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In this paper we are concerned with the existence of discrete series for semisimple homogeneous spaces. This leads to the definition of polar homogeneous spaces. We first study the harmonic analysis on compact polar homogeneous spaces. The proof of the existence of disrete series and the Plancherel formula for non-compact and non-symmetric polar homogeneous spaces will be given in a consequent paper.

1. Introduction.

One of the greatest achievements of mathematics in twentieth century is Harish-Chandra's work on discrete series for semisimple Lie groups. Suppose G is a semisimple Lie group with a maximal compact Lie group K. Harish-Chandra proved that G has discrete series if and only if the real rank of G is equal to the rank of K ([**HC**]). In other words, G has discrete series if and only if G has a compact Cartan subgroup. Harish-Chandra also obtained a classification of the discrete series for semisimple Lie groups. Harish-Chandra's results on discrete series have been generalized to semisimple symmetric spaces. Flensted-Jensen ([**FJ**]) proved that a semisimple