Kadison states that $C_r^*(\Gamma)$ has no idempotent except 0 and 1 (see [Val] for a survey). We explore here some consequences of this conjecture for one-relator groups.

Proposition 5.1. *Let $\Gamma = \langle X : r \rangle$ be a torsion-free one-relator group satisfying the Kaplansky-Kadison conjecture. Denote by Σ the sum of all exponents in r . Then:*

- (i) *If $\Sigma = 0$, then $\text{Sp}(h_X)$ is either a disk or an annulus centered at 0;*
- (ii) *If S is even, then $\text{Sp}(h_S)$ is an interval symmetric with respect to 0.*

Proof. Any element in $C_r^*(\Gamma)$ has a connected spectrum (otherwise, by holomorphic functional calculus, we construct non-trivial idempotents in $C_r^*(\Gamma)$).

If $\Sigma = 0$, Proposition 2.1 says that $\text{Sp}(h_X)$ is a union of concentric circles; by connectedness, it is either a disk or an annulus. Similarly, by connectedness $\text{Sp}(h_S)$ must be an interval, symmetric with respect to 0 if Σ is even. \square

Corollary 5.2. *Let Γ_g be the surface group $\langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^g [a_i, b_i] \rangle$ with $X = \{a_1, b_1, \dots, a_g, b_g\}$ and $g \geq 2$; then $\text{Sp}(h_S) = [-r, r]$ with $r \leq \frac{\sqrt{2g-1}}{g}$.*

Proof. The fact that Γ_g satisfies the Kaplansky-Kadison conjecture was proved by Kasparov [Kas]. The result then follows by combining Proposition 5.1 with Theorem 4.6.

Of course, in the case of Γ_g , it is somewhat frustrating that we are able to compute explicitly the spectrum of h_X (see Theorem 1.2), but not the spectrum of its real part h_S . \square

Example 4.3 revisited.

We consider again the group $\Gamma = \langle a, b, c : [ac^{-1}, bc^{-1}] \rangle$ with $X = \{a, b, c\}$. We claim that $\text{Sp}(h_X)$ is the disk $\{z \in \mathbb{C} : |z| \leq \frac{1}{\sqrt{3}}\}$. That the spectral radius is $\frac{1}{\sqrt{3}}$ follows from Proposition 4.7. Now, Γ is the free product $\mathbb{Z}^2 * \mathbb{Z}$, so Γ satisfies the Kaplansky-Kadison conjecture as a corollary of a result of Rosenberg ([Ros], Proposition 2.10). By Proposition 5.1, $\text{Sp}(h_X)$ is either a disk or an annulus centered at 0. To prove that it is a disk, we just have to prove that h_X is not invertible. But since $h_X = \frac{1}{3}(\rho(ac^{-1}) + \rho(bc^{-1}) + 1)\rho(c)$, it is enough to show that $\rho(ac^{-1}) + \rho(bc^{-1}) + 1$ is not invertible. Since ac^{-1} and bc^{-1} generate a subgroup isomorphic to \mathbb{Z}^2 , this follows from the fact that the function $(z_1, z_2) \rightarrow z_1 + z_2 + 1$ is not invertible on the 2-torus.

Appendix : An upper bound for the norms of powers of normalised adjacency operators.

By Paul Jolissaint

The aim of the present note is to prove a result needed in Proposition 3 of the previous article.

Let Γ be a group, let X be a finite subset of Γ containing 1, and let G be the free product $\Gamma * \langle y \rangle$ of Γ with an infinite cyclic group generated by y .

Set also $G_1 = \Gamma$, $G_2 = \langle y \rangle$, $X^* = X - \{1\}$ and $G_j^* = G_j - \{1\}$ for $j = 1, 2$.

Recall that any element ω of G can be uniquely written as a product $\omega = \omega_1 \dots \omega_n$ with $\omega_j \in G_{i_j}^*$, and $i_j \neq i_{j+1}$ for $j \leq n - 1$. The integer n is called the length of ω , and we denote by Λ_n the set of words of length n .

