RAMANUJAN'S MASTER THEOREM FOR SYMMETRIC CONES

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Dedicated to R. A. Kunze, on the occasion of his sixty-fifth birthday

The Master Theorem of Ramanujan (1913), so named because of its centrality in much of Ramanujan's work on definite integrals, hypergeometric functions, and series expansions, relates coefficients in the Taylor's expansion of a function to the Mellin transform of the function over the interval $(0, \infty)$. In this paper we extend the setting of this classical theorem to apply to spherical series and spherical transforms on symmetric cones (also known as domains of positivity). To illustrate the range of applications of this theorem we obtain higher dimensional analogues of Carlson's uniqueness theorem for holomorphic functions, Newton's interpolation formula, and Mellin-Barnes integrals for certain hypergeometric functions.

Introduction.

Srinivasa Ramanujan Aiyangar, otherwise known as Ramanujan, needs no introduction, either to professional mathematicians or mathematical historians. He was born in poverty in India in 1887 and died not far from his place of birth at age 32. He was a self-taught mathematical genius possessing exceptional mathematical powers and a special originality and insight that defies comparison. Upon his death Ramanujan left a mountain of unpublished work, a great amount of which was contained in three Notebooks, together with three Quarterly Reports that were written in 1913 and communicated to the Board of Studies at University of Madras, where he was supported by a small research scholarship. Remarkably, only in the last decade, with the appearance of the wonderful book [1] by Berndt, has the material contained in these notebooks and reports been organized, analyzed, and published.

The subject of this paper is the generalization to symmetric cones (also known as domains of positivity) of Ramanujan's fundamental discovery described in his first quarterly report. Because this result was a touchstone for Ramanujan throughout his work on definite integrals, hypergeometric functions, and series expansions, it is known as *Ramanujan's Master Theorem*.

Here is what Ramanujan wrote of his discovery in the cover letter to his first Quarterly Report [1, p. 297]:

"The Progress Report is merely the exposition of a new theorem I have discovered in Integral Calculus. At present there are many definite integrals the values of which we know to be finite but still not possible of evaluation by the present known methods. This method will be an instrument by which at least some of the definite integrals whose values are at present not known can be evaluated. ... The investigations I have made on the basis of this theorem are not all contained in the attached paper. There is ample scope for new and interesting results out of this theorem. ... I beg to submit this, my maiden attempt, and humbly request that the Members of the Board will make allowance for any defect which they may notice to my want of usual training which is now undergone by college students and view sympathetically my humble effort in the attached paper."

With the above as historical introduction, let us begin to describe our work in higher dimensions by first stating Ramanujan's Master Theorem in the classical case of one dimension. The objects of concern are real-analytic functions of a real variable having a power series expansion

$$(1) \qquad F(x) = \sum_m \frac{(-1)^m}{m!} \, q(m) x^m$$

that converges on some interval about zero, where q is an analytic function on a domain in $\mathbf{C}$ containing the non-negative integers. The thrust of the theorem is to recapture the analytic function q of the complex variable λ from the Mellin transform

$$(2) \qquad \tilde{F}(\lambda) = \int_0^\infty F(x) x^{-\lambda-1} dx$$

of F, or equivalently by Mellin inversion

$$(3) \qquad F(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{F}(\lambda) x^\lambda d\lambda$$

where $\sigma \in \mathbf{R}$. One can view Ramanujan's Master Theorem as providing the "interpolation" formula

$$(4) \qquad q(\lambda) = \frac{\tilde{F}(\lambda)}{\Gamma(-\lambda)}$$

for the coefficients $q(m)$ in (1).

In his Quarterly Report, Ramanujan gave literally scores of applications in which he computed definite integrals and found series expansions. His "proof" of the Master Theorem, however, was a formal procedure, in the

words of Berndt [1], "fraught with numerous difficulties," and the hypotheses he listed were far too weak and simplicial for its validity. As one can see immediately from the case $q(\lambda) = \sin \pi\lambda$, for which the function F in (1) vanishes identically, there exist analytic functions q for which (4) fails. Thus, the problem is to prescribe classes of functions q for which the Mellin transform of the function F given by (1) exists and satisfies Ramanujan's formula (4). In 1920, using complex methods, G. H. Hardy [11] provided a rigorous proof for a class of functions F of at most exponential growth. One should note, however, that the Hardy class, although robust, does not exhaust all of the situations for which (4) is true.

In this paper we generalize Hardy's method to apply to symmetric cones. Roughly speaking, an open cone Ω in a real Euclidean space is said to be *symmetric* if it is a symmetric space under its (connected) group G of automorphisms. That is, $\Omega = G/K$ where G is a noncompact reductive Lie group and K is its maximal compact subgroup. Symmetric cones are intimately related to formally real Jordan algebras. More specifically, irreducible symmetric cones are in one-to-one correspondence with simple formally real Jordan algebras, and up to isomorphism an irreducible symmetric cone Ω is the interior of the set of squares in a simple formally real Jordan algebra J. As an example, the cone Ω of positive-definite real symmetric $r \times r$ matrices is an irreducible symmetric cone, the ambient Jordan algebra being the vector space of all $r \times r$ real symmetric matrices with the anticommutator $\frac{1}{2}(xy + yx)$ as Jordan product. The case $r = 1$ is the classical setting of Ramanujan's original work.

Consider an irreducible symmetric cone Ω realized in a simple Jordan algebra J. Let r be the rank of Ω, n the dimension of J, set $\mu = n/r$ and $\nu = 2(\mu - 1)/(r - 1)$, and denote G-invariant measure on Ω by d_*x. Let ε be the identity in J, write the inner product in J as $\langle \cdot | \cdot \rangle$, and define the trace in J by $\operatorname{tr} x = \langle x | \varepsilon \rangle$. We say that an r-tuple $m = (m_1, \ldots, m_r)$ of non-negative integers is a *partition* if $m_1 \geq m_2 \geq \cdots \geq m_r$, and we call the number $|m| = m_1 + \cdots + m_r$ the *length* of m. To each partition m is associated a K-invariant polynomial function Z_m on Ω known among statisticians as a *zonal polynomial* and by mathematicians as a *normalized spherical polynomial*. We will call an infinite series of the form

$$(5) \qquad \sum_m \frac{a_m}{|m|!} Z_m(x)$$

where the summation is over all partitions, a *spherical series*. When $r = 1$, a spherical series reduces to a Taylor series. The analogue for the cone Ω of the classical Mellin transform is the *spherical transform* (also known as the Harish-Chandra transform) $F \to \tilde{F}$, defined for integrable K-invariant

functions F on Ω by the integral

$$(6) \qquad \tilde{F}(\lambda) = \int_\Omega F(x)\phi_\lambda(x^{-1})d_*x$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$ is an r-tuple of complex variables and ϕ_λ is the corresponding spherical function. Formally, the inversion formula is given by

$$(7) \qquad F(x) = c_0 \int_{\sigma+i\mathbf{R}^r} \tilde{F}(\lambda)\phi_\lambda(x)\, \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

where $\sigma \in \mathbf{R}^r$, c_0 is a constant depending only upon Ω, and $\lambda \mapsto c(\lambda)$ is Harish-Chandra's c-function for Ω. Finally, the gamma function Γ_Ω for Ω is defined by

$$(8) \qquad \Gamma_\Omega(\lambda) = \int_\Omega e^{-\operatorname{tr}x}\Delta_\lambda(x)d_*x$$

for each $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbf{C}^r$ for which the integral converges, and elsewhere by analytic continuation. The function Δ_λ in (8), called the *generalized power function* for Ω, reduces to the ordinary power function x^λ in the classical case $r = 1$.

We can now state the main result of this paper. In what follows, we set $\rho = (\rho_1, \ldots, \rho_r)$ where $\rho_j = (2j - r - 1)\nu/4$, and we write the complex variable λ_j as $\lambda_j = \sigma_j + it_j$ where $\sigma_j = \operatorname{Re} \lambda_j$ and $t_j = \operatorname{Im} \lambda_j$.

Ramanujan's Master Theorem For The Cone Ω: *Let δ be a real number such that $\delta > \frac{1}{2}(\mu - 1)$ and let q be a function of $\lambda = (\lambda_1, \ldots, \lambda_r)$ with the following three properties:*

(i) *q is defined and holomorphic on the right half-space $H(\delta)$ in $\mathbf{C}^r$ given by $\operatorname{Re} \lambda_j > -\delta$ for $j = 1, \ldots, r$.*

(ii) *q is symmetric in $\lambda_1, \ldots, \lambda_r$.*

(iii) *There exist positive constants M, P and A with $A < \pi$ such that*

$$(9) \qquad |q(\lambda)| \le M\,|\Gamma_\Omega(\lambda + \rho + \mu)| \prod_{j=1}^{r} e^{P\sigma_j}\,e^{A|t_j|}$$

for all $\lambda \in H(\delta)$.

For $x \in \Omega$ set

$$(10) \qquad F(x) = \sum_m \frac{(-1)^{|m|}}{|m|!}\, q(m - \rho)Z_m(x)$$

and (with redundancy of notation)

$$(11) \qquad F(x) = c_0 \int_{\sigma+i\mathbf{R}^r} \Gamma_\Omega(-\lambda+\rho)q(\lambda)\phi_\lambda(x)\frac{d\lambda}{c(\lambda)c(-\lambda)}.$$

Then the following four properties hold.

(a) *The spherical series in* (10) *converges on a neighborhood of zero in* J, *and* (10) *defines a function* F *that is real-analytic on that neighborhood of zero.*

(b) *Let* $-\delta < \sigma_j < -\frac{1}{2}(\mu-1)$ *for* $j = 1,\dots,r$. *Then for any* $x \in \Omega$ *the integral in* (11) *is absolutely convergent and is independent of* $\sigma = (\sigma_1,\dots,\sigma_r)$, *and the function* F *defined by* (11) *is a continuous extension to all of* Ω *of the function* F *in* (10).

(c) *For a real number* σ_0, *set* $\vec{\sigma}_0 = (\sigma_0,\dots,\sigma_0) \in \mathbf{R}^r$. *Then* $\tilde{F}(\lambda)$ *exists in the* L^2-*sense for all* λ *of the form* $\lambda = \vec{\sigma}_0 + it$ *with* $-\delta < \sigma_0 < -\frac{1}{2}(\mu-1)$ *and* $t \in \mathbf{R}^r$, *and*

$$(12) \qquad\qquad \tilde{F}(\lambda) = \Gamma_\Omega(-\lambda+\rho)q(\lambda).$$

Moreover, formula (12) *is valid for any domain in* $\mathbf{C}^r$ *containing such a point* $\lambda = \vec{\sigma}_0 + it$ *and on which* $\tilde{F}$ *exists and is holomorphic.*

(d) *Assume now that* $\delta > \frac{3}{2}(\mu-1)$; *let* C_ρ *denote the convex hull of the the points* $w\rho \in \mathbf{R}^r$ *where* w *varies through the permutation group on the indices* $1,\dots,r$; *and let* $E_{\delta,\rho}$ *be the union of all translates* $\vec{\sigma}_0 + C_\rho$ *of* C_ρ *for* $-\delta + \frac{1}{2}(\mu-1) < \sigma_0 < -(\mu-1)$. *Then the integral* (6) *for* $\tilde{F}(\lambda)$ *converges absolutely and formula* (12) *holds for all* λ *in a tube domain* $E + i\mathbf{R}^r$, *where* $E \supset E_{\delta,\rho}$.

We derive several consequences of this theorem.

The first corollary is the following uniqueness theorem: *Let* q *be a holomorphic function on a half-space* $H(\delta)$ *with* $\delta > \frac{1}{2}(\mu-1)$, *and assume that* q *is symmetric and satisfies an estimate of the form* (9) *with* $A < \pi$. *If* $q(m - \rho) = 0$ *for all partitions* m, *then* q *vanishes identically on* $H(\delta)$. The specialization of this result to the one-dimensional classical setting is known as *Carlson's theorem* [3, 11]. Next, we refine this uniqueness theorem by providing an explicit construction of the function q from its values on the partitions. This generalizes the classical *Newton's interpolation formula*. Finally, we derive some *Mellin-Barnes integrals* for the hypergeometric functions $_pF_p$ and $_{p+1}F_p$ [5, 6, 7, 8, 9] on Ω: *Under suitable restrictions on the*

parameters a_i and b_i and on the real variable σ,

$$_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; -x) = c_0 \frac{\Gamma_\Omega(b_1) \cdots \Gamma_\Omega(b_p)}{\Gamma_\Omega(a_1) \cdots \Gamma_\Omega(a_p)}$$

$$(13) \quad \times \int_{\vec{\sigma}+i\mathbf{R}^r} \Gamma_\Omega(-\lambda + \rho) \frac{\Gamma_\Omega(a_1 + \lambda) \cdots \Gamma_\Omega(a_p + \lambda)}{\Gamma_\Omega(b_1 + \lambda) \cdots \Gamma_\Omega(b_p + \lambda)} \phi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

and

$$_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; -x) = c_0 \frac{\Gamma_\Omega(b_1) \cdots \Gamma_\Omega(b_p)}{\Gamma_\Omega(a_1) \cdots \Gamma_\Omega(a_{p+1})}$$

$$(14) \quad \times \int_{\vec{\sigma}+i\mathbf{R}^r} \Gamma_\Omega(-\lambda + \rho) \frac{\Gamma_\Omega(a_1 + \lambda) \cdots \Gamma_\Omega(a_{p+1} + \lambda)}{\Gamma_\Omega(b_1 + \lambda) \cdots \Gamma_\Omega(b_p + \lambda)} \phi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

for all $x \in \Omega$. Other applications of the Master Theorem, especially to integral formulas and series expansions for hypergeometric functions, will be treated in a subsequent paper.

This paper is organized into five sections, as follows. Section 1 brings together necessary background material on the structure of symmetric cones, Jordan algebras, and the relationship between them. Section 2 is a summary of the needed harmonic analysis on symmetric cones. The gamma function, its properties, and related constructs are discussed in paragraphs 2.1 through 2.5. The decomposition of the polynomial algebra on J under the group G, zonal and spherical polynomials, and spherical series appear in paragraphs 2.6 through 2.9. Invariant differential operators, spherical functions, and the spherical transform form the subject matter of paragraphs 2.10 through 2.12. Section 2 concludes with paragraphs 2.13 – 2.21 that describe Harish-Chandra's c-function, spherical inversion, and the Plancherel and Paley-Wiener theorems. In Section 3 we prove Ramanujan's Master Theorem for symmetric cones, as stated above. The argument ultimately reduces to the special case in which $q(\lambda) \equiv 1$. As an immediate corollary we prove the uniqueness theorem described above. The remaining two sections of the paper treat applications: Newton's interpolation formula for symmetric cones appears in Section 4, and Section 5 is devoted to Mellin-Barnes integrals.

