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EVERY STATIONARY POLYHEDRAL SET IN R^n IS AREA MINIMIZING UNDER DIFFEOMORPHISMS

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It is shown that every stationary polyhedral set in the Euclidean space is area minimizing under diffeomorphisms leaving the boundary fixed. Similar theorems are also proved for crystals and immiscible fluids.

There are infinitely many minimal cones in R^3. Of these only three are area minimizing under Lipschitz maps: the plane; the three half planes meeting along their common boundary line at an angle of 120 degrees; the cone over the one-skeleton of the regular tetrahedron (see [**T**]). A recent work of Lawlor and Morgan [**LM**] has produced a generalization to higher dimensional cones in R^n. Namely, the hypercone over the $(n-2)$-skeleton of the regular simplex in R^n has the least area among all surfaces separating the $(n-1)$-dimensional faces of the simplex. Consequently this cone is area minimizing under Lipschitz maps. Moreover, Brakke has proved that the hypercone over the $(n-2)$-skeleton of the cube in R^n is area minimizing under Lipschitz maps when and only when $n \geq 4$ [**B**].

In this paper we prove that every stationary polyhedral set in R^n is area minimizing under *diffeomorphisms* leaving the boundary fixed. Therefore if we only consider competing surfaces of diffeomorphic images of a minimal cone C in R^3, C has the least area. Hence in R^3 all minimal cones are *stable*.

We wish to thank Frank Morgan for helpful comments on the extension of the main theorem.

1. Terminology.

An m-dimensional set $C \subset R^n$ is said to be *polyhedral* if there exist m-dimensional planes $\{\prod_i\}_{i\in I}$ in R^n such that $C \subset \bigcup_{i\in I}\prod_i$. Each m-dimensional set $F_i = C\cap\prod_i$ is called a *face* of C. The *singular* set S of C is the largest $(m-1)$-dimensional subset of C which lies inside $\bigcup_{i\neq j}\left(\prod_i\cap\prod_j\right)$. A *singular edge* of C is the $(m-1)$-dimensional subset E of S defined by $E = C\cap\left(\prod_i\cap\prod_j\right)$ for each pair $i, j \in I$. So each singular edge of C is the intersection of two faces of C. For each face F_i of C the *boundary edge* B_i of C in F_i is the closure of $\partial F_i \sim S$. The union of all boundary edges of C is called the *boundary* ∂C of C.

A polyhedral set $C \subset R^n$ is said to be *area minimizing under diffeomorphisms* (*Lipschitz maps*, respectively) if

$$\text{Volume}(C) \leq \text{Volume}(\varphi(C))$$

for any diffeomorphism (Lipschitz map, respectively) φ of R^n leaving the boundary of C fixed.

A polyhedral set $C \subset R^n$ is *stationary* if

$$\frac{d}{dt}\text{Volume}(\phi_t(C))\,|_{t=0} = 0$$

for any 1-parameter family of diffeomorphisms $\{\phi_t\}_{-1<t<1}$ of R^n with $\phi_0 =$ id and leaving ∂C fixed. A stationary polyhedral set C is said to be *stable* if

$$\frac{d^2}{dt^2}\text{Volume}(\phi_t(C))\,|_{t=0} \geq 0$$

for any $\{\phi_t\}$ as above.

2. Main theorem.

Theorem 1. *Every m-dimensional stationary polyhedral set C in R^n is area minimizing under diffeomorphisms.*

Proof. Let $C \subset R^n$ be an m-dimensional stationary polyhedral set with faces $\{F_i\}_{i\in I}$, boundary edges $\{B_i\}_{i\in I}$, and singular edges $\{E_j\}_{j\in J}$. Then $\bigcup_{i\in I} B_i$ is the boundary of C, $\bigcup_{j\in J} E_j$ is the singular set S of C, and $\bigcup_{i\in I} F_i$ becomes C itself.

In order to simplify computations we shall frequently classify the singular edges of C in terms of the faces. For this purpose let us double-index $\{E_j\}_{j\in J}$ using two indices: Given a face F_i, define a reindexed set $\{E_{i1}, E_{i2}, \cdots, E_{im_i}\}$ to be the set of all singular edges E_j such that $E_j \subset \partial F_i$. Hence

$$\partial F_i = B_i \cup E_{i1} \cup \cdots \cup E_{im_i}.$$

Similarly we classify the faces of C in terms of the singular edges: Given a singular edge E_j, define

$$\{F_{j1}, F_{j2}, \cdots, F_{jn_j}\} = \{F_i : \partial F_i \supset E_j\}.$$

Then

$$E_j = \bigcap_{k=1}^{n_j} \partial F_{jk}.$$

Since a face contains at least two singular edges, each face is double-indexed at least two times. Likewise each singular edge is double-indexed at least three times because a singular edge is part of the boundary of at least three faces.