The main result of this appendix is :

Proposition 5.3. *With the notation above, set*

$$h = \frac{1}{|X|} \sum_{x \in X} \rho(xy).$$

Then there exists a positive constant C such that for every positive integer k , one has :

$$\|h^k\| \leq C(1+k)^3 \|h^k\|_2,$$

where $\| \cdot \|$ in the left hand side is the operator norm and h^k is the k^{th} convolution power of h .

$$\text{Let } \chi = \frac{1}{|X|} \sum_{x \in X} \delta_{xy}, \text{ so that } h = \rho(\chi).$$

It turns out that it will be more convenient to prove the inequality in Proposition 5.3 for $\lambda(\chi)$ instead of h , where λ denotes the left regular representation of G . (As λ and ρ are equivalent representations, this will prove Proposition 5.3, as well.)

If k is a positive integer, it is easy to check by induction on k that the function $\chi^k (= \chi * \dots * \chi, k \text{ times})$ is supported in $\bigcup_{1 \leq j \leq 2k} \Lambda'_j$, where Λ'_l is the set of reduced words $\omega = \omega_1 \dots \omega_l$ such that

- (a) either $\omega_j \in X^*$ or $\omega_j = y^{\mu_j}$ with $\mu_j > 0$;
- (b) $\omega_l = y^{\mu_l}$;
- (c) $\sum \mu_j \leq l$.

Hence Proposition 5.3 follows immediately from the slightly more general:

Proposition 5.4. *There exists a positive constant C such that for any finitely supported function φ on G whose support lies in $\bigcup_{1 \leq j \leq k} \Lambda'_j$ for some k , then*

$$\|\lambda(\varphi)\| \leq C(1+k)^3 \|\varphi\|_2.$$

The proof of Proposition 5.4 is similar to that of Theorem 2.2.2 of [Jol] and is based on an idea due to U. Haagerup in the case of the free group [Haa].

Using the same arguments as in the proof of Proposition 1.2.6 in [Jol], Proposition 5.4 is a consequence of the following result:

Proposition 5.5. *There exists a positive constant c such that for non-negative integers k, l and m satisfying: $|k-l| \leq m \leq k+l$, and for functions φ and ψ on G supported in Λ'_k and Λ_l respectively, one has*

$$\|(\varphi * \psi)\chi_{\Lambda_m}\|_2 \leq c(1+k) \|\varphi\|_2 \|\psi\|_2.$$

Let us recall the following result which is a special case of Lemma 2.2.1 of [Jol]:

Lemma 5.6. *Let k, l, m and q be non-negative integers such that $m = k + l - q$.*

If $\omega \in \Lambda_m$, let $\omega = \omega_1 \dots \omega_m$ be its reduced form.

Set also: $E_{k,l}(\omega) = \{(u, v) \in \Lambda_k \times \Lambda_l \mid uv = \omega\}$.

Then

- (1) *If $q = 2p$ is even, set $u_\omega = \omega_1 \dots \omega_{k-p}$ and $v_\omega = \omega_{k-p+1} \dots \omega_m$; then $E_{k,l}(\omega)$ is the set of pairs $(u, v) \in \Lambda_k \times \Lambda_l$ such that there exists $a \in \Lambda_p$ with*

$$u = u_\omega a \text{ and } v = a^{-1} v_\omega.$$

- (2) *If $q = 2p + 1$ is odd, set $u_\omega = \omega_1 \dots \omega_{k-p-1}$ and $v_\omega = \omega_{k-p+1} \dots \omega_m$; then $E_{k,l}(\omega)$ is the set of pairs $(u, v) \in \Lambda_k \times \Lambda_l$ such that there exists $a \in \Lambda_p$ and $b_1, b_2 \in \Lambda_1$ satisfying:*

$$b_1 b_2 = \omega_{k-p}, \quad u = u_\omega b_1 a \text{ and } v = a^{-1} b_2 v_\omega.$$

Proof of Proposition 5.5.

Set $m = k + l - q$; then q is an integer such that $0 \leq q \leq \min(k, l)$. We divide the proof into two cases:

Case 1. Assume that q is even. Using the first part of the above lemma, one shows more generally that if φ is supported in Λ_k then:

$$\|(\varphi * \psi)\chi_{\Lambda_m}\|_2 \leq \|\varphi\|_2 \|\psi\|_2.$$

The proof is exactly the same as that of Lemma 1.3 in [Haa].

Case 2. Assume that $q = 2p + 1$ is odd.