We close this introduction by noting some points of contact of our work with existing literature. The special case of the Mellin-Barnes integral for the Gaussian hypergeometric function $_2F_1$ is derived in [6] by an argument completely different from ours. The generalized binomial coefficients used in the proof of Newton's formula were first introduced in multivariate statistics for the purposes of analysis on the cone of positive-definite real symmetric $r \times r$ matrices. See, for example, the book [17]. In the general setting these

coefficients have been studied in [15, 16, 20]. A precursor to formula (12), in the special case in which the function F is in the Schwartz class on Ω, appears in [5, XIV.4.3]. In [4] the problem of analytically continuing a spherical series of the form (5) is approached rather differently by generalizing to symmetric cones a classical theorem by Leroy in complex function theory. Quite recently – in work carried out contemporaneously with and independently of ours – in his doctoral dissertation, announced in [2], W. Bertram gives an elegant formulation of Ramanujan's Master Theorem for the special case of Schwartz class functions. We wish to thank J. Faraut for calling our attention to [2]. Of course, the applications in our paper require the full generality of the Master Theorem as stated above.

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1. The structure of symmetric cones.

We review the definition and structure of symmetric cones, listing only those properties that are needed for the analysis that follows. For a presentation in more detail we refer to [5, 18].

1.1. Symmetric cones. Let Ω be a non-empty open convex cone in a finite-dimensional inner product space $(J, \langle \cdot | \cdot \rangle)$, and denote by Ω^* the *dual cone* of elements $x \in \Omega$ such that $\langle x | y \rangle > 0$ for all y in the closure of Ω. Let $G(\Omega)$ denote the *automorphism group* of Ω, consisting of all invertible linear transformations of J which preserve the cone Ω. The cone Ω is said to be *symmetric* if it is self-dual and homogeneous under $G(\Omega)$. That is to say, $\Omega = \Omega^*$ and $G(\Omega)$ acts transitively on Ω. An example of a symmetric cone is the set $\Omega(r, \mathbf{R})$ of all positive-definite $r \times r$ real symmetric matrices endowed with the inner product $\langle x | y \rangle = \mathrm{tr}(xy)$.

1.2. Formally real Jordan algebras. A *Jordan algebra J* over a field $\mathbf{F}$ is a finite-dimensional commutative algebra over $\mathbf{F}$ having an identity element ε and satisfying the Jordan identity

$$(1) \qquad x^2 \circ (x \circ y) = (x^2 \circ x) \circ y$$

for all $x, y \in J$. Here, $\circ$ denotes the multiplication in J. If $\mathbf{F}$ is the real field

$\mathbf{R}$ and J is equipped with an inner product $\langle \cdot | \cdot \rangle$, we say J is *formally real* (or *Euclidean*) if

$$(2) \qquad \langle x \circ y | z \rangle = \langle y | x \circ z \rangle$$

for all $x, y, z \in J$. The trace in a formally real Jordan algebra is the linear functional

$$(3) \qquad \operatorname{tr} x = \langle x | \varepsilon \rangle.$$

As examples of Jordan algebras, let $\mathcal{A}$ be an associative algebra with unit ε and J be a vector subspace of $\mathcal{A}$ closed under squares. Then with the anticommutator $x \circ y = \frac{1}{2}(xy + yx)$ as multiplication, J is a Jordan algebra. Such Jordan algebras are called *special*. A special formally real Jordan algebra to keep in mind is the space $S(r, \mathbf{R})$ of all $r \times r$ real symmetric matrices.

1.3. The cone of squares in J. Let J be a formally real Jordan algebra, and denote by Q the subset of all elements x^2 where $x \in J$. The interior of Q is a symmetric cone, and any symmetric cone is isomorphic to a cone of this kind [5, III.3.1]. Consequently, the study of symmetric cones reduces to the study of formally real Jordan algebras. In the case in which $J = S(r, \mathbf{R})$, for example, $\Omega = \Omega(r, \mathbf{R})$ is the interior of the set of squares in J.

1.4. The symmetric space structure. Let G be the connected component of the identity in the automorphism group $G(\Omega)$. Then G acts transitively on Ω, and $\Omega \simeq G/K$ where K is the stability group of the identity element ε in Ω; i.e., K consists of all $k \in G$ such that $k \cdot \varepsilon = \varepsilon$. The group G is a connected non-compact reductive Lie group, K is its maximal compact subgroup, and considered as the homogeneous space G/K, the symmetric cone Ω becomes a Riemannian symmetric space. In the example in which $\Omega = \Omega(r, \mathbf{R})$, the group $G = \mathrm{GL}_+(r, \mathbf{R})$ of real $r \times r$ matrices having positive determinant acts on Ω by $g \cdot x = gxg^t$ for $x \in \Omega$ and $g \in G$. Then $K = \mathrm{SO}(r)$ is the special orthogonal group and $\Omega(r, \mathbf{R}) \simeq \mathrm{GL}_+(r, \mathbf{R})/\mathrm{SO}(r)$.

1.5. Invariant measure on Ω. Denote by r the real rank of the symmetric space G/K, and let n be the dimension of J. Set

$$(1) \qquad \mu = n/r.$$

The number μ is either an integer or a half-integer. The *characteristic function* ψ of Ω is defined by

$$(2) \qquad \psi(x) = \int_\Omega e^{-(x|y)} dy$$

for all $x \in \Omega$, and the *Koecher norm function* Δ is given by

$$(3) \qquad\qquad \Delta(x) = c\psi(x)^{-1/\mu}$$

where c is a constant determined by the normalization $\Delta(\varepsilon) = 1$. The function Δ extends to a K-invariant polynomial on J such that

$$(4) \qquad\qquad \Delta(g \cdot x) = (\det g)^{1/\mu}\Delta(x)$$

for all $g \in G$, from which it follows that the measure

$$(5) \qquad\qquad d_*x = \Delta(x)^{-\mu}dx$$

on Ω is G-invariant. Note that when $\Omega = \Omega(r, \mathbf{R})$ is the cone of real $r \times r$ positive-definite symmetric matrices, Δ is just the ordinary determinant and $\mu = (r+1)/2$.

1.6. Irreducible symmetric cones. A Jordan algebra is *simple* if it contains no proper ideals, and a symmetric cone is *irreducible* if it is not the direct product of nonzero symmetric cones. A formally real Jordan algebra is simple if and only if the associated symmetric cone is irreducible. In general, any formally real Jordan algebra is a direct sum of simple ideals, and correspondingly a symmetric cone is the direct product of irreducible symmetric cones. The cone $\Omega(r, \mathbf{R})$, for example, is irreducible.

In the remainder of this section, and throughout the rest of the paper, we will assume that Ω is an irreducible symmetric cone in the simple Jordan algebra J.

1.7. Idempotents. An element $e \in J$ is *idempotent* if $e^2 = e$. The scalar β is an *eigenvalue* of e if there exists nonzero $x \in J$ such that $e \circ x = \beta x$. If e is idempotent, then the eigenvalues must be equal to 1, $\frac{1}{2}$, or 0. An idempotent is *primitive* if it is nonzero and not the sum of two nonzero idempotents. Two idempotents e and f are *orthogonal* if $e \circ f = 0$. A maximal system of orthogonal primitive idempotents is called a *Jordan frame*, and any Jordan frame has r elements where r is the rank of Ω. If $\{e_1, \ldots, e_r\}$ is a Jordan frame, then $e_1 + \cdots + e_r = \varepsilon$, where ε is the identity in J. When $J = \Omega(r, \mathbf{R})$, for example, we may take e_j to be the $r \times r$ matrix whose (j, j)-th entry is 1 and all other entries are zero.

1.8. The constant ν. Fix a Jordan frame $\{e_1, \ldots, e_r\}$ in J and define the following subspaces: $V_j = \{x \in J : e_j \circ x = x\}$ and $V_{ij} = \{x \in J : e_i \circ x = \frac{1}{2}x$ and $e_j \circ x = \frac{1}{2}x\}$. Then $V_j = \mathbf{R}e_j$ for $j = 1, \ldots, r$ are 1-dimensional subalgebras, while the subspaces V_{ij} for $i, j = 1, \ldots, r$ with $i \neq j$ all have a common dimension. We denote the common dimension of the subspaces V_{ij}

by the symbol ν. Since J decomposes as the orthogonal direct sum

$$(1) \qquad J = \left(\sum_{j=1}^{r} \oplus V_j\right) \oplus \left(\sum_{i\neq j} \oplus V_{ij}\right)$$

it follows that

$$(2) \qquad n = r + \frac{\nu}{2}r(r-1)$$

and from (1.5.1) that

$$(3) \qquad \mu - 1 = \frac{\nu}{2}(r-1)$$

or

$$(4) \qquad \nu = \frac{2(\mu-1)}{(r-1)}.$$

The decomposition (1) is known as the *Peirce decomposition* of J relative to the given Jordan frame.

1.9. The principal minors. Fix a Jordan frame $\{e_1, \ldots, e_r\}$ in J. For $j = 1, \ldots, r$, let $E_j = e_1 + \cdots + e_j$, and set $J_j = \{x \in J : E_j \circ x = x\}$. Then E_j is an idempotent and J_j is a subalgebra of J of rank j. Denote by P_j the orthogonal projection of J on J_j, and define

$$(1) \qquad \Delta_j(x) = \delta_j(P_j x)$$

for $x \in J$, where δ_j denotes the Koecher norm function for J_j. Then Δ_j is a polynomial on J that is homogeneous of degree j. We call $\Delta_j(x)$ the jth *principal minor* of x.

Let $\lambda_1, \ldots, \lambda_r$ be complex numbers, and define the function Δ_λ on J by

$$(2) \qquad \Delta_\lambda(x) = \Delta(x)^{\lambda_r} \prod_{j=1}^{r-1} \Delta_j(x)^{\lambda_j - \lambda_{j+1}}.$$

The function Δ_λ is the *generalized power function* on J. When $\lambda_j = m_j$ is an integer and $m_1 \geq \cdots \geq m_r \geq 0$, (2) defines a polynomial function Δ_m that is homogeneous of degree $m_1 + \cdots + m_r$.

1.10. The polar decomposition of J. Denote by $\mathcal{R}$ the subset of J of elements of the form

$$(1) \qquad a = \xi_1 e_1 + \cdots + \xi_r e_r$$

with $\xi_1, \ldots, \xi_r \in \mathbf{R}$. Then for any $x \in J$ there exists $k \in K$ and $\xi_1, \ldots, \xi_r \in \mathbf{R}$ such that

$$(2) \qquad\qquad x = k \cdot (\xi_1 e_1 + \cdots + \xi_r e_r).$$

Moreover, the scalars $\xi_1, \ldots, \xi_r$ are uniquely determined by x up to permutations, and the element $k \in K$ is uniquely determined by x modulo the centralizer M of x in K. We refer to (2) as the *polar decomposition* of x. When x is written in the form (2),

$$(3) \qquad\qquad \mathrm{tr}(x) = \sum_{j=1}^{r} \xi_j \qquad \text{and} \qquad \Delta(x) = \prod_{j=1}^{r} \xi_j.$$

When $J = S(r, \mathbf{R})$ is the algebra of $r \times r$ real symmetric matrices, $K = \mathrm{SO}(r)$ is the special orthogonal group and (2) is the spectral decomposition $x = k\xi k^{-1}$ in which $k \in K$ and ξ is the diagonal matrix $\xi = \mathrm{diag}(\xi_1, \ldots, \xi_r)$ whose entries $\xi_1, \ldots, \xi_r$ are the eigenvalues of x.

1.11. The quadratic representation. Denote by L the regular representation $L(x)y = x \circ y$ of J, and set

$$(1) \qquad\qquad P(x) = 2L(x)^2 - L(x^2)$$

for $x \in J$. The mapping $x \mapsto P(x)$ is called the *quadratic representation* of J. For example, when $J = S(r, \mathbf{R})$, the algebra of all $r \times r$ symmetric real matrices, $P(x)$ is given by matrix multiplication as $P(x)y = xyx$.

In general, if $x^{1/2}$ denotes the square root of $x \in \Omega$, then $P(x^{1/2})\varepsilon = x$, $P(x^{1/2}) \in G$, and $\langle y|x \rangle = \langle P(x^{1/2})y|\varepsilon \rangle$ for all $y \in \Omega$. Moreover, [5, XIV.1.2], for x and y in Ω there exists $k \in K$ such that

$$(2) \qquad\qquad P(x^{1/2})y = k \cdot P(y^{1/2})x.$$

1.12. Classification. The simple formally real Jordan algebras, or equivalently the irreducible symmetric cones, were classified in 1934 by Jordan, von Neumann, and Wigner [14] into four families of *classical* algebras together with a single *exceptional* algebra. The first three families of classical algebras are *matrix spaces* and the fourth is a *Minkowski space*. For the matrix spaces the Jordan product is the anti-commutator $x \circ y = \frac{1}{2}(xy + yx)$.

1. Real matrix space. $J = S(r, \mathbf{R})$ is the space of all $r \times r$ symmetric real matrices, Ω is the cone in $S(r, \mathbf{R})$ of positive-definite matrices, the rank of $S(r, \mathbf{R})$ is r, the dimension is $n = \frac{1}{2}r(r+1)$, $\mu = \frac{1}{2}(r+1)$, and $\nu = 1$. As noted earlier, the group G is isomorphic to the general linear group $\mathrm{GL}_+(r, \mathbf{R})$ of non-singular $r \times r$ real matrices with positive determinant, K

is isomorphic to the special orthogonal group $SO(r)$, and as a symmetric space, $\Omega \simeq GL_+(r, \mathbf{R})/SO(r)$.

2. Complex matrix space. $J = S(r, \mathbf{C})$ is the space of all $r \times r$ Hermitian matrices, Ω is the cone of $r \times r$ positive-definite matrices, the dimension of $S(r, \mathbf{C})$ is r^2, the rank is r, $\mu = r$, and $\nu = 2$. The group G is isomorphic to the general linear group $GL(r, \mathbf{C})$, K is isomorphic to the unitary subgroup $U(r)$, and $\Omega \simeq GL(r, \mathbf{C})/U(r)$.

3. Quaternionic matrix space. $J = S(r, \mathbf{H})$ is the vector space of all $r \times r$ Hermitian matrices over the quaternion division algebra $\mathbf{H}$, Ω is the cone of positive-definite such matrices, the dimension of $S(r, \mathbf{H})$ is $r(2r - 1)$, the rank is r, $\mu = 2r - 1$, and $\nu = 4$. Here, G is isomorphic to the general linear group $GL(r, \mathbf{H})$, K is isomorphic to the compact symplectic group $Sp(r)$, and $\Omega \simeq GL(r, \mathbf{H})/Sp(r)$.