Let φ be a diffeomorphism of R^n leaving ∂C fixed. Since φ is homotopic to the identity map in R^n, the singular set S is *homologous* to $\varphi(S)$. More precisely, let φ_t be the homotopy from the identity to φ. For each singular edge E_j, there exists an m-dimensional smooth submanifold G_j of R^n such that G_j is the set swept out by $\varphi_t(E_j)$. Clearly $\partial G_j \supset E_j \cup \varphi(E_j)$. $\{G_j\}_{j\in J}$ can be double-indexed in the same manner as $\{E_j\}$: We write $G_j = G_{ik}$ if $E_j = E_{ik}$. Similarly, given a face F_i, there exists an $(m+1)$-dimensional smooth submanifold D_i swept out by $\varphi_t(F_i)$ such that

$$\partial D_i = F_i \cup \varphi(F_i) \cup G_{i1} \cup \cdots \cup G_{im_i}.$$

Now let us equip submanifolds F_i, $\varphi(F_i)$, G_{ik}, and D_i with appropriate orientations in such a way that

$$\partial D_i = F_i - \varphi(F_i) + G_{i1} + \cdots + G_{im_i}. \tag{1}$$

One can find a coordinate system $\{x_1, \cdots, x_n\}$ in R^n such that the coordinate frame fields $\partial/\partial x_1, \cdots, \partial/\partial x_n$ are orthonormal and $\partial/\partial x_1, \cdots, \partial/\partial x_m$ are parallel to F_i. Define

$$\omega_i = dx_1 \wedge \cdots \wedge dx_m.$$

Then $d\omega_i = 0$. Reordering $\{x_1, \cdots, x_m\}$ according to the orientation of F_i, if necessary, one sees that

$$\int_{F_i} \omega_i = \text{Volume}(F_i). \tag{2}$$

Then

$$0 = \int_{D_i} d\omega_i = \int_{\partial D_i} \omega_i = \text{Volume}(F_i) - \int_{\varphi(F_i)} \omega_i + \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i. \tag{3}$$

Let ξ be the volume form of $\varphi(F_i)$ and ξ^* the m-vector on $\varphi(F_i)$ with $\xi(\xi^*) = 1$. Note that $\omega_i(\xi^*) \leq 1$ and

$$\int_{\varphi(F_i)} \omega_i = \int_{\varphi(F_i)} \omega_i(\xi^*)\xi \leq \text{Volume}(\varphi(F_i)).$$

Then summing up (3) for all $i \in I$, we have

$$\text{Volume}(C) = \sum_{i\in I} \text{Volume}(F_i) \leq \text{Volume}(\varphi(C)) - \sum_{i\in I}\sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i.$$

Here let us double-index $\{\omega_i\}_{i\in I}$ such that $\omega_i = \omega_{jk}$ if $F_i = F_{jk}$. Then rearranging the summation in terms of the singular edges gives

$$\text{Volume}(C) \leq \text{Volume}(\varphi(C)) - \sum_{j\in J}\sum_{k=1}^{n_j}\int_{G_j}\omega_{jk}. \tag{4}$$

In the integral in (4), however, the orientation of G_j is ambiguous since it depends on the orientation of F_{jk} for each k subject to (1). But since $\int_{G_j}\omega_{jk} = \int_{-G_j}-\omega_{jk}$, one can fix the orientation of G_j by taking the negative of ω_{jk} if necessary. Then through (1) the orientation of G_j determines that of F_{jk}, which in turn determines ω_{jk} through (2). Now m-forms $\omega_{j1},\cdots,\omega_{jn_j}$ can be expressed more explicitly as follows. Let ν_k be a unit constant vector field in R^n parallel to F_{jk} and perpendicular to E_j. Assume further that $\nu_k \mid F_{jk}$ points inward along E_j. Define θ_k to be the 1-form in R^n dual to ν_k, i.e., $\theta_k(v) = \nu_k \cdot v$ for any vector field v. Let η_j be a volume form of E_j for an appropriate orientation of E_j. Then one can easily check that

$$\omega_{jk} = \eta_j \wedge \theta_k.$$

The stationarity of C states that

$$\sum_{k=1}^{n_j}\nu_k = 0 \quad \text{and} \quad \sum_{k=1}^{n_j}\theta_k = 0.$$