For every $\omega \in \Lambda_m$, set $E'_{k,l}(\omega) = \{(u, v) \in \Lambda'_k \times \Lambda_l \mid uv = \omega\}$. Let $\omega \in \Lambda_m$ be such that $E'_{k,l}(\omega) \neq \emptyset$ and let us write ω as $\omega = \omega_1 \dots \omega_{k-p-1} \omega_{k-p} \omega_{k-p+1} \dots \omega_m$ in its reduced form. Set $u_\omega = \omega_1 \dots \omega_{k-p-1}$ and $v_\omega = \omega_{k-p+1} \dots \omega_m$ as in the second part of the lemma. If $(u, v) \in E'_{k,l}(\omega)$, then $u = u_1 \dots u_{k-1} y^{\mu_k}$ and $v = v_1 \dots v_l$ for some u_i, v_i and $\mu_k > 0$.

Moreover there exists $a \in \Lambda_p$ and $b_1, b_2 \in \Lambda_1 = G_1^* \cup G_2^*$ such that $u = u_\omega b_1 a, v = a^{-1} b_2 v_\omega$ and $b_1 b_2 = \omega_{k-p}$.

Then two cases may occur:

- (i) $\omega_{k-p} \in G_1^*$: Thus $b_1, b_2 \in G_1^*$ and, since $u \in \Lambda'_k$, we must have $b_1 \in X^*, a \in \Lambda'_p$ and a begins with y^ν for some $\nu > 0$. (We will write: $a \in \Lambda'_p, a_1 = y^{\nu_1}$.)

- (ii) $\omega_{k-p} \in G_2^*$: Then $b_1, b_2 \in G_2^*$ and $a \in \Lambda'_p$ begins with some $x \in X^*$.

One has:

$$\begin{aligned}
 (1) \quad \|\varphi * \psi\|_{\chi_{\Lambda_m}}^2 &= \sum_{\omega \in \Lambda_m} |(\varphi * \psi)(\omega)|^2 \\
 &= \sum_{\omega \in \Lambda_m} \left| \sum_{(u,v) \in E'_{k,i}(\omega)} \varphi(u)\psi(v) \right|^2 \\
 &= \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_1^*}} \left| \sum_{(u,v) \in E'_{k,i}(\omega)} \varphi(u)\psi(v) \right|^2 \\
 &\quad + \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_2^*}} \left| \sum_{(u,v) \in E'_{k,i}(\omega)} \varphi(u)\psi(v) \right|^2.
 \end{aligned}$$

Denote by Σ_1 (resp. Σ_2) the first (resp. the second) sum in the right handside of (1).

Let us estimate Σ_1 first:

$$\begin{aligned}
 \Sigma_1 &= \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_1^*}} \left| \sum_{\substack{a \in \Lambda'_p \\ a_1 = y^\nu}} \sum_{x \in X^*} \varphi(u_\omega x a) \psi(a^{-1}(x^{-1}\omega_{k-p})v_\omega) \right|^2 \\
 &\leq \sum_{\substack{u \in \Lambda_{k-p-1} \\ u_{k-p-1} \in G_2^*}} \sum_{\substack{v \in \Lambda_{l-p-1} \\ v_1 \in G_2^*}} \sum_{\omega_{k-p} \in G_1^*} \left| \sum_{\substack{a \in \Lambda'_p \\ a_1 = y^\nu}} \sum_{x \in X^*} \varphi(uxa) \psi(a^{-1}(x^{-1}\omega_{k-p})v) \right|^2 \\
 &\leq \sum_{\substack{u \dots \\ v \dots \\ \omega \dots}} \sum_{\omega \in G_1^*} \left(\sum_{\substack{a \in \Lambda'_{p+1} \\ a_1 \in X^*}} |\varphi(ua)| |\psi(a^{-1}\omega v)| \right)^2 \\
 &\leq \sum_{\substack{u \dots \\ v \dots \\ \omega \dots}} \left(\sum_{a \dots} |\varphi(ua)|^2 \right) \left(\sum_{a \dots} |\psi(a^{-1}\omega v)|^2 \right) \\
 &= \|\varphi\|_2^2 \sum_{\substack{v \in \Lambda_{l-p} \\ v_1 \in G_1^*}} \sum_{\substack{a \in \Lambda'_{p+1} \\ a_1 \in X^*}} |\psi(a^{-1}v)|^2.
 \end{aligned}$$