4. Minkowski space. $J = \mathbf{R}(1, n)$ is the space $\mathbf{R} \times \mathbf{R}^n$ with the Lorentz metric of signature $(1, n)$. The corresponding cone is the light cone, composed of all elements $(\lambda, x) \in \mathbf{R} \times \mathbf{R}^n$ such that $|\lambda|^2 - \|x\|^2 > 0$ and $\lambda > 0$. The Jordan product is given by $(\lambda_1, x_1) \circ (\lambda_2, x_2) = (\lambda_1 \lambda_2 + (x_1|x_2), \lambda_2 x_1 + \lambda_1 x_2)$ for $\lambda_1, \lambda_2 \in \mathbf{R}$ and $x_1, x_2 \in \mathbf{R}^n$. The dimension of $\mathbf{R}(1, n)$ is $n + 1$, the rank is 2, $\mu = \frac{1}{2}(n + 1)$, and $\nu = n - 1$. The group G is the product of the connected component of the Lorentz group $SO_0(1, n)$ with the multiplicative group $\mathbf{R}^+$ of positive real numbers, K is the special orthogonal subgroup $SO(n)$, and $\Omega \simeq SO_0(1, n)/SO(n) \times \mathbf{R}_+$.

5. The exceptional algebra. This is a certain space of 3×3 matrices over the Cayley algebra, endowed with an anti-commutator as Jordan product. The dimension of J is 27, the rank is 3, $\mu = 9$, and $\nu = 8$. Here, $\Omega \simeq G/K$ with $G \simeq E_6 \times \mathbf{R}^+$ and K is isomorphic to compact F_4.

2. Harmonic analysis on symmetric cones.

In this section we bring together a summary of the required background from harmonic analysis. For a more detailed discussion, we refer to [5]. Throughout, Ω is an irreducible symmetric cone in a simple Jordan algebra J, and $\Omega = G/K$.

2.1. The gamma function on Ω. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbf{C}^r$, and let Δ_λ be given by (1.9.2). We define the *gamma function* Γ_Ω *for the cone* Ω by

$$(1) \qquad \Gamma_\Omega(\lambda) = \int_\Omega e^{-\operatorname{tr} x} \Delta_\lambda(x) d_* x$$

whenever the integral converges absolutely. By [5, VII.1.1], the integral in (1) converges absolutely if and only if

$$(2) \qquad \operatorname{Re} \lambda_j > (j - 1)\frac{\nu}{2}$$

where ν is given by (1.8.4). Moreover, in the range (2) of the variable λ, Γ_Ω is evaluated in terms of the classical gamma function as

$$(3) \qquad \Gamma_\Omega(\lambda) = (2\pi)^{\frac{1}{2}(n-r)} \prod_{j=1}^{r} \Gamma\left(\lambda_j - (j-1)\frac{\nu}{2}\right)$$

and (3) defines the meromorphic continuation of Γ_Ω to all of $\mathbf{C}^r$. Note that $1/\Gamma_\Omega(\lambda)$ is an entire function on $\mathbf{C}^r$.

2.2. The Pochhammer symbol for Ω. We say that an r-tuple

$$(1) \qquad m = (m_1, \ldots, m_r)$$

of integers m_j is a *partition* if

$$(2) \qquad m_1 \geq m_2 \geq \cdots \geq m_r \geq 0.$$

We set

$$(3) \qquad |m| = m_1 + \cdots + m_r$$

and call $|m|$ the *length* of the partition. Recall the classical Pochhammer symbol (also known as the truncated factorial), defined for $\alpha \in \mathbf{C}$ and non-negative integer j by

$$(4) \qquad (\alpha)_j = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)} = \alpha(\alpha + 1) \cdots (\alpha + j - 1).$$

In analogy to (4) we define the (generalized) *Pochhammer symbol for Ω* by

$$(5) \qquad [\lambda]_m = \frac{\Gamma_\Omega(\lambda + m)}{\Gamma_\Omega(\lambda)}$$

where $\lambda \in \mathbf{C}^r$ and m is any partition. From (2.1.3)

$$(6) \qquad [\lambda]_m = \prod_{j=1}^{r} \left(\lambda_j - (j-1)\frac{\nu}{2}\right)_{m_j}.$$

2.3. Connection with the sine function. From the classical identity

$$(1) \qquad \Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi}{\sin \pi \alpha}$$

and from (2.1.3), we obtain

$$(2) \qquad \Gamma_\Omega(\lambda)\Gamma_\Omega(\mu - \lambda') = \frac{2^{n-r}\pi^n}{\prod_{j=1}^{r} \sin \pi \left(\lambda_j - \frac{\nu}{2}(j-1)\right)}$$

where

(3) $$\lambda' = (\lambda_r, \lambda_{r-1}, \dots, \lambda_1)$$

and, as always, $\mu = n/r$.

2.4. The ρ-shift. The formula (2.1.3) for $\Gamma_\Omega(\lambda)$ lacks a certain degree of symmetry, due to the shifting of the variable λ_j by the amount $(j-1)\nu/2$. To compensate for that shift we introduce the r-tuple

(1) $$\rho = (\rho_1 \dots, \rho_r)$$

defined by

(2) $$\rho_j = \frac{\nu}{4}(2j - r - 1) = (j-1)\frac{\nu}{2} - \frac{1}{2}(\mu - 1)$$

where we have utilized the identity (1.8.3). Then

(3) $$\Gamma_\Omega(\lambda + \rho) = (2\pi)^{\frac{1}{2}(n-r)} \prod_{j=1}^{r} \Gamma\left(\lambda_j - \frac{\mu - 1}{2}\right).$$

In terms of ρ we can rewrite (2.3.2) in the more symmetrical form

(4) $$\Gamma_\Omega(\lambda + \rho)\Gamma_\Omega(\mu - \lambda + \rho) = \frac{2^{n-r}\pi^n}{\displaystyle\prod_{j=1}^{r} \sin \pi \left(\lambda_j - \frac{\mu - 1}{2}\right)}.$$

Note that

(5) $$\sum_{j=1}^{r} \rho_j = 0.$$

The importance of ρ in the harmonic analysis of G/K is due to its appearance in the integral formula for the Iwasawa decomposition of the group G [13].

2.5. Notation. Because we will often make use of the constant elements in $\mathbf{C}^r$, we introduce the notation

(1) $$\vec{\alpha} = (\alpha, \dots, \alpha) \in \mathbf{C}^r$$

for $\alpha \in \mathbf{C}$. Thus

(2) $$\Gamma_\Omega\left(\vec{\alpha}\right) = (2\pi)^{\frac{1}{2}(n-r)} \prod_{j=1}^{r} \Gamma\left(\alpha - (j-1)\frac{\nu}{2}\right)$$

(3) $$\Gamma_\Omega\left(\vec{\alpha} + \rho\right) = (2\pi)^{\frac{1}{2}(n-r)} \left[\Gamma\left(\alpha - \frac{\mu - 1}{2}\right)\right]^r$$

(4) $$\left[\vec{\alpha}\right]_m = \prod_{j=1}^{r}\left(\alpha - (j-1)\frac{\nu}{2}\right)_{m_j}$$

and

$$(5) \qquad \left[\vec{\alpha} + \rho\right]_m = \prod_{j=1}^{r} \left(\alpha - \frac{\mu - 1}{2}\right)_{m_j}.$$

When there is no cause for confusion, we drop the vector notation and write $\Gamma_\Omega(\alpha)$, $\Gamma_\Omega(\alpha + \rho)$, $[\alpha]_m$, and $[\alpha + \rho]_m$ for (2), (3), (4), and (5), respectively.

2.6. Decomposition of the polynomial algebra. The algebra of all polynomial functions on J, denoted by $\mathcal{P}(J)$, decomposes as

$$(1) \qquad \mathcal{P}(J) = \sum_{d=0}^{\infty} \oplus \mathcal{P}_d(J)$$

where $\mathcal{P}_d(J)$ is the collection of polynomials homogeneous of degree d. The space $\mathcal{P}_d(J)$ is finite-dimensional and the formula

$$(2) \qquad \dim \mathcal{P}_d(J) = \binom{n + d - 1}{d}$$

gives its dimension as a binomial coefficient. Under the action of the group G, the space $\mathcal{P}_d(J)$ is *multiplicity free* and decomposes as follows: For any partition m such that $|m| = d$, there exists an irreducible subspace $\mathcal{P}^m(J)$ of $\mathcal{P}_d(J)$ such that

$$(3) \qquad \mathcal{P}_d(J) = \sum_{|m|=d} \oplus \mathcal{P}^m(J)$$

relative to which distinct partitions m (irrespective of length) correspond to inequivalent subspaces $\mathcal{P}^m(J)$. The polynomial Δ_m defined in paragraph 1.9 lies in $\mathcal{P}^m(J)$, and since G acts irreducibly on that space, $\mathcal{P}^m(J)$ may be characterized as the space spanned by the translates of Δ_m under G. We set

$$(4) \qquad d_m = \dim \mathcal{P}^m(J)$$

and from (2) deduce the estimate

$$(5) \qquad d_m \leq C(1 + |m|)^{n-1}$$

where C is a constant depending only upon n. From (1) and (3)

$$(6) \qquad \mathcal{P}(J) = \sum_m \oplus \mathcal{P}^m(J)$$

where the summation is over all partitions.

2.7. Zonal polynomials. Let $\mathcal{P}(J)^K$ denote the algebra of all K-invariant polynomials, i.e., elements $p \in \mathcal{P}(J)$ such that

$$(1) \qquad\qquad p(k \cdot x) = p(x)$$

for all $k \in K$ and $x \in J$. From (2.6.6),

$$(2) \qquad\qquad \mathcal{P}(J)^K = \sum_m \oplus \mathcal{P}^m(J)^K$$

where $\mathcal{P}^m(J)^K = \mathcal{P}^m(J) \cap \mathcal{P}(J)^K$. Since the decomposition (2.6.6) is multiplicity-free, it follows that $\mathcal{P}^m(J)^K$ is one dimensional. That is to say, for each partition m there exists a non-zero K-invariant polynomial in $\mathcal{P}^m(J)$, and up to scalar multiples that polynomial is unique. Let Φ_m denote the unique element in $\mathcal{P}^m(J)^K$ such that

$$(3) \qquad\qquad \Phi_m(\varepsilon) = 1.$$

We call Φ_m the *spherical polynomial of weight m*. The decomposition (2) then takes the form

$$(4) \qquad\qquad \mathcal{P}(J)^K = \sum_m \oplus \mathbf{C}\Phi_m.$$

The spherical polynomial has the integral representation

$$(5) \qquad\qquad \Phi_m(x) = \int_K \Delta_m(k \cdot x) dk$$

where $x \in J$ and $\Delta_m \in \mathcal{P}^m(J)$ is given by (1.9.2). From (3) and homogeneity,

$$(6) \qquad\qquad \Phi_m(\xi\varepsilon) = \xi^{|m|}$$

for all $\xi \in \mathbf{R}$. More generally, [5, XII.1.1], if $x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_r e_r \in \Omega$

$$(7) \qquad\qquad 0 < \Phi_m(x) \le \prod_{j=1}^r \xi_j^{m_j}.$$

A normalization different from (3) arises from the exponential function. We set

$$(8) \qquad\qquad Z_m(x) = \varpi_m \Phi_m(x)$$

where

$$(9) \qquad\qquad \varpi_m = \frac{d_m |m|!}{[\mu]_m}.$$

Here, as always, $\mu = n/r$ and $[\mu]_m$ is the Pochhammer symbol (2.5.4). With this normalization for the K-invariant polynomial in $\mathcal{P}^m(J)$,

$$(10) \qquad (\operatorname{tr} x)^d = \sum_{|m|=d} Z_m(x)$$

for all $x \in J$ and all non-negative integers d. Hence,

$$(11) \qquad e^{\operatorname{tr} x} = \sum_{d=0}^{\infty} \frac{1}{d!} \left(\sum_{|m|=d} Z_m(x) \right) = \sum_m \frac{1}{|m|!} Z_m(x)$$

for all $x \in J$.

Of course, all of the considerations in this section apply not only to J itself, but by analytic continuation to the complexification $J^{\mathbf{C}}$ as well. In particular, Φ_m and Z_m can be viewed as polynomial functions on $J^{\mathbf{C}}$, and the expansions (10) and (11) are valid for all $x \in J^{\mathbf{C}}$. However, we are primarily interested in the restriction to Ω.

2.8. Symmetric polynomials on $\mathbf{R}^r$. From the polar decomposition (1.10.2), any K-invariant function on J is determined by its restriction to the subspace $\mathcal{R}$ composed of elements of the form (1.10.1). Moreover, since the symmetric group on the elements $e_1, \ldots, e_r$ can be realized as a subgroup of K, it follows that the restriction to $\mathcal{R}$ of a K-invariant function on J is a symmetric function of the r variables $\xi_1, \ldots, \xi_r$. Let $W = W_r$ denote the symmetric (Weyl) group acting on the variables $\xi_1, \ldots, \xi_r$, let $\mathcal{P}(\mathbf{R}^r)^W$ denote the algebra of all symmetric polynomials on $\mathbf{R}^r$, and in the obvious way identify a polynomial function on $\mathcal{R}$ with a polynomial in $\xi_1, \ldots, \xi_r$. Then the restriction mapping

$$(1) \qquad q \mapsto q|_{\mathcal{R}}$$

is an algebra isomorphism of $\mathcal{P}(J)^K$ with $\mathcal{P}(\mathbf{R}^r)^W$. Thus, any K-invariant function on J, or on the cone Ω, is essentially a symmetric function of the r real variables $\xi_1, \ldots, \xi_r$.

As any symmetric polynomial on $\mathbf{R}^r$ extends uniquely by analytic continuation to a symmetric polynomial on $\mathbf{C}^r$, it will be convenient to utilize the algebra $\mathcal{P}(\mathbf{C}^r)^W$ of symmetric polynomials in the complex variables $\lambda_1, \ldots, \lambda_r$ rather than the algebra $\mathcal{P}(\mathbf{R}^r)^W$ in the real variables $\xi_1, \ldots, \xi_r$.

2.9. Spherical series. We refer to a series of the form

$$(1) \qquad \sum_m \frac{a_m}{|m|!} Z_m(x) = \sum_m a_m \frac{d_m}{[\mu]_m} \Phi_m(x)$$

as a *spherical series.* For example, (2.7.11) is a spherical series converging for all $x \in J^{\mathbf{C}}$. In general, the following estimate is useful in analyzing the

convergence of a spherical series. Let $x = \xi_1 e_1 + \cdots + \xi_n e_n$ with $\xi_j \in \mathbf{R}$. By (1.8.3), (2.5.4), (2.6.5), and (2.7.7),

$$(2) \qquad \left| \frac{d_m}{[\mu]_m} \Phi_m(x) \right| \leq C(1 + |m|)^{n-1} \prod_{j=1}^{r} \left(\frac{\xi_j}{1 + \frac{\nu}{2}(r - j)} \right)^{m_j}$$

where C is a constant depending only upon n.