Therefore

$$\sum_{k=1}^{n_j}\omega_{jk} = 0.$$

Thus it follows from (4) that

$$\text{Volume}(C) \leq \text{Volume}(\varphi(C)).$$

□

Corollary 1. *Every stationary cone in R^3 is area minimizing under diffeomorphisms.*

Proof. All stationary cones in R^3 are polyhedral. □

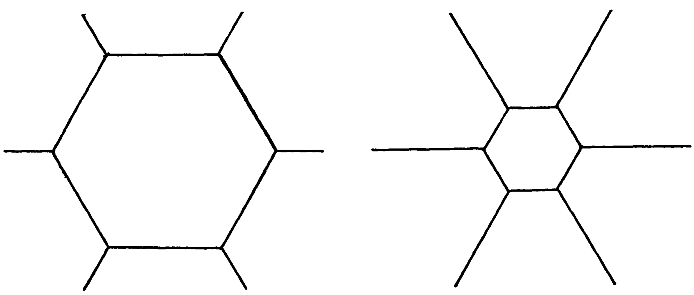
Corollary 2. *Every m-dimensional stationary polyhedral set in R^n is stable.*

Remarks. i) In fact, the set of competing surfaces in Theorem 1 can be enlarged from diffeomorphic images $\varphi(C)$ of C to the surfaces *homologous* to C: If $\tilde{C} = \bigcup_{i \in I} \tilde{F}_i$, $\tilde{C} \supset \partial C$, and if each $\tilde{F}_i$ satisfies $\partial D_i = F_i \cup \tilde{F}_i \cup G_{i1} \cup \cdots \cup G_{im_i}$, then Volume $(\tilde{C}) \geq$ Volume(C).

ii) There are nonpolyhedral stationary cones that are unstable: e.g. the cones over $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ in R^4 and over $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$ in R^6 [S].

iii) B. White has also shown that stationary polyhedral cones are always stable.

iv) It should be mentioned that not every stationary polyhedral set is a unique minimizer. Figure below illustrates two diffeomorphic 1-dimensional stationary polyhedral sets of equal length. However, if we assume that each face of the m-dimensional stationary polyhedral set C has nonempty intersection with the boundary of C, then Volume$(\varphi(C))$ = Volume(C) for a diffeomorphism φ leaving ∂C fixed if and only if $\varphi(C) = (C)$. This is because Volume$(\varphi(C))$ = Volume(C) if and only if $\omega_i(\xi^*) = 1$ on $\varphi(F_i)$ for all $i \in I$ if and only if $\varphi(F_i) = F_i$ or $\varphi(C) = C$. But is there an m-dimensional stationary polyhedral set which is not a unique minimizer? Indeed it seems to be an interesting problem to find an m-dimensional ($m \geq 2$) stationary polyhedral set C which has an interior face, i.e., a face disjoint from the boundary of C (like the edges of the hexagons in the 1-dimensional polyhedral sets of figure below)



Two diffeomorphic stationary sets of equal length.

3. Extensions to crystals and immiscible fluids.

Crystals tend to minimize the surface energe which is given by an integral $\int_S \Psi(n)$ in which the weighting of area depends on the unit normal n at each point. Immiscible fluids try to minimize the total interface energy. This energy is proportional to area, but the constant of proportionality depends on a pair of fluids separated by the interface. In this section we extend Theorem 1 to the stationary polyhedral hypersurfaces (interfaces) of crystals and immiscible fluids.

Definition. A norm Ψ in R^n is a homogeneous convex function on R^n, positive except at 0. The dual norm Ψ^* is defined by

$$\Psi^*(w) = \sup\{w \cdot v : \Psi(v) = 1\}.$$

It follows immediately that

$$v \cdot w \leq \Psi(v)\Psi^*(w).$$

If equality holds, we say that w is dual to v. One can easily see that w is dual to a Ψ-unit vector v when w is an outward-pointing normal to the unit Ψ-ball at v.