But

$$\sum_{\substack{v \in \Lambda_{l-p} \\ v_1 \in G_1^*}} \sum_{\substack{a \in \Lambda'_{p+1} \\ a_1 \in X^*}} |\psi(a^{-1}v)|^2 \leq (|X| - 1) \|\psi\|_2^2.$$

Let us finally estimate Σ_2 :

We are going to use the fact that if f is a finitely supported function on $G_2 (= \mathbb{Z})$, then $\|\lambda(f)\| \leq \frac{\pi}{\sqrt{3}} \|f\|_{2,1}$, where $\|f\|_{2,1}^2 = \sum_{x \in \mathbb{Z}} |f(x)|^2 (1 + |x|)^2$. (See [Jol], Example 1.2.3.)

Then:

$$\Sigma_2 = \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_2^*}} \left| \sum_{\substack{a \in \Lambda'_p \\ a_1 \in X^*}} \lambda_{G_2}(\varphi_{u_\omega, a}) \psi_{v_\omega, a}(\omega_{k-p}) \right|^2,$$

where we set, for $u \in \Lambda_{k-p-1}$, $a \in \Lambda'_p$ and $v \in \Lambda_{l-p-1}$: $\varphi_{u,a}(x) = \varphi(uxa)$ and $\psi_{v,a}(x) = \psi(a^{-1}xv)$ for $x \in G_2$.

Hence

$$\begin{aligned} \Sigma_2 &\leq \sum_{\substack{u \in \Lambda_{k-p-1} \\ u_{k-p-1} \in X^*}} \sum_{\substack{v \in \Lambda_{l-p-1} \\ v_1 \in G_1^*}} \left\| \sum_{\substack{a \in \Lambda'_p \\ a_1 \in X^*}} \lambda_{G_2}(\varphi_{u,a}) \psi_{v,a} \right\|_2^2 \\ &\leq \sum_{\substack{u \dots \\ v \dots}} \left(\sum_{a \dots} \|\lambda_{G_2}(\varphi_{u,a})\|^2 \right) \left(\sum_{a \dots} \|\psi_{v,a}\|_2^2 \right) \\ &\leq \frac{\pi^2}{3} \left(\sum_{\substack{u \dots \\ a \dots}} \|\varphi_{u,a}\|_{2,1}^2 \right) \left(\sum_{\substack{v \dots \\ a \dots}} \|\psi_{v,a}\|_2^2 \right). \end{aligned}$$

But

$$\sum_{\substack{v \dots \\ a \dots}} \|\psi_{v,a}\|^2 = \|\psi\|_2^2 \quad \text{and}$$

$$\begin{aligned} \sum_{\substack{u \dots \\ a \dots}} \|\varphi_{u,a}\|_{2,1}^2 &= \sum_{u \dots} \sum_{a \dots} \sum_{x \in \{y^\nu; 1 \leq \nu \leq k\}} |\varphi(uxa)|^2 (1 + |x|)^2 \\ &\leq (1 + k)^2 \|\varphi\|_2^2. \end{aligned}$$

□

References

[AkO] C.A. Akemann and P.A. Ostrand, *Computing norms in group C*-algebras*, Amer. J. Math., **98** (1976), 1015-1047.