Denote by the expression $0 \leq x < x_0$ the set of all $x \in \bar{\Omega}$ such that $x_0 - x \in \Omega$. If a spherical series converges at $x_0 \in \Omega$, then it converges absolutely on $0 \leq x < x_0$. (More generally, [5, XII.1.2], the series converges for all $z \in J^{\mathbf{C}}$ with polar decomposition $z = ux$ such that $0 \leq x < x_0$.)

2.10. Invariant differential operators. Let $\mathcal{D}(\Omega)$ denote the algebra of all *invariant differential operators* on Ω; that is, differential operators which commute with the action of G. The following three algebras are mutually isomorphic: $\mathcal{D}(\Omega)$, $\mathcal{P}(J)^K$, and $\mathcal{P}(\mathbf{C}^r)^W$. The isomorphism of $\mathcal{P}(J)^K$ with $\mathcal{P}(\mathbf{C}^r)^W$ is outlined in paragraph 2.8 above. The isomorphism of $\mathcal{P}(\mathbf{C}^r)^W$ with $\mathcal{D}(\Omega)$ can be described as follows: Let ρ be defined by (2.4.2) and for $\lambda \in \mathbf{C}^r$ let Δ_λ be given by (1.9.2). Then to each $q \in \mathcal{P}(\mathbf{C}^r)^W$ there corresponds an operator $D_q \in \mathcal{D}(\Omega)$ such that

$$(1) \qquad D_q \Delta_{\lambda+\rho} = q(\lambda) \Delta_{\lambda+\rho}$$

for all $\lambda \in \mathbf{C}^r$, and the mapping $q \mapsto D_q$ is an isomorphism of $\mathcal{P}(\mathbf{C}^r)^W$ with $\mathcal{D}(\Omega)$.

2.11. Spherical functions. A *spherical function* on Ω is a C^∞-function which is normalized to have the value 1 at the identity ε of J and which is an eigenfunction for every operator in $\mathcal{D}(\Omega)$. For $\lambda \in \mathbf{C}^r$ define the function ϕ_λ by

$$(1) \qquad \phi_\lambda(x) = \int_K \Delta_{\lambda+\rho}(kx)\,dk$$

for all $x \in \Omega$. Then ϕ_λ is a spherical function, and if ϕ is any spherical function there exists $\lambda \in \mathbf{C}^r$ such that $\phi = \phi_\lambda$. By (1) and (2.10.1)

$$(2) \qquad D_q \phi_\lambda = q(\lambda) \phi_\lambda$$

for all $q \in \mathcal{P}(\mathbf{C}^r)^W$ and $\lambda \in \mathbf{C}^r$. The spherical functions satisfy the following properties:

    (i) $\phi_\lambda = \phi_{\lambda'}$ if and only if there exists a permutation w such that $\lambda' = w\lambda$.

    (ii) For all $x \in \Omega$,

$$(3) \qquad \phi_\lambda(x^{-1}) = \phi_{-\lambda}(x).$$

(iii) From (1) and (2.7.5), $\Phi_m = \phi_{m-\rho}$ for any partition m, and consequently

$$(4) \qquad\qquad D_q Z_m = q(m - \rho) Z_m$$

for all G-invariant polynomials q and all partitions m.

(iv) For any $g \in G$ and $x \in \Omega$,

$$(5) \qquad\qquad \int_K \phi_\lambda(gk \cdot x)dk = \phi_\lambda(g \cdot \varepsilon)\phi_\lambda(x).$$

Equation (5) is known as the *functional equation* for the spherical function ϕ_λ. In terms of the quadratic representation P (see paragraph 1.11), (5) may also be written as

$$(6) \qquad\qquad \int_K \phi_\lambda(P(x^{1/2})k \cdot y)dk = \phi_\lambda(x)\phi_\lambda(y)$$

for $x, y \in \Omega$.

2.12. The spherical transform. Let F be a K-invariant measurable function on Ω and $\lambda \in \mathbf{C}^r$. We define the *spherical transform* of F by

$$(1) \qquad\qquad \tilde{F}(\lambda) = \int_\Omega F(x)\phi_\lambda(x^{-1})d_*x$$

whenever the integral converges absolutely. Following [5, Ch. XIV] we denote by $E(\tilde{F})$ the subset of $\mathbf{R}^r$ on which the integral in (1) converges absolutely. The function $\tilde{F}$ is defined on the tube $E(\tilde{F}) + i\mathbf{R}^r$ and is holomorphic on its interior. From the definition (2.11.1) of ϕ_λ and the K-invariance of F,

$$(2) \qquad\qquad \tilde{F}(\lambda) = \int_\Omega F(x)\Delta_{\lambda+\rho}(x^{-1})d_*x$$

for $\lambda \in E(\tilde{F}) + i\mathbf{R}^r$. If $\tilde{F}$ and $(D_q F)^\sim$ exist and are C^∞ for any polynomial $q \in \mathcal{P}(\mathbf{C}^r)^W$ then [5, XIV.4.2]

$$(3) \qquad\qquad (D_q F)^\sim(\lambda) = q(\lambda)\tilde{F}(\lambda).$$

2.13. The c-function on Ω. Inversion of the spherical transform, as first constructed by Harish-Chandra [12] in his monumental work on the harmonic analysis of semi-simple Lie groups, is described by what is called *Harish-Chandra's c-function*, or just "the c-function" for short. In the case of symmetric spaces G/K, the explicit calculation of the c-function is due to

Gindikin and Karpelevich [7]. For the symmetric cone Ω, their formula can be calculated (see [5, XIV.5.2]) as

$$(1) \qquad c(\lambda) = \frac{\displaystyle\prod_{k<j} B\left(\lambda_j - \lambda_k, \frac{\nu}{2}\right)}{\displaystyle\prod_{k<j} B\left((j-k)\frac{\nu}{2}, \frac{\nu}{2}\right)}$$

where $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbf{C}^r$ and

$$(2) \qquad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the classical beta function.

In the following theorem we list a variety of properties of the c-function, all of which follow directly from properties of the classical gamma function.

2.14. Theorem. (i) *Let ν be even. Then $1/c(\lambda)$ is a polynomial on $\mathbf{C}^r$ that vanishes on the hyperplanes $\lambda_j - \lambda_k = 0, -1, -2, \ldots, -\frac{\nu}{2} + 1$ with $k < j$.*

(ii) *Let ν be odd. Then $1/c(\lambda)$ has singularities along the hyperplanes $\lambda_j - \lambda_k = -\left(\frac{1}{2} + \ell\right)$, where $1 \leq k < j \leq r$ and ℓ is any integer such that $\ell \geq (\nu - 1)/2$; and $1/c(\lambda)$ vanishes where $\lambda_j - \lambda_k$ is a non-positive integer (and $k < j$).*

(iii) *For $\alpha \in \mathbf{C}$ and $\lambda \in \mathbf{C}^r$,*

$$(1) \qquad c(\lambda + \vec{\alpha}) = c(\lambda)$$

where $\vec{\alpha}$ is given by (2.5.1).

(iv) *For $\lambda = it$ purely imaginary (i.e., $t \in \mathbf{R}^r$), $1/c(it)$ is a C^∞-function of t with polynomial growth at infinity.*

Proof. (i) When ν is even, we write (2.13.1) in terms of the Pochhammer symbol as

$$(2) \qquad \frac{1}{c(\lambda)} = \left(\prod_{k<j} \left((j-k)\frac{\nu}{2}\right)_{\nu/2}\right)^{-1} \prod_{k<j} (\lambda_j - \lambda_k)_{\nu/2}$$

which is polynomial since ν is even.

(ii) For ν odd,

$$(3) \qquad \frac{1}{c(\lambda)} = \left(\prod_{k<j} \frac{\Gamma\left((j-k)\frac{\nu}{2}\right)}{\Gamma\left((j-k+1)\frac{\nu}{2}\right)}\right) \prod_{k<j} \frac{\Gamma\left(\lambda_j - \lambda_k + \frac{\nu}{2}\right)}{\Gamma(\lambda_j - \lambda_k)}$$

from which one can read off the zeros and singularities.

(iii) This result is an obvious consequence of the dependence of the c-function on $\lambda_j - \lambda_k$.

(iv) Using the classical estimate

$$(4) \qquad \lim_{z \to \infty} \frac{\Gamma(z+a)}{z^a \Gamma(z)} = 1$$

for $|\arg z| < \pi - \delta$ with $\delta > 0$, we see from (2) and (3) that

$$\frac{1}{|c(it)|} \leq M|t|^{r(r-1)\nu/4},$$

where $|t| = \max\{|t_j| : j = 1, \ldots, r\}$. By (2) and (3), $1/c(it)$ has no singularities with $t \in \mathbf{R}^r$.

2.15. Inversion of the spherical transform. The inversion of the spherical transform is stated as follows in [5]. The proof can be found in [13, Ch. IV]. Suppose f is continuous on Ω, K-invariant, and integrable with respect to invariant measure $d_* x$. Then $\tilde{f}(it)$ exists for all $t \in \mathbf{R}^r$; and if

$$(1) \qquad \int_{\mathbf{R}^r} |\tilde{f}(it)| \frac{dt}{|c(it)|^2} < \infty$$

then there exists a positive constant c_0 that depends only upon the cone Ω and not upon f, such that

$$(2) \qquad f(x) = c_0 \int_{\mathbf{R}^r} \tilde{f}(it) \phi_{it}(x) \frac{dt}{|c(it)|^2}$$

for all $x \in \Omega$.

2.16. Shifting the axis of integration. In the inversion formula (2.15.2) the integration takes place over the purely imaginary "axis" in $\mathbf{C}^r$. For our purposes we require a parallel shift of the axis of integration, as follows. Let $\sigma_0 \in \mathbf{R}$, set $\vec{\sigma}_0 = (\sigma_0, \ldots, \sigma_0) \in \mathbf{R}^r$, and write $\lambda = \vec{\sigma}_0 + it$ with $t \in \mathbf{R}^r$. From the invariance property (2.14.1) of the c-function,

$$(3) \qquad |c(\lambda)|^2 = c(\lambda)c(-\lambda) = |c(it)|^2$$

for λ of this form.

2.17. Theorem. *Suppose f is continuous on Ω and K-invariant, and assume that*

$$(1) \qquad \int_{\Omega} |f(x)| \Delta(x)^{-\sigma_0} d_* x < \infty.$$

Then $\tilde{f}(\vec{\sigma}_0 + it)$ exists for all $t \in \mathbf{R}^r$; and if

$$(2) \qquad \int_{\vec{\sigma}_0 + i\mathbf{R}^r} |\tilde{f}(\lambda)| \, \frac{d\lambda}{c(\lambda)c(-\lambda)} < \infty$$

where $\lambda = \vec{\sigma}_0 + it$, then

$$(3) \qquad f(x) = c_0 \int_{\vec{\sigma}_0 + i\mathbf{R}^r} \tilde{f}(\lambda)\phi_\lambda(x) \, \frac{d\lambda}{c(\lambda)c(-\lambda)}.$$

Proof. Set $F(x) = \Delta(x)^{-\sigma_0} f(x)$. Then from paragraph 2.12, $\tilde{F}(it)$ exists for all $t \in \mathbf{R}^r$. By the definitions (2.11.1) of ϕ_λ and (1.9.2) of $\Delta_{\lambda+\rho}$, it follows that

$$(4) \qquad \phi_{\vec{\sigma}_0 + it}(x) = \Delta(x)^{\sigma_0}\phi_{it}(x)$$

for all $x \in \Omega$ and $t \in \mathbf{R}^r$. Thus, $\tilde{F}(it) = \tilde{f}(\vec{\sigma}_0 + it)$, and (3) follows from (2.15.2).

2.18. The Plancherel Theorem. Denote by $L^2(\Omega)^K$ the space of all K-invariant measurable functions on Ω that are square-integrable with respect to d_*x; and let $L^2(i\mathbf{R}^r)^W$ denote the space of all W-invariant measurable functions on $i\mathbf{R}^r$ that are square-integrable with respect to the *Plancherel measure for Ω*

$$(1) \qquad d_*\lambda = c_0 \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

where $\lambda = it$, $t \in \mathbf{R}^r$. If F is in $L^2(\Omega)^K$ and also integrable, then the integral (2.12.1) for the spherical transform exists for all $\lambda \in i\mathbf{R}^r$, and the *Plancherel formula*

$$(2) \qquad \int_\Omega |F(x)|^2 d_*x = \int_{i\mathbf{R}^r} |\tilde{F}(\lambda)|^2 d_*\lambda$$

holds. The *Plancherel theorem* then states that the mapping $F \to \tilde{F}$ extends uniquely from the integrable functions in $L^2(\Omega)^K$ to a unitary operator, the *Plancherel transform*, from $L^2(\Omega)^K$ onto $L^2(iR^r)^W$.

In particular, if a K-invariant function F is square-integrable on Ω but is not integrable, then the spherical transform (2.12.1) exists as a limit-in-mean; e.g.,

$$(3) \qquad \tilde{F}(\lambda) = \lim_{R \to \infty, \delta \to 0} \int_{\delta\varepsilon < x < R\varepsilon} f(x)\phi_\lambda(x^{-1}) d_*x$$

where the integration takes place over all $x \in \Omega$ such that $x - \delta\varepsilon \in \Omega$ and $R\varepsilon - x \in \Omega$.

2.19. A Paley-Wiener Theorem. In [19], Wallach has proved a strong version of the Paley-Wiener theorem for the spherical transform, which we adapt to our needs. Let δ be a real number such that

$$(1) \qquad\qquad \delta > \frac{3}{2}(\mu - 1)$$

and set $E_\delta = \{\sigma \in \mathbf{R}^r : -\delta < \sigma_j < -\frac{1}{2}(\mu - 1)$ for $j = 1, \ldots, r\}$. Denote by C_ρ the convex hull of the points $w\rho \in \mathbf{R}^r$, where w runs through W (cf., paragraph 2.8), and let $E_{\delta,\rho}$ be the subdomain of E_δ consisting of all translates $\vec{\sigma}_0 + C_\rho$ of the set C_ρ where

$$(2) \qquad\qquad -\delta + \frac{1}{2}(\mu - 1) < \sigma_0 < -(\mu - 1).$$

The following Paley-Wiener theorem is a consequence of Theorem 3.3 of [19] and the fact that the spherical functions ϕ_λ are bounded for $\lambda \in C_\rho + i\mathbf{R}^r$ [13, Theorem 8.1].