For a hypersurface S in R^n with a unit normal n, the *energy* $\Psi(S)$ of S associated with the norm Ψ is defined by

$$\Psi(S) = \int_S \Psi(n).$$

Theorem 2. *Let Ψ be a norm in R^n , and let C be an $(n-1)$-dimensional polyhedral set in R^n which is stationary with respect to the Ψ-energy. Then C is energy minimizing under diffeomorphisms, i.e., for any diffeomorphism φ leaving ∂C fixed,*

$$\Psi(C) \leq \Psi(\varphi(C)).$$

Proof. For the faces and edges of C and their "swept-out" sets, we use the same notations $F_i, F_{jk}, E_j, D_i, G_j, G_{ik}$ as used in the proof of Theorem 1. Also we employ the same double-indexing convention as used there. Let n_i be a unit normal to F_i. Extend n_i to a constant unit vector field n_i in R^n. Let n_i^* be the Ψ^*-unit vector field dual to n_i, that is,

$$v \cdot n_i^* \leq \Psi(v)$$

with equality for $v = n_i$. Let dV be the volume form of R^n and define

$$\omega_i = n_i^* \lrcorner\, dV.$$

Then $d\omega_i = 0$. Hence

$$\begin{aligned}\Psi(F_i) &= \int_{F_i} \Psi(n_i) = \int_{F_i} n_i \cdot n_i^* = \int_{F_i} \omega_i \\ &= \int_{\varphi(F_i)} \omega_i - \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i = \int_{\varphi(F_i)} \nu \cdot n_i^* - \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i\end{aligned}$$

where ν is a unit normal to $\varphi(F_i)$. Therefore

$$\text{(5)} \qquad \Psi(F_i) \leq \int_{\varphi(F_i)} \Psi(\nu) - \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i = \Psi(\varphi(F_i)) - \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i .$$

Adding up (5) for all i gives

$$\text{(6)} \qquad \Psi(C) \leq \Psi(\varphi(C)) - \sum_{i \in I} \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i .$$

Now, for a singular edge E_j, the faces $F_{j1}, F_{j2}, \cdots, F_{jn_j}$ are assumed to be indexed in the order they appear around E_j. Also the unit normals n_{jk} to F_{jk} are chosen in such a way that n_{jk} points from F_{jk} to $F_{j(k+1)}$ (to F_{j1} if $k = n_j$). Then the stationarity of C implies that

$$\sum_{k=1}^{n_j} n_{jk}^* = 0.$$

(See [**LM**, Theorem 4.2].) Hence we have

$$\sum_{k=1}^{n_j} \omega_{jk} = 0.$$

Therefore the last term in (6) vanishes by the same reason as in the proof of Theorem 1. □

Definition. Let S be a union of hypersurfaces S_i of R^n and a_i the multiplicity constant (interface energy) of S_i. Given a diffeomorphism φ of R^n, define the *total interface energy* $M(\varphi(S))$ of $\varphi(S)$ by

$$M(\varphi(S)) = \sum_i a_i \operatorname{Volume}(\varphi(S_i)).$$

Theorem 3. *Given an $(n-1)$-dimensional polyhedral set C in R^n with faces F_i of multiplicity a_i, suppose C is stationary with respect to the total interface energy. Then C is energy minimizing in its diffeomorphism class.*

Proof. Employing the same notations as in the proof of Theorem 2, we define an $(n-1)$-form ω_i in R^n by

$$\omega_i = a_i n_i \lrcorner \, dV.$$

Then $d\omega_i = 0$ and so

$$M(F_i) = \int_{F_i} a_i = \int_{F_i} \omega_i = \int_{\varphi(F_i)} \omega_i - \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i$$
$$= \int_{\varphi(F_i)} a_i n_i \cdot \nu - \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i \leq M(\varphi(F_i)) - \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i .$$

Hence

$$M(C) \leq M(\varphi(C)) - \sum_{i \in I} \sum_{k=1}^{m_i} \int_{G_{ik}} \omega_i .$$

Since C is stationary, we have

$$\sum_{k=1}^{n_j} a_{jk} n_{jk} = 0,$$

where a_{jk} is the multiplicity of the face F_{jk}, and n_{jk} the unit normal to F_{jk}. Hence

$$\sum_{k=1}^{n_j} \omega_{jk} = 0$$

and we get

$$M(C) \leq M(\varphi(C)),$$

for any diffeomorphism φ leaving ∂C fixed. □

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