- [Car] D.I. Cartwright, *Singularities of the Green function of a random walk on a discrete group*, Mh. Math., **113** (1992), 183-188.
- [CS1] D.I. Cartwright and P.M. Soardi, *Harmonic analysis on the free product of two cyclic groups*, J. Funct. Anal., **65** (1986), 147-171.
- [CS2] ———, *Random walk on free products, quotients and amalgams*, Nagoya Math. J., **102** (1986), 163-180.
- [Ch1] C. Champetier, *Cocroissance des groupes à petite simplification*, Bull. London Math. Soc., **25** (1993), 438-444.
- [Ch2] ———, *Propriétés statistiques des groupes de présentation finie*, Adv. in Maths., **116** (1995), 197-262.
- [Day] M.M. Day, *Convolutions, means and spectra*, Illinois J. Math., **8** (1964), 100-111.
- [GhH] E. Ghys and P. de la Harpe eds., *Sur les groupes hyperboliques d'après M. Gromov*, Progress in Maths., **83**, Birkhäuser, 1990.
- [Gri] R.I. Grigorchuk, *Symmetrical random walks on discrete groups*, in "Multicomponent random systems", Nauk, Moscow, 1978.
- [Gro] M. Gromov, *Hyperbolic groups*, in "Essays in Group Theory", ed. S.M. Gersten, M.S.R.I. Publ., **8**, Springer, 1987, 75-263.
- [Haa] U. Haagerup, *An example of a non nuclear C^* -algebra which has the metric approximation property*, Inventiones Mathematicae, **50** (1979), 279-293.
- [HRV1] P. de la Harpe, A.G. Robertson and A. Valette, *On the spectrum of the sum of generators for a finitely generated group*, Israel J. of Maths., **81** (1993), 65-96.
- [HRV2] ———, *On the spectrum of the sum of generators for a finitely generated group II*, Colloquium Math., **LXV** (1993), 87-102.
- [IoP] A. Iozzi and M.A. Picardello, *Spherical functions on symmetric graphs*, in Conference in Harmonic Analysis, Lecture Notes in Math., **992**, Springer (1983), 87-104.
- [Jol] P. Jolissaint, *Rapidly decreasing functions in reduced C^* -algebras of groups*, Transactions of Amer. Math. Soc., **317** (1990), 167-196.
- [Juh] A. Juhasz, *On the hyperbolicity and non-hyperbolicity of one-relator groups I*, preprint, Haifa Technion, 1993.
- [Kas] G.G. Kasparov, *Lorentz groups: K -theory of unitary representations and crossed products*, Dokl. Akad. Nauk SSSR, **275** (1984), 541-545 (in Russian).
- [Ke1] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc., **92** (1959), 336-354.
- [Ke2] ———, *Full Banach mean values on countable groups*, Math. Scand., **7** (1959), 146-156.
- [KeS] H. Kesten and F. Spitzer, *Random walk on countably infinite abelian groups*, Acta. Math., **114** (1965), 237-265.
- [KuS] G. Kuhn and P.M. Soardi, *The Plancherel measure for polygonal graphs*, Ann. Mat. Pura Appl., **134** (1983), 393-401.
- [Lyn] R.C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math., **52** (1950), 650-665.
- [LyS] R.C. Lyndon and P.E. Schupp, *Combinatorial group theory*, Ergebnisse der Math., **89**, Springer, 1977.

- [Mlo] W. Mlotkowski, *Positive definite functions on free products of groups*, Boll. Un. Mat. Ital., **3-B** (1989), 343-355.
- [Moh] B. Mohar, *Some relations between analytic and geometric properties of infinite graphs*, Discrete Maths., **95** (1991), 193-219.
- [New] B.B. Newman, *Some results on one-relator groups*, Bull. Amer. Math. Soc., **74** (1968), 568-571.
- [Pas] W.L. Paschke, *Lower bound for the norm of a vertex-transitive graph*, Math. Z., **213** (1993), 225-239.
- [Ro] J. Rosenberg, *C^* -algebras, positive scalar curvature and the Novikov conjecture*, Publ. Math. IHES, **58** (1983), 197-212.
- [Ser] J.-P. Serre, *Arbres, amalgames, SL_2* , Astérisque, **46**, Soc. Math. France, 1977.
- [She] A. Shenitzer, *Decomposition of a group with a single defining relation into a free product*, Proc. Amer. Math. Soc., **6** (1955), 273-279.
- [Sta] J.R. Stallings, *Groups of dimension 1 are locally free*, Bull. Amer. Math. Soc., **74** (1968), 361-364.
- [Str] R. Strelbel, *Small cancellation groups*, Appendix of [GhH], pp. 227-273.
- [Val] A. Valette, *The conjecture of idempotents: a survey of the C^* -algebraic approach*, Bull. Soc. Math. Belgique, **XLI** (1989), 485-521.
- [Woe] W. Woess, *A short computation of the norms of free convolution operators*, Proc. Amer. Math. Soc., **96** (1986), 167-170.

Received May 15, 1994 and revised March 5, 1995.