2.20. Theorem. *Let the function g be holomorphic on the tube domain $E_\delta + i\mathbf{R}^r$, symmetric (which means that $g(\lambda) = g(w\lambda)$ for all $w \in W$), and assume that there exist positive constants M and Q such that*

$$(1) \qquad\qquad |g(\lambda)| \leq M e^{-Q\sum_{j=1}^r |t_j|}$$

for all $\lambda = \sigma + it \in E_\delta + i\mathbf{R}^r$. Then there exists a function F on Ω such that

$$(2) \qquad\qquad \int_\Omega |F(x)| \Delta(x)^{\sigma_0} d_* x < \infty$$

for all σ_0 satisfying (2.19.2), for which the integral (2.12.1) defining $\tilde{F}(\lambda)$ converges absolutely and

$$(3) \qquad\qquad \tilde{F}(\lambda) = g(\lambda)$$

for all λ in a tube domain $E + i\mathbf{R}^r$, where $E \supset E_{\delta,\rho}$.

2.21 Remarks. Wallach's Theorem 3.3 actually implies the following stronger conclusions than we have stated in the Theorem.

    (a) The function F in Theorem 2.20 is infinitely differentiable. Moreover, properties (2.20.2) and (2.20.3) of F can be drawn from a weaker estimate (polynomial decay) than (2.20.1). However, these results are not needed in this paper.

(b) The domain $E \subset \mathbf{R}^r$ in the Theorem contains a domain $E'_{\delta,\rho}$ even larger than $E_{\delta,\rho}$, but that domain is a little more complicated to describe. For a non-negative real number α, let $\alpha C_\rho = \{\alpha \rho' : \rho' \in C_\rho\}$. Then $E'_{\delta,\rho}$ is the union of all translates $\vec{\sigma}_0 + \alpha C_\rho$ where σ_0 satisfies (2.19.2) and $\vec{\sigma}_0 + \alpha C_\rho \subset E_\delta$.

In the case in which Ω is of rank 2, the set $E'_{\delta,\rho}$ is pictured in the diagram below. Thus, the picture is an accurate visualization of $E'_{\delta,\rho}$ for all the light cones, as well as the 2×2 matricial cones over the real numbers, complex numbers, and quaternions.
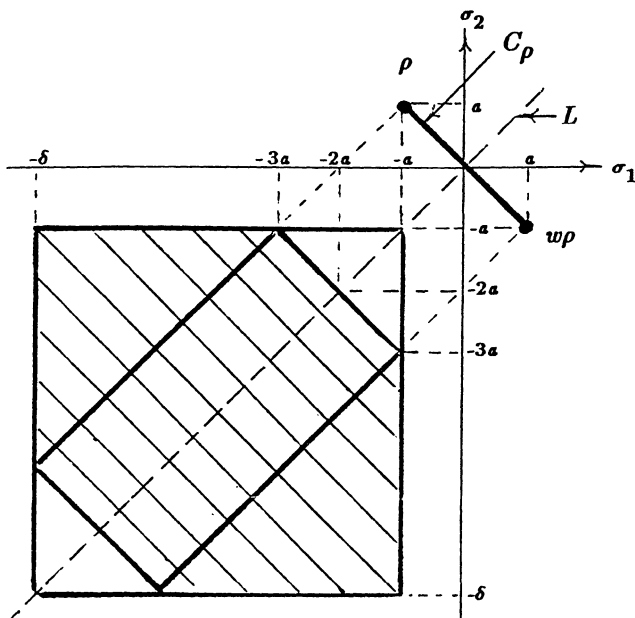
In the diagram $a = \frac{1}{2}(\mu - 1)$, and the points $\vec{\sigma}_0 = (\sigma_0, \sigma_0)$ lie on the 45°–line L. The large square is E_δ; the shaded hexagonal region inside E_δ is $E'_{\delta,\rho}$; and the rectangle positioned diagonally inside $E'_{\delta,\rho}$ is $E_{\delta,\rho}$. The complement $E_\delta - E'_{\delta,\rho}$ of $E'_{\delta,\rho}$ inside E_δ is the union of two triangles. In $E'_{\delta,\rho}$ the spherical transform $\tilde{F}(\lambda)$ is given by the absolutely convergent integral (2.12.1), while in $E_\delta - E'_{\delta,\rho}$ the spherical transform $\tilde{F}(\lambda)$ is defined as the L^2 limit-in-mean (2.18.3).

Note that only in the classical case of the real line, where $\rho = 0$ and $\mu = 1$, do E_δ and $E_{\delta,\rho}$ (or $E'_{\delta,\rho}$) coincide.

3. Ramanujan's Master Theorem.

We first state the Master Theorem formally, in the spirit of Ramanujan, without concern for mathematical rigor, after which we give a careful proof.

The object of interest is a symmetric function q holomorphic on a domain in $\mathbf{C}^r$ that contains all points of the form $m - \rho$ where m is a partition.

3.1. The formal statement. Suppose that the spherical Fourier series

$$(1) \qquad F(x) = \sum_m \frac{(-1)^{|m|}}{|m|!}\, q(m - \rho) Z_m(x)$$

converges on some neighborhood of zero in $J^{\mathbf{C}}$, and assume that the spherical transform

$$(2) \qquad \tilde{F}(\lambda) = \int_\Omega F(x)\phi_\lambda(x^{-1}) d_* x$$

exists on some domain in $\mathbf{C}^r$. Then Ramanujan's Master Theorem for Ω asserts that

$$(3) \qquad \tilde{F}(\lambda) = \Gamma_\Omega(-\lambda + \rho) q(\lambda)$$

or, by inversion of the spherical transform (and holomorphic arguments),

$$(4) \qquad F(x) = c_0 \int_{\sigma + i\mathbf{R}^r} \Gamma_\Omega(-\lambda + \rho) q(\lambda) \phi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}.$$

The problem is to prescribe conditions on the symmetric function q such that these formulas are valid.

3.2. The Hardy class $\mathcal{H}$ on Ω. Throughout this section we adopt the notation $\lambda = \sigma + it$ with $\sigma = (\sigma_1, \ldots, \sigma_r)$, $t = (t_1, \ldots, t_r) \in \mathbf{R}^r$. Let

$$(1) \qquad \delta > \frac{1}{2}(\mu - 1)$$

where, as usual, $\mu = n/r$, and denote by $H(\delta)$ the closed half-space of all $\lambda \in \mathbf{C}^r$ such that $\sigma_j \geq -\delta$. We will say that a function q of r complex variables is of class $\mathcal{H} = \mathcal{H}(\delta)$ if three properties hold:

(i) q is holomorphic on $H(\delta)$.
(ii) q is invariant under the symmetric group.
(iii) There exist positive constants M, P, and A with $A < \pi$, such that

$$(2) \qquad |q(\lambda)| \leq M\, |\Gamma_\Omega(\lambda + \rho + \mu)| \prod_{j=1}^{r} e^{P\sigma_j + A|t_j|}$$

for all $\lambda \in H(\delta)$.

3.3. The constant function $q(\lambda) \equiv 1$. If in (3.1.1) we set $q(m) = 1$ for all m, then

$$(1) \qquad e^{-\operatorname{tr} x} = \sum_m \frac{(-1)^{|m|}}{|m|!} Z_m(x)$$

defines an entire function on $J^{\mathbf{C}}$. In this case, (3.1.3) takes the form

$$(2) \qquad (e^{-\operatorname{tr} x})^{\sim}(\lambda) = \Gamma_\Omega(-\lambda + \rho)$$

for all $\lambda \in \mathbf{C}^r$ such that $\operatorname{Re} \lambda_j < -\frac{1}{2}(\mu - 1)$. For by (2.1.1), (2.11.1), and (2.11.3),

$$\Gamma_\Omega(-\lambda + \rho) = \int_\Omega e^{-\operatorname{tr} x} \Delta_{-\lambda+\rho}(x) d_* x = \int_\Omega e^{-\operatorname{tr} x} \phi_{-\lambda}(x) d_* x$$

$$= \int_\Omega e^{-\operatorname{tr} x} \phi_\lambda(x^{-1}) d_* x = (e^{-\operatorname{tr} x})^{\sim}(\lambda).$$

We verify that the function $q(\lambda) \equiv 1$ is in $\mathcal{H}(\delta)$ for any $\delta > 0$. That is, we must exhibit constants $M > 0$, $P > 0$, and $0 < A < \pi$ such that

$$(3) \qquad \left| \frac{1}{\Gamma_\Omega(\lambda + \rho + \mu)} \right| \le M \prod_{j=1}^r e^{P\sigma_j + A|t_j|}$$

for $\lambda = \sigma + it$ with $\sigma_j > -\delta$; or from (2.4.3) that the estimate

$$(4) \qquad \left| \frac{1}{\Gamma\left(\lambda_j + \frac{\mu+1}{2}\right)} \right| \le M e^{P\sigma_j + A|t_j|}$$

for the classical gamma function holds for all $j = 1, \ldots, r$. But from standard classical estimates, for any $\delta > 0$,

$$(5) \qquad \left| \frac{1}{\Gamma(x + iy)} \right| \le M e^{\left(\frac{\pi}{2} + \epsilon\right)|y|}$$

uniformly for $x > -\delta$, where $\epsilon > 0$ is arbitrary. Since (4) follows from (5), this completes the proof that $q(\lambda) \equiv 1$ is in $\mathcal{H}(\delta)$ for any $\delta > 0$.

We remark that our proof of Ramanujan's Master Theorem ultimately reduces to the special case in which $q(\lambda) \equiv 1$.

3.4. Theorem. (Ramanujan's Master Theorem for Ω.) *Let* $\delta > \frac{1}{2}(\mu - 1)$ *and suppose that* $q \in \mathcal{H}(\delta)$. *Then*

(a) *The spherical series in (3.1.1) converges on a neighborhood of zero in J and defines a real-analytic function F on that domain.*

(b) *For any $\sigma \in \mathbf{R}^r$ such that $-\delta < \sigma_j < -\frac{1}{2}(\mu - 1)$ for $j = 1, \ldots, r$, and for any $x \in \Omega$, the integral in (3.1.4) converges absolutely and is independent of*

σ, and the function F defined by (3.1.4) is a continuous extension to Ω of the function defined by (3.1.1).

(c) Let σ_0 be a real number such that $-\delta < \sigma_0 < -\frac{1}{2}(\mu - 1)$. Then

$$(1) \qquad \int_\Omega |F(x)|^2 \Delta(x)^{2\sigma_0} d_* x < \infty$$

and $\tilde{F}(\lambda)$ exists in the L^2-sense for $\lambda = \vec{\sigma}_0 + it$ (for all $t \in \mathbf{R}^r$) and satisfies (3.1.3). Moreover, (3.1.3) holds on any domain in $\mathbf{C}^r$ which contains a point, as above, of the form $\lambda = \vec{\sigma} + it$ and on which $\lambda \mapsto \tilde{F}(\lambda)$ exists and is holomorphic.

(d) Assume now that $\delta > \frac{3}{2}(\mu - 1)$, and let $-\delta + \frac{1}{2}(\mu - 1) < \sigma_0 < -(\mu - 1)$. Then

$$(2) \qquad \int_\Omega |F(x)| \Delta(x)^{\sigma_0} d_* x < \infty$$

and $\tilde{F}(\lambda)$ exists for all $\lambda \in E_{\delta,\rho} + i\mathbf{R}^r$, where $E_{\delta,\rho}$ is defined in paragraph 2.19, and (3.1.3) holds on a tube domain $E + i\mathbf{R}^r$ where $E \supset E_{\delta,\rho}$.

3.5. The classical case $n = 1$. To illustrate the proof of Theorem 3.4, shorn of the complications inherent in higher dimensions, we outline a modification of Hardy's argument in the classical case $n = 1$ in which Ω is the positive real axis $0 < x < \infty$. Here, $\lambda = \sigma + it$ is a single complex variable and q is a holomorphic function on the half-plane $\sigma > -\delta$ that satisfies the growth condition

$$(1) \qquad |q(\lambda)| \le M |\Gamma(\lambda + 1)| e^{P\sigma + A|t|}$$

for constants $M > 0$, $P > 0$, and $0 < A < \pi$. In this setting (3.1.1) has the form

$$(2) \qquad F(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!} q(m) x^m$$

and one proves that this power series converges for all $0 < x < e^{-P}$. Formula (3.1.4) takes the form

$$(3) \qquad F(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(-\lambda) q(\lambda) x^\lambda d\lambda.$$

One proves that the integral on the right side of (3) converges absolutely for all $x > 0$ (indeed, uniformly on any compact subset of $(0, \infty)$) and all $-\delta < \sigma < 0$, and that the function F on the left side of (3) coincides with the function F in (2) for all $0 < x < e^{-P}$. Finally, (3.1.3) states that

$$(4) \qquad \tilde{F}(\lambda) = \Gamma(-\lambda) q(\lambda)$$

for $-\delta < \sigma < 0$. One proves that

$$(5) \qquad \int_0^\infty |F(x)| x^{-\sigma-1} dx < \infty$$

from which $\tilde{F}(\lambda)$ exists and is equal to $\Gamma(-\lambda)q(\lambda)$ for all $\lambda = \sigma + it$. The proof proceeds as follows.

Let C be the closed rectangular contour joining the vertices $\sigma_0 - it_0$, $\sigma_0 + it_0$, $\sigma_0' + it_0$, and $\sigma_0' - it_0$ in the order indicated. Let $-\delta < \sigma_0 < 0$ and $N < \sigma_0' < N+1$ where N is a (large) positive integer. Set

$$(6) \qquad \psi(\lambda) = \Gamma(-\lambda)q(\lambda)x^\lambda$$

for Re $\lambda > -\delta$, and apply the residue theorem to obtain

$$(7) \qquad \sum_{m=0}^N \frac{(-1)^m}{m!} q(m)x^m = \frac{1}{2\pi i} \oint_C \psi(\lambda)\, d\lambda.$$

From (1) with $\lambda = m$,

$$(8) \qquad \left| \frac{(-1)^m}{m!} q(m)x^m \right| \le M(e^P x)^m,$$

which implies the convergence for $|x| < e^{-P}$ of the power series on the left of (7). By (1) and the identity (2.3.1) for the classical gamma function,

$$(9) \qquad |\psi(\lambda)| \le M \frac{\pi}{|\sin \pi\lambda|} x^\sigma e^{P\sigma + A|t|} \le M'(xe^P)^\sigma e^{-(\pi-A)|t|}.$$

From the estimate (9), by straightforward analysis, the contribution to the integral on the right of (7) attributable to the top, bottom, and right sides of the rectangle C goes to 0 as these sides go to infinity. In symbols,

$$(10) \qquad \lim_{t_0 \to \infty} \lim_{\sigma_0' \to \infty} \oint_C \psi(\lambda)\, d\lambda = \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \Gamma(-\lambda)q(\lambda)x^\lambda\, d\lambda.$$

Moreover, also from (9) the integral on the right converges absolutely for all $x > 0$, and uniformly on compact subsets of $(0, \infty)$. This proves (3).

As for (4), the function g_σ is in both $L^1(-\infty, \infty)$ and $L^2(-\infty, \infty)$, where $g_\sigma(t) = \Gamma(-\sigma - it)q(\sigma + it)$. To establish (4), we invoke a Paley-Wiener type argument. We observe, as in the estimate (9), that

$$(11) \qquad |\Gamma(-\lambda)q(\lambda)| \le M e^{P\sigma} e^{-Q|t|}$$

for all $\lambda = \sigma + it$ with $-\delta < \sigma < 0$, where $Q = \pi - A > 0$. It follows from (11) that

$$(12) \qquad \int_0^\infty |F(x)|x^{\epsilon-1}dx < \infty$$

for all $0 < \epsilon < \delta$. Indeed, for any σ such that $-\epsilon < \sigma < 0$, by (3) and (11)

$$\int_0^1 |F(x)|x^{\epsilon-1}dx \le Me^{P\sigma}\left(\int_0^1 x^{\sigma+\epsilon-1}dx\right)\left(\int_{-\infty}^\infty e^{-Q|t|}dt\right)$$

which is finite since $\sigma + \epsilon > 0$. Similarly, for $-\delta < \sigma < -\epsilon$

$$\int_1^\infty |F(x)|x^{\epsilon-1}dx \le Me^{P\sigma}\left(\int_1^\infty x^{-(1-\sigma-\epsilon)}dx\right)\left(\int_{-\infty}^\infty e^{-Q|t|}dt\right)$$

which is finite since $-\sigma - \epsilon > 0$. By (12), $\tilde{F}(\lambda)$ exists for λ in the strip $-\delta > \sigma > 0$; and by (3) and Mellin inversion, $\tilde{F}(\lambda) = \Gamma(-\lambda)q(\lambda)$. This completes the proof of Ramanujan's Master Theorem in the classical setting.

The estimate in the following lemma is required in the proof of Ramanujan's Master Theorem in the general setting of symmetric cones. By way of notation we write $0 < x < y$ to mean that $x, y \in \Omega$ and $y - x \in \Omega$.

3.6. Lemma. *For* $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbf{C}^r$, *set* $\sigma_j = \text{Re } \lambda_j$ *and assume that* $\sigma_j > \frac{1}{2}(\mu - 1)$ *for all* j. *Let* c *be any positive real number. Then* $|\phi_\lambda(x)| \le c^{\sigma_1 + \cdots + \sigma_r}$ *for all* $0 < x < c\varepsilon$.

Proof. Since (1.9.2) and (2.11.1) imply that $|\phi_\lambda(x)| \le \phi_\sigma(x)$ where $\sigma = (\sigma_1, \ldots, \sigma_r)$, it is enough to prove the result for σ. Set

$$(1) \qquad I_\sigma(x) = \int_\Omega e^{-\langle y|x^{-1}\rangle}\phi_\sigma(y)d_*y.$$

We will prove that

$$(2) \qquad I_\sigma(x) = \Gamma_\Omega(\sigma + \rho)\phi_\sigma(x)$$

in the range $\sigma_j > \frac{1}{2}(\mu - 1)$ for all j. To this end note from paragraph $1.\overset{..}{1}\overline{1}$ that $\langle y|x^{-1}\rangle = \langle P(x^{1/2})^{-1}y|\varepsilon\rangle = \text{tr}(P(x^{1/2})^{-1}y)$, and from (1.11.2) and the K-invariance of ϕ_σ,

$$\phi_\sigma(P((k \cdot x)^{1/2})y) = \phi_\sigma(P(y^{1/2})k \cdot x)$$

for all $x, y \in \Omega$. Finally, note that $I_\sigma(k \cdot x) = I_\sigma(x)$ for all $k \in K$. Then

$$
\begin{aligned}
I_\sigma(x) &= \int_K I_\sigma(k \cdot x)\, dk \\
&= \int_K \left(\int_\Omega e^{-\operatorname{tr}(P((k \cdot x)^{-1/2})y)} \phi_\sigma(y)\, d_* y \right) dk \\
&= \int_K \left(\int_\Omega e^{-\operatorname{tr} y} \phi_\sigma(P((k \cdot x)^{1/2})y)\, d_* y \right) dk \\
&= \int_K \left(\int_\Omega e^{-\operatorname{tr} y} \phi_\sigma(P(y^{1/2})k \cdot x)\, d_* y \right) dk \\
&= \int_\Omega e^{-\operatorname{tr} y} \left(\int_K \phi_\sigma(P(y^{1/2})k \cdot x)\, dk \right) d_* y \\
&= \int_\Omega e^{-\operatorname{tr} y} \phi_\sigma(y)\phi_\sigma(x)\, d_* y \\
&= \phi_\sigma(x) \int_\Omega e^{-\operatorname{tr} y} \phi_\sigma(y)\, d_* y
\end{aligned}
$$

by (2.11.6). But by (2.11.1), the last integral above is precisely $\Gamma_\Omega(\sigma + \rho)$, and we have proved (2). (We should remark that the right-hand side of (2) is finite if and only if $\sigma_j > \frac{1}{2}(\mu - 1)$ for all j, but (2) holds otherwise as the statement that both sides of (2) are infinite.)

Finally, if $0 < x < c\varepsilon$, then for all $y \in \Omega$ we have $-\langle y|x^{-1}\rangle < -\langle y|c^{-1}\varepsilon\rangle = -c^{-1}\operatorname{tr} y$; from which we conclude that

$$
\begin{aligned}
\Gamma_\Omega(\sigma + \rho)\phi_\sigma(x) = I_\sigma(x) &= \int_\Omega e^{-\langle y|x^{-1}\rangle} \phi_\sigma(y)\, d_* y \\
&< \int_\Omega e^{-c^{-1}\operatorname{tr} y} \phi_\sigma(y)\, d_* y = \Gamma_\Omega(\sigma + \rho)\phi_\sigma(c\varepsilon).
\end{aligned}
$$

Thus, $\phi_\sigma(x) < \phi_\sigma(c\varepsilon) = c^{\sigma_1 + \cdots + \sigma_r}\phi_\sigma(\varepsilon)$ by (2.11.1), and since $\phi_\sigma(\varepsilon) = 1$ the proof of the lemma is complete.

3.7. The general case. We turn to the proof of Theorem 3.4 for the general cone Ω. Let $q \in \mathcal{H}(\delta)$. By (2.7.8) and (2.7.9)

$$
\frac{(-1)^{|m|}}{|m|!}\, q(m - \rho)Z_m(x) = (-1)^{|m|}\frac{d_m}{[\mu]_m}\, q(m - \rho)\Phi_m(x)
$$

for $x \in \Omega$. Let $x = \xi\varepsilon$ where $\xi \in \mathbf{R}$ such that $0 < \xi < e^{-P}$, and ε is the identity in J. By (3.2.2), (2.2.5), (2.6.5), and (2.7.6), there exists $M' > 0$ such that

$$
\left| \frac{(-1)^{|m|}}{|m|!}\, q(m - \rho)Z_m(x) \right| \le C\, M\Gamma_\Omega(\mu)(1 + |m|)^{n-1}e^{P|m|}\xi^{|m|}
$$

$$
\le M'|m|^{n-1}(e^P \xi)^{|m|}.
$$

This estimate implies that the spherical series (3.1.1) converges absolutely for all $x \in \Omega$ such that $x < e^{-P}\varepsilon$. This proves part (a) of the theorem.

Next we consider the statement (3.1.4), which we write as

(1)
$$\sum_m \frac{(-1)^{|m|}}{|m|!} q(m-\rho)Z_m(x) = c_0 \int_{\sigma+i\mathbf{R}^r} \Gamma_\Omega(-\lambda+\rho)q(\lambda)\phi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}.$$

We establish (1) for all $x \in \Omega$ such that $x < e^{-P}\varepsilon$ and all $\sigma \in \mathbf{R}^r$ such that $-\delta < \sigma_j < -\frac{1}{2}(\mu-1)$. In what follows we iterate the classical argument of the previous section r times, a procedure that is complicated by the presence of the c-function in the inversion formula and the difficulty of computing the r-fold residues.

In analogy to (3.5.6), set

(2)
$$\psi_q(\lambda) = \psi(\lambda)q(\lambda)$$

where

(3)
$$\psi(\lambda) = c_0' \frac{\Gamma_\Omega(-\lambda+\rho)\phi_\lambda(x)}{c(\lambda)c(-\lambda)}.$$

Here, $c_0' = (2\pi i)^r c_0$ and c_0 is the constant that appears in the inversion formula for the spherical transform. Because the two cases, ν even and ν odd, are distinct in character, we separate the rest of the proof of (1) into these two cases.

3.8. The proof for ν even. By Theorem 2.14 the function $1/c(\lambda)c(-\lambda)$ is a polynomial that vanishes for $\lambda_j - \lambda_k = 0, -1, -2, \ldots, -(\frac{\nu}{2}-1)$. Here, $1 \le j,\ k \le r$ and $j \ne k$. In particular, the singularities of ψ_q all result from singularities of the factor $\Gamma_\Omega(-\lambda+\rho)$ in the numerator of (3.7.3). Hence, by (2.4.3), the singularities of ψ_q occur when $-\lambda_j - \frac{1}{2}(\mu-1)$ is a non-positive integer; i.e.,

(1)
$$\lambda_j = \ell_j - \frac{1}{2}(\mu-1)$$

with $\ell_j \ge 0$ an integer.

Fix real numbers $t_r > 0$ and σ_r, σ_r' such that

(2)
$$-\delta < \sigma_r < -\frac{1}{2}(\mu-1)$$

and

(3)
$$N < \sigma_r' + \frac{1}{2}(\mu-1) < N+1$$

with N a positive integer, and define a closed rectangular contour C_r in the λ_r-plane (with $\lambda_1, \ldots, \lambda_{r-1}$ fixed) by joining the vertices $\sigma_r - it_r$, $\sigma_r + it_r$, $\sigma_r' + it_r$, and $\sigma_r' - it_r$ in that order. If we make sure that none of the variables $\lambda_1, \ldots, \lambda_{r-1}$ are fixed at a value of the form $\ell - \frac{1}{2}(\mu - 1)$ with $\ell \geq 0$ and integral, then ψ_q has no singularities on this contour and by the residue theorem

$$(4) \qquad \sum_{\ell_r=0}^{N} (\mathrm{Res}_r \ \psi_q)(\lambda_1, \ldots, \lambda_{r-1}; \ell_r) = \frac{1}{2\pi i} \oint_{C_r} \psi_q(\lambda) d\lambda_r$$

where

$$(5) \quad (\mathrm{Res}_r \ \psi_q)(\lambda_1, \ldots, \lambda_{r-1}; \ell_r) = \lim_{\lambda_r \to \ell_r - \frac{1}{2}(\mu-1)} \left(\lambda_r - \ell_r + \frac{\mu - 1}{2} \right) \psi_q(\lambda)$$

is the residue of the function $\lambda_r \mapsto \psi(\lambda)$ at $\lambda_r = \ell_r - \frac{1}{2}(\mu - 1)$.

We proceed to the next variable λ_{r-1} and perform the same contour integration in that variable, and so on through all of the variables. After having taken r contour integrals we arrive at the iterated form of (4)

$$(6) \quad \sum_{\ell_1=0}^{N} \cdots \sum_{\ell_r=0}^{N} q\left(\ell - \frac{\mu - 1}{2} \right) (\mathrm{Res}\,\psi)\left(\ell - \frac{\mu - 1}{2} \right)$$

$$= \left(\frac{1}{2\pi i} \right)^r \oint_{C_1} \cdots \oint_{C_r} \psi_q(\lambda) d\lambda_r \ldots d\lambda_1$$

where $\ell = (\ell_1, \ldots, \ell_r)$ is an r-tuple of non-negative integers and

$$(7) \qquad (\mathrm{Res}\,\psi)\left(\ell - \frac{\mu - 1}{2} \right) = \lim_{\lambda \to \ell - \frac{1}{2}(\mu-1)} \prod_{j=1}^{r} \left(\lambda_j - \ell_j + \frac{\mu - 1}{2} \right) \psi(\lambda)$$

is the r-fold residue of ψ at $\lambda = \ell - \frac{1}{2}(\mu - 1)$. Here, by $\lim\limits_{\lambda \to \ell - \frac{1}{2}(\mu-1)}$ we mean the iterated limit as $\lambda_j \to \ell_j - \frac{1}{2}(\mu - 1)$ with $j = 1, 2, \ldots, r$.

Since ψ and q are both invariant under the symmetric group, we may assume that

$$(8) \qquad\qquad\qquad \ell_1 \geq \ell_2 \geq \cdots \geq \ell_r \geq 0$$

and write the left side of (6) as

$$(9) \qquad\qquad r! \sum_{\ell} q\left(\ell - \frac{\mu - 1}{2} \right) (\mathrm{Res}\,\psi)\left(\ell - \frac{\mu - 1}{2} \right)$$

where ℓ ranges over all partitions. We turn to estimates for the right side of (6).

By (3.2.2) and (2.4.4),

(10)

$$|\psi_q(\lambda)| \leq M 2^{n-r} \pi^n \left(\prod_{j=1}^{r} \frac{1}{|\sin \pi (\lambda_j + \frac{\mu-1}{2})|} \right) \frac{|\phi_\lambda(x)|}{|c(\lambda)c(-\lambda)|} \prod_{j=1}^{r} e^{P\sigma_j + A|t_j|}.$$

From Lemma 3.6 with $c < e^{-P}$, together with the fact that $1/c(\lambda)$ is polynomial, we obtain the estimate

(11)
$$|\psi_q(\lambda)| \leq M' \prod_{j=1}^{r} \left(\frac{1}{|\sin \pi (\lambda_j + \frac{\mu-1}{2})|} c^{\sigma_j} e^{P\sigma_j + A|t_j|} \right)$$

which is an r-fold variation of the classical estimate (3.5.9). Consequently, an iterated variant of the reasoning that led to (3.5.10) in the classical case implies that

(12)
$$\lim_{\sigma' \to \infty, t \to \infty} \oint_{C_1} \cdots \oint_{C_r} \psi(\lambda) d\lambda_r \cdots d\lambda_1$$
$$= c_0' \int_{\sigma+i\mathbf{R}^r} \Gamma_\Omega(-\lambda + \rho) q(\lambda) \phi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

where $\lambda = \sigma + it$, and also that the integral converges uniformly in x on any compact subset of Ω. From (6), (9), and (12),

(13)
$$r! \sum_\ell q \left(\ell - \frac{\mu-1}{2} \right) (\text{Res } \psi) \left(\ell - \frac{\mu-1}{2} \right)$$
$$= c_0 \int_{\sigma+i\mathbf{R}^r} \Gamma_\Omega(-\lambda + \rho) q(\lambda) \phi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

where c_0 is the constant that appears in the inversion formula (2.15.2) and ℓ ranges over all partitions.