RUE EMILE-ARGAND 11
CH-2007 NEUCHÂTEL
SWITZERLAND

Added on proof:

1) It has been pointed out to us by P. de la Harpe that, by combining Kesten's results [Ke1] with the Freiheitssatz, the upper bound on $r(h_S) = \|h_S\|$ in Proposition 4.2 can be improved, to the effect that

$$r(h_S) \leq \frac{\sqrt{2(\#X) - 3} + 1}{\#X}.$$

In particular, for $\#X \rightarrow \infty$, one sees that $r(h_S)$ behaves like $\sqrt{\frac{2}{\#X}}$.

2) The first author has recently extended the genericity result, Theorem 3.12, to all finitely presented groups; see P.-A. Cherix, *Generic result for the existence of a free semi-group*, Séminaire de théorie spectrale et géométrie, **13** (1994-95), Institut Fourier, Grenoble.

3) The Kaplansky-Kadison conjecture has now been proved for the class of torsion-free one-relator groups (see C. Béguin, H. Bettaieb & A. Valette, *The Baum-Connes conjecture for torsion-free one-relator groups*, preprint Neuchâtel, 1996). So this extra assumption can be dropped from Proposition 5.1.

Mogens L. Hansen and Richard V. Kadison , Banach algebras with unitary norms	535
Xin-hou Hua , Sharing values and a problem due to C.C. Yang	71
Jing-Song Huang , Harmonic analysis on compact polar homogeneous spaces	553
Min-Jei Huang , Commutators and invariant domains for Schrödinger propagators	83
Hisao Kato , Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs	93
Oliver Küchle , Some properties of Fano manifolds that are zeros of sections in homogeneous vector bundles over Grassmannians	117
Xin Li and Francisco Marcellan , On polynomials orthogonal with respect to Sobolev inner product on the unit circle	127
Steven Liedahl , Maximal subfields of $Q(i)$ -division rings	147
Alan L.T. Paterson , Virtual diagonals and n -amenability for Banach algebras	161
Claude Schochet , Rational Pontryagin classes, local representations, and K^G -theory	187
Sandra L. Shields , An equivalence relation for codimension one foliations of 3-manifolds	235
D. Siegel and E. O. Talvila , Uniqueness for the n -dimensional half space Dirichlet problem	571
Aleksander Simonič , A Construction of Lomonosov functions and applications to the invariant subspace problem	257
Endre Szabó , Complete intersection subvarieties of general hypersurfaces ..	271

PACIFIC JOURNAL OF MATHEMATICS

Volume 175 No. 2 October 1996

Mean-value characterization of pluriharmonic and separately harmonic functions	295
LEV ABRAMOVICH AĪZENBERG, CARLOS A. BERENSTEIN and L. WERTHEIM	
Convergence for Yamabe metrics of positive scalar curvature with integral bounds on curvature	307
KAZUO AKUTAGAWA	
Generalized modular symbols and relative Lie algebra cohomology	337
AVNER DOLNICK ASH and DAVID GINZBURG	
Convolution and limit theorems for conditionally free random variables	357
MAREK BOŹEJKO, MICHAEL LEINERT and ROLAND SPEICHER	
L^p -bounds for hypersingular integral operators along curves	389
SHARAD CHANDARANA	
On spectra of simple random walks on one-relator groups. With an appendix by Paul Jolissain	417
PIERRE-ALAIN CHERIX, ALAIN J. VALETTE and PAUL JOLISSAINT	
Every stationary polyhedral set in \mathbf{R}^n is area minimizing under diffeomorphisms	439
JAIGYOUNG CHOE	
Ramanujan's master theorem for symmetric cones	447
HONGMING DING, KENNETH I. GROSS and DONALD RICHARDS	
On norms of trigonometric polynomials on $SU(2)$	491
ANTHONY H. DOOLEY and SANJIV KUMAR GUPTA	
On the symmetric square. Unit elements	507
YUVAL ZVI FLICKER	
Stable constant mean curvature surfaces minimize area	527
KARSTEN GROSSE-BRAUCKMANN	
Banach algebras with unitary norms	535
MOGENS LEMVIG HANSEN and RICHARD VINCENT KADISON	
Harmonic analysis on compact polar homogeneous spaces	553
JING-SONG HUANG	
Uniqueness for the n -dimensional half space Dirichlet problem	571
DAVID SIEGEL and ERIK O. TALVILA	