All that remains to prove (3.1.4) is to calculate the residues of ψ on the left of (13). First, we cast out those residues which are zero. To this end, from (3.7.3) we see that $(\text{Res } \psi)(\lambda) = 0$ whenever $1/c(\lambda) = 0$; that is, when $\lambda_j - \lambda_k = \ell_j - \ell_k = 0, -1, -2, \ldots, -\frac{\nu}{2} + 1$. Thus, the summation on the left of (13) takes place over all partitions $\ell = (\ell_1, \ldots, \ell_r)$ such that

(14)
$$\ell_1 \geq \ell_2 + \frac{\nu}{2} \geq \cdots \geq \ell_j + \frac{\nu}{2}(j-1) \geq \cdots \geq \ell_r + \frac{\nu}{2}(r-1).$$

If we set

(15)
$$m_j = \ell_j + \rho_j - \frac{\mu-1}{2} = \ell_j - \frac{\nu}{2}(r-j)$$

then in light of the definition (2.4.2) of ρ, (14) can be rewritten as

$$m_1 \geq m_2 \geq \cdots \geq m_r \geq 0.$$

In terms of m, rather than ℓ, the left side of (13) is

(16)
$$\sum_m q(m - \rho)\{r!(\operatorname{Res}\psi)(m - \rho)\}$$

where the summation is over all partitions m. Finally, we evaluate the factor $r!(\operatorname{Res}\psi)(m - \rho)$ by reference to the special case of (3.7.3) in which $q(\lambda) = 1$ for all λ. (Note from paragraph 3.3 that $q(\lambda) \equiv 1$ is in $\mathcal{H}(\delta)$.) From paragraph 3.3, we know that $q(\lambda) \equiv 1$ corresponds to the spherical series for $e^{-\operatorname{tr} x}$, and from (3.3.1) and (3.3.2) we see that

(17)
$$r!(\operatorname{Res}\psi)(m - \rho) = \frac{(-1)^{|m|}}{|m|!}Z_m(x).$$

Substituting (17) and (16) into (13) we arrive at the desired formula (3.7.1). This completes the proof of part (b) of the theorem when ν is even.

3.9. The proof for ν odd. We describe the modifications of the preceding argument that are required to prove (3.7.1) when ν is odd.

By Theorem 2.14 the function $1/c(\lambda)$ now has singularities when $\lambda_j - \lambda_k = -\left(\frac{\nu}{2} + \ell\right)$ for integers $\ell \geq 0$ and for $1 \leq k \leq j \leq r$. In contrast, therefore, to the case of ν even, we see that the function ψ defined by (3.7.3) has its singularities determined not by $\Gamma_\Omega(-\lambda + \rho)$ itself, but rather by $\Gamma_\Omega(-\lambda + \rho)/c(\lambda)c(-\lambda)$. In this setting, the singularities of $\Gamma_\Omega(-\lambda + \rho)$ are cancelled by the zeros of $1/c(\lambda)c(-\lambda)$ (see Theorem 2.14 (ii)). In short, the values $\lambda = \ell - \frac{1}{2}(\mu - 1)$ are no longer singularities of ψ. Thus, we must look to $1/c(\lambda)c(-\lambda)$ for the singularities of ψ.

As in the case when ν is even, we may assume by symmetry under the symmetric group that $\sigma_1 \geq \cdots \geq \sigma_r$ where $\sigma_j = \operatorname{Re}\lambda_j$. For that reason we need only consider the singularities of $1/c(\lambda)$ and can ignore those of $1/c(-\lambda)$. The singularities of $1/c(\lambda)$ occur when

(1)
$$\lambda_j - \lambda_{j+1} = m'_j + \frac{\nu}{2}$$

with m'_j a non-negative integer. By recursion

(2)
$$\lambda_{r-j} = \lambda_r + \left(\sum_{k=1}^{j} m'_{r-k}\right) + \frac{\nu}{2}j$$

for $j = 1, \ldots, r-1$. The singularities for the variable λ_r arise from the factor $\Gamma(-\lambda_r - \frac{\mu-1}{2})$ in the numerator of the expression (3.7.3) for ψ, i.e., for

$$(3) \qquad \lambda_r = m'_r - \frac{1}{2}(\mu - 1) = m_r - \rho_r$$

where $m_r = m'_r \geq 0$ is an integer. If we insert (3) into (2), make the substitution $\frac{\mu-1}{2} = \frac{\nu}{4}(r-1)$, and let $\ell = r - j$, we obtain

$$(4) \qquad \lambda_\ell = m_\ell - \rho_\ell$$

for $\ell = 1, 2, \ldots, r$, where

$$(5) \qquad m_\ell = \sum_{k=\ell}^{r} m'_k.$$

Thus, the singularities of ψ are given by

$$(6) \qquad \lambda = m - \rho$$

where m runs through all partitions. The rest of the proof is identical to the proof for ν even, with the understanding that the integration must avoid the singular hyperplanes described in Theorem 2.14, (ii).

3.10. The proof of parts (c) and (d). To conclude the proof of Ramanujan's Master Theorem, we show that (3.1.3) follows from (3.1.4). Let $-\delta < \sigma_0 < \frac{1}{2}(\mu - 1)$ and set $g_{\sigma_0}(t) = \Gamma_\Omega(-\vec{\sigma}_0 - it + \rho)q(\vec{\sigma}_0 + it)$. Note from (3.2.2) and (2.4.4) that

$$(1) \qquad |g_{\sigma_0}(t)| \leq M \prod_{j=1}^{r} e^{-Q|t_j|}$$

for all $t \in \mathbf{R}^r$, where M depends only on δ and $Q = \pi - A$. Thus, by (iv) of Theorem 2.14

$$\int_{\vec{\sigma}_0 + i\mathbf{R}^r} |g_{\sigma_0}(t)|^2 \frac{dt}{|c(\sigma_0 + it)|^2} < \infty$$

and the Plancherel theorem (paragraph 2.18) applies. Thus, (3.4.1) is valid, $\tilde{F}$ exists in the L^2 sense, and (3.1.3) follows from (3.1.4). The remainder of part (c) follows by analytic continuation.

As for part (d), we let $g(\lambda) = \Gamma_\Omega(-\lambda + \rho)q(\lambda)$ for $\lambda = \sigma + it$ and $-\delta < \sigma_j < \frac{1}{2}(\mu - 1)$. Then, again from (3.2.2) and (2.4.4), the estimate (2.20.1) is valid with $Q = \pi - A$, and part (d) is a consequence of the Paley-Wiener Theorem in paragraph 2.20. This completes the proof of Theorem 3.4.

The following corollary generalizes the classical uniqueness theorem of M. Carlson [**3**, **11**] to the setting of symmetric cones. The proof is immediate from the Master Theorem.

3.11. Corollary. *Suppose that $q \in \mathcal{H}(\delta)$ and $q(m - \rho) = 0$ for all partitions m. Then $q(\lambda)$ vanishes identically.*

3.12. Two remarks. (1) In the next section (cf., Theorem 4.5) we will apply Ramanujan's Master Theorem when q is an invariant polynomial. As the following argument shows, in that context (3.1.3) holds for all $\lambda \in \mathbf{C}^r$ such that Re $\lambda_j < -\frac{1}{2}(\mu - 1)$.

Let q be an invariant polynomial and D_q the corresponding invariant differential operator on Ω. Let $E(x) = e^{-\operatorname{tr} x}$ and set $F = D_q E$. From (3.3.1) and (2.11.4)

$$(1) \qquad F(x) = \sum_m \frac{(-1)^{|m|}}{|m|!} \, q(m - \rho) Z_m(x)$$

for all $x \in \Omega$; and by (2.12.3) we have

$$(2) \qquad \tilde{F}(\lambda) = (D_q E)^{\sim}(\lambda) = q(\lambda)\tilde{E}(\lambda) = q(\lambda)\Gamma_\Omega(-\lambda + \rho)$$

for all $\lambda \in \mathbf{C}^r$ such that Re $\lambda_j < -\frac{1}{2}(\mu - 1)$.

(2) Ramanujan's Master Theorem can be generalized slightly, as follows. Suppose that $q \in \mathcal{H}(\delta)$ and

$$F(x) = \sum_m \frac{(-1)^{|m|}}{|m|!} \, q(m - \rho) Z_m(x)$$

is the corresponding spherical series. Let P denote the quadratic representation of the Jordan algebra J (see paragraph 1.11). Then for $y \in \Omega$, formula (3.1.3) can be rephrased as

$$(3) \qquad \int_\Omega F(P(y^{1/2})x)\phi_\lambda(x^{-1})d_*x = \Gamma_\Omega(-\lambda + \rho)q(\lambda)\phi_\lambda(y).$$

The proof is quite straightforward. Simply multiply both sides of (3.1.4) by $\phi_\lambda(y)$, use the functional equation (2.11.6) to replace the product $\phi_\lambda(x)\phi_\lambda(y)$ by an integral over K, and (since the integrals in question converge absolutely) interchange the order of integration.

4. Newton's interpolation formula for symmetric cones.

In this section we generalize the classical interpolation theorem of Newton to the setting of symmetric cones, thereby providing an explicit reconstruction of a holomorphic function $q \in \mathcal{H}(\delta)$ from its values on the set of partitions.

4.1. Newton's formula. The classical interpolation formula of Newton (cf. [**3**, p. 171], [**10**, (11.6.1)]) states that for any entire function $q : \mathbf{C} \to \mathbf{C}$ satisfying the hypotheses of Carlson's theorem,

$$(1) \qquad q(\lambda) = \sum_{j=0}^{\infty} \binom{\lambda}{j} \Delta^j q(0)$$

for $\lambda \in \mathbf{C}$, where

$$(2) \qquad \binom{\lambda}{j} = \frac{\lambda(\lambda - 1) \cdots (\lambda - j + 1)}{j!}$$

and

$$(3) \qquad \Delta^j q(0) = \sum_{\ell=0}^{j} (-1)^{j-\ell} \binom{j}{\ell} q(\ell)$$

is the *j-th difference of q at* 0. Newton's formula permits the interpolation of q from its values on the nonnegative integers.

In [**10**, §11.6], Hardy deduced Newton's formula by application of Ramanujan's Master Theorem. Here, we utilize Theorem 3.4 to obtain the analogue of Newton's formula for a symmetric cone. As preparation for the results to follow, we first study a generalization of the binomial coefficients $\binom{\lambda}{j}$ in the setting of symmetric cones.

4.2. The generalized binomial coefficients. For any two partitions m and m', define the generalized binomial coefficient $\binom{m'}{m}$ by the binomial expansion [**15**, **16**, **20**]

$$(1) \qquad \frac{Z_{m'}(e + x)}{Z_{m'}(e)} = \sum_{m} \binom{m'}{m} \frac{Z_m(x)}{Z_m(e)}.$$

From [**15**, **20**] we know that $\binom{m'}{m} = 0$ unless $m \subseteq m'$; that is, if $m' = (m'_1, \ldots, m'_r)$ and $m = (m_1, \ldots, m_r)$, then $m_j \leq m'_j$ for $j = 1, \ldots, r$. Therefore, the sum in (1) takes place over all $m \subseteq m'$. It is also known [**15**, **16**, **20**] that the generalized binomial coefficients satisfy the identity

$$(2) \qquad e^{\operatorname{tr} x} Z_{m'}(x) = |m'|! \sum_{m} \binom{m}{m'} \frac{Z_m(x)}{|m|!}.$$

Moreover, we deduce from the estimates (2.9.2) for $Z_m(x)$ that the series (2) converges absolutely for all $x \in \Omega$.

It will be useful to have a second interpretation for the generalized binomial coefficients $\binom{m'}{m}$. Thus, recall from (1.10.2) that each $x \in \Omega$ is of

the form $x = k \cdot (\xi_1 e_1 + \cdots + \xi_r e_r)$ where $\{e_1, \ldots, e_r\}$ is a Jordan frame and $\xi_1, \ldots, \xi_r \in \mathbf{R}$. Then, by [15, 20], for each partition m, there exists a unique polynomial $P_m \in \mathcal{P}(\mathbf{C}^r)^W$ such that P_m is homogeneous of degree $|m|$; and for any partition m',

$$(3) \qquad P_m \left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_r} \right) \frac{Z_{m'}(x)}{Z_{m'}(e)} \bigg|_{x=0} = \begin{cases} 1, \, m = m', \\ 0, \, m \neq m'. \end{cases}$$

Then the coefficients $\binom{m'}{m}$ are also given by the formula

$$(4) \qquad \binom{m'}{m} = P_m \left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_r} \right) \frac{Z_{m'}(x)}{Z_{m'}(e)} \bigg|_{x=e}.$$

Applying the differential operator $P_m(\partial/\partial \xi_1, \ldots, \partial/\partial \xi_r)$ to the integral formula (2.7.5), we deduce that (4) also implies $\binom{m'}{m} = 0$ if $m \not\subseteq m'$.

In preparation for the statement and proof of Newton's formula for Ω, we need three preliminary results on the generalized binomial coefficients.

4.3. Lemma. *Let $\{q(m - \rho)\}$ be a sequence of coefficients indexed by the set of partitions. Then*

$$(1) \qquad e^{\operatorname{tr} x} \sum_m \frac{(-1)^{|m|}}{|m|!} q(m - \rho) \, Z_m(x) = \sum_m p(m) \frac{Z_m(x)}{|m|!}$$

where

$$(2) \qquad p(m) = \sum_{m'} (-1)^{|m'|} \binom{m}{m'} q(m' - \rho)$$

as long as both series in (1) are absolutely convergent.

Proof. By (4.2.2),

$$e^{\operatorname{tr} x} \sum_{m'} \frac{(-1)^{|m'|}}{|m'|!} q(m' - \rho) \, Z_{m'}(x) = \sum_{m'} \frac{(-1)^{|m'|}}{|m'|!} q(m' - \rho) e^{\operatorname{tr} x} Z_{m'}(x)$$

$$= \sum_{m'} (-1)^{|m'|} q(m' - \rho) \sum_m \binom{m}{m'} \frac{Z_m(x)}{|m|!}$$

$$= \sum_m \left[\sum_{m'} (-1)^{|m'|} \binom{m}{m'} q(m' - \rho) \right] \frac{Z_m(x)}{|m|!}$$

which is the desired result.

4.4. Lemma. *Let m' be a fixed partition. Then the function $m \mapsto \binom{m}{m'}$, m a partition, is the restriction (to the space of partitions) of an entire function on $\mathbf{C}^r$. Indeed, this function is a polynomial on $\mathbf{C}^r$.*

Proof. We claim that the function

$$
\text{(1)} \qquad \lambda \to P_{m'}\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_r}\right) \phi_{\lambda-\rho}(x)\Big|_{x=e}
$$

is the required polynomial. Indeed, applying the operator

$$
P_{m'}\left(\partial/\partial\xi_1, \dots, \partial/\partial\xi_r\right)
$$

to (2.11.1) we see that (1) is a polynomial function. Moreover, from (2.7.5), (2.7.8), and (4.2.4) we deduce that when λ is a partition m, then (1) is the binomial coefficient $\binom{m}{m'}$.

In the sequel, we denote the polynomial identified in Lemma 4.4 by $\binom{\lambda}{m'}$, $\lambda \in \mathbf{C}^r$. This polynomial is the required generalization of the binomial coefficients (4.1.2).

We next use the spherical transform to obtain another interpretation for the polynomial $\binom{\lambda}{m'}$.

4.5. Proposition. *Let m' be a partition and $\lambda \in \mathbf{C}^r$ such that* $\mathrm{Re}(\lambda_j) < \frac{1}{2}(\mu - 1)$, $j = 1, 2, \dots, r$. *Then*

$$
\left(e^{-\operatorname{tr} x} Z_{m'}(x)\right)^{\sim}(\lambda) = |m'|!\,(-1)^{|m'|}\binom{\lambda+\rho}{m'}\Gamma_\Omega(-\lambda+\rho).
$$

Proof. On replacing x by $-x$ in (4.2.2), we obtain

$$
e^{-\operatorname{tr} x} Z_{m'}(x) = \sum_m q(m-\rho)\frac{Z_m(-x)}{|m|!}
$$

where

$$
q(\lambda) = |m'|!(-1)^{|m'|}\binom{\lambda+\rho}{m'}.
$$

Applying the Master Theorem for polynomials (see Remark 3.12), we obtain

$$
\left(e^{-\operatorname{tr} x} Z_{m'}(x)\right)^{\sim}(\lambda) = \Gamma_\Omega(-\lambda+\rho)q(\lambda)
$$
$$
= |m'|!(-1)^{|m'|}\binom{\lambda+\rho}{m'}\Gamma_\Omega(-\lambda+\rho)
$$

and the transform exists for all λ in the stated range.

The following result generalizes the classical interpolation formula of Newton to symmetric cones.

4.6. Theorem. *Suppose that $q \in \mathcal{H}(\delta)$ is entire. Then*

$$(1) \qquad q(\lambda) = \sum_m (-1)^{|m|} \binom{\lambda + \rho}{m} \Delta^m q(-\rho)$$

for $\lambda \in \mathbf{C}^r$, where

$$(2) \qquad \Delta^m q(-\rho) = \sum_{m'} (-1)^{|m|-|m'|} \binom{m}{m'} q(m' - \rho).$$

Proof. For $x \in \Omega$, let

$$F(x) = \sum_m \frac{(-1)^{|m|}}{|m|!} q(m - \rho) Z_m(x)$$

be the spherical series corresponding to q. Applying the Master Theorem to F, we obtain $\tilde{F}(\lambda) = \Gamma_\Omega(-\lambda + \rho) q(\lambda)$ for all $\lambda \in \mathbf{C}^r$ such that $-\delta < \mathrm{Re}\, \lambda_j < -\frac{1}{2}(\mu - 1)$ for all $j = 1, \ldots, r$. On the other hand, by Lemma 4.3,

$$\tilde{F}(\lambda) = \int_\Omega e^{-\mathrm{tr}\, x} \phi_\lambda(x^{-1}) e^{\mathrm{tr}\, x} F(x) d_* x$$

$$= \int_\Omega e^{-\mathrm{tr}\, x} \phi_\lambda(x^{-1}) \left[\sum_m p(m) \frac{Z_m(x)}{|m|!} \right] d_* x.$$

Since this last series is absolutely convergent, we may integrate term-by-term. Applying Proposition 4.5, we obtain

$$\tilde{F}(\lambda) = \sum_m \frac{p(m)}{|m|!} \int_\Omega \phi_\lambda(x^{-1}) e^{-\mathrm{tr}\, x} Z_m(x) d_*(x)$$

$$= \Gamma_\Omega(-\lambda + \rho) \sum_m (-1)^{|m|} \binom{\lambda + \rho}{m} p(m).$$

Comparing the two expressions for $\tilde{F}(\lambda)$ and writing $\Delta^m q(-\rho)$ for $(-1)^{|m|} p(m)$ we obtain (1) and (2).

5. Mellin-Barnes integrals.

Classical formulas for the Mellin transform of hypergeometric functions, known as *Mellin-Barnes integrals*, are integral representations that relate to many areas of applications. For the binomial theorem and both the confluent and Gaussian hypergeometric functions, and more generally for $_pF_p$ and $_{p+1}F_p$ for any p, these results carry over to symmetric cones as immediate applications of the Master Theorem.

5.1. Hypergeometric series. Let $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ be complex parameters and define [6, 8, 9] the *hypergeometric function* $_pF_q$ on J, or via analytic continuation on the complexification of J, by the spherical series

$$(3) \qquad _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_m \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m} \frac{Z_m(x)}{|m|!}.$$

The denominator coefficients b_j are restricted by the condition that $-b_i + (j-1)\nu/2$ is not a non-negative integer (for $1 \le i \le q$ and $1 \le j \le r$). The convergence properties are as follows: (i) If $-a_i + (j-1)\nu/2$ is a non-negative integer for some i and j, then the series terminates and $_pF_q$ is a polynomial; (ii) If $p \le q$, then the series defines an entire function $_pF_q$ on J; (iii) If $p = q+1$ the series has "radius of convergence" 1; i.e., the series converges for all $x \in J$ with polar decomposition (1.10.2) such that $|\xi_j| < 1$ for all $j = 1, \ldots, r$. (iv) If $p > q+1$ the series diverges unless it terminates.

As special cases

$$(4) \qquad _0F_0(x) = e^{-\operatorname{tr} x}$$

is the *exponential function*;

$$(5) \qquad _1F_0(a; x) = \Delta(1-x)^{-a}$$

is the (generalized) *binomial theorem*; *Bessel functions* are defined in terms of $_0F_1$; $_1F_1$ is the *confluent hypergeometric function*; and $_2F_1$ is the *Gaussian hypergeometric function*. For the fine structure of these special functions see [5, 6, 8, 9].

We establish Mellin-Barnes integrals for $_pF_p$ and $_{p+1}F_p$, p arbitrary, by showing that the appropriate functions $q(\lambda)$ are in the Hardy class $\mathcal{H}(\delta)$. Note in the case of $_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; -x)$ that the Mellin-Barnes integral extends the domain of definition to all $x \in J$.

5.2. Lemma. *Let* $a_1, \ldots, a_p, b_1, \ldots, b_p \in \mathbf{C}$, *and set*

$$(1) \qquad q(\lambda) = \prod_{i=1}^{p} \frac{\Gamma_\Omega(a_i + \lambda + \rho)\Gamma_\Omega(b_i)}{\Gamma_\Omega(a_i)\Gamma_\Omega(b_i + \lambda + \rho)}.$$

If $\operatorname{Re} a_i > \delta$ *for all* i, *then* $q \in \mathcal{H}(\delta)$.

Proof. By (2.1.3) and (2.4.3)

$$(2) \qquad q(\lambda) = \prod_{i=1}^{q} \prod_{j=1}^{r} \left(\frac{\Gamma\left(\lambda_j + a_i - \frac{\mu-1}{2}\right)\Gamma\left(b_i - (j-1)\frac{\nu}{2}\right)}{\Gamma\left(a_i - (j-1)\frac{\nu}{2}\right)\Gamma\left(\lambda_j + b_i - \frac{\mu-1}{2}\right)} \right)$$

from which we see that (i) and (ii) of the definition of $\mathcal{H}(\delta)$ in paragraph 3.2 are satisfied by q. It remains to prove that there exist constants $M > 0$, $P > 0$, and $0 < A < \pi$ such that

$$(3) \qquad \left| \frac{q(\lambda)}{\Gamma_\Omega(\lambda + \rho + \mu)} \right| \leq M \prod_{j=1}^{r} e^{P\sigma_j + A|t_j|}.$$

By (2) and (2.4.3), (3) is equivalent to the estimates

$$|\gamma_j(\lambda_j)| \leq M e^{P\sigma_j + A|t_j|}$$

where

$$(4) \qquad \gamma_j(\lambda_j) = \prod_{i=1}^{p} \frac{\Gamma\left(\lambda_j + a_i - \frac{\mu-1}{2}\right)}{\Gamma\left(\lambda_j + \frac{\mu-1}{2}\right)\Gamma\left(\lambda_j + b_i - \frac{\mu-1}{2}\right)}.$$

From Stirling's formula, or the well-known classical estimate

$$\lim_{z \to \infty} \frac{\Gamma(z+\alpha)}{z^\alpha \Gamma(\alpha)} = 1$$

for $|\operatorname{Arg} z| < \pi$, we see that

$$\left| \frac{\Gamma\left(\lambda_j + a_i - \frac{\mu-1}{2}\right)}{\Gamma\left(\lambda_j + b_i - \frac{\mu-1}{2}\right)} \right| \leq M e^{\epsilon(\sigma_j + |t_j|)}$$

for any $\epsilon > 0$. Hence, from (3.3.5) with $x + iy$ replaced by $\lambda_j + \frac{\mu+1}{2}$,

$$|\gamma_j(\lambda_j)| \leq M e^{\epsilon_1 \sigma_j + \left(\frac{\pi}{2} + \epsilon_2\right)|t_j|}$$

for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, and the proof is complete.

5.3. Corollary. *Let p be arbitrary, let $\delta > \frac{1}{2}(\mu - 1)$, and assume that $\operatorname{Re} a_i > \delta$ for $i = 1, \ldots, p$. Then*

$$_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_j; -x) = c_0 \frac{\Gamma_\Omega(b_1) \cdots \Gamma_\Omega(b_p)}{\Gamma_\Omega(a_1) \cdots \Gamma_\Omega(a_p)}$$

$$(1) \qquad \times \int_{\sigma + i\mathbf{R}^r} \Gamma_\Omega(-\lambda + \rho) \frac{\Gamma_\Omega(a_1 + \lambda) \cdots \Gamma_\Omega(a_p + \lambda)}{\Gamma_\Omega(b_1 + \lambda) \cdots \Gamma_\Omega(b_p + \lambda)} \phi_\lambda(x) \frac{d\lambda}{c(\lambda)c(-\lambda)}$$

for all $x \in \Omega$ and all σ such that $\delta < \sigma_j < -\frac{1}{2}(\mu - 1)$ for $j = 1, \ldots, r$.

The corollary follows by applying the form (3.7.1) of the Master Theorem to the function q given by (5.2.1). Indeed, from (5.2.1) and (2.2.5)

$$(2) \qquad q(m - \rho) = \frac{[a_1]_m \cdots [a_p]_m}{[b_1]_m \cdots [b_q]_m}.$$

Hence, we obtain the Mellin-Barnes formula (1) by substituting (2) into (3.7.1).

By entirely analogous arguments, one obtains the following Mellin-Barnes integral for $_{p+1}F_p$.

5.4. Corollary. *Let p be arbitrary, and assume that $\operatorname{Re} a_i > \delta$, $i = 1, \dots, p+1$, and $\operatorname{Re} b_i > \delta$, $i = 1, \dots, p$, where $\delta > \frac{1}{2}(\mu - 1)$. Then*

$$
\begin{aligned}
{}_{p+1}F_p(a_1, \dots, a_{p+1}; b_1, \dots, b_p; -x) = c_0 \, \frac{\Gamma_\Omega(b_1) \cdots \Gamma_\Omega(b_p)}{\Gamma_\Omega(a_1) \cdots \Gamma_\Omega(a_{p+1})} \\
\text{(1)} \quad \times \int_{\sigma + i\mathbf{R}^r} \Gamma_\Omega(-\lambda + \rho) \, \frac{\Gamma_\Omega(a_1 + \lambda) \cdots \Gamma_\Omega(a_{p+1} + \lambda)}{\Gamma_\Omega(b_1 + \lambda) \cdots \Gamma_\Omega(b_p + \lambda)} \, \phi_\lambda(x) \, \frac{d\lambda}{c(\lambda) c(-\lambda)}
\end{aligned}
$$

for all $x \in \Omega$ and σ such that $\delta < \sigma_j < -\frac{1}{2}(\mu - 1)$ for $j = 1, \dots, r$.

5.5. Remarks. (1) In a suitable range of values of the variables and the parameters, one can rewrite the Mellin-Barnes integrals (5.3.1) and (5.4.1) directly as spherical transforms, rather than inverse transforms.

(2) When $p < q$, the estimate (3.2.2) generally fails for the $_pF_q$ hypergeometric functions as the constant A is outside the range $(0, \pi)$. It should be noted, however, that at the end point, $A = \pi$, other Mellin-Barnes integrals can still be proved but they require a refinement of the estimate (3.3.2). For the $_0F_1$ hypergeometric function, for example,

$$
\text{(2)} \qquad {}_0F_1(b; -x) = c_0 \, \Gamma_\Omega(b) \int_{\sigma + i\mathbf{R}^r} \frac{\Gamma_\Omega(-\lambda + \rho)}{\Gamma_\Omega(b + \lambda)} \phi_\lambda(x) \, \frac{d\lambda}{c(\lambda) c(-\lambda)}
$$

for all $x \in \Omega$ and all σ such that $\delta < \sigma_j < -\frac{1}{2}(\mu - 1)$ for $j = 1, \dots, r$. The corresponding formula

$$
\text{(3)} \qquad \left({}_0F_1(b; -x)\right)^\sim(\lambda) = \frac{\Gamma_\Omega(b) \Gamma_\Omega(-\lambda + \rho)}{\Gamma_\Omega(\lambda + b)}
$$

is proved in [**5**, XV.4.5] for $\operatorname{Re} b > \delta$ and $\operatorname{Re} \lambda_j > 2\mu - 1$. The proof is based upon the convolution property of the spherical transform.

(3) We will address the subject of Mellin-Barnes integrals more fully in a later paper.

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ST. LOUIS UNIVERSITY
ST. LOUIS, MO 63103

AND

UNIVERSITY OF VERMONT
BURLINGTON, VT 05405
E-mail address: gross@uvm-gen.emba.uvm.edu

AND

UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VA 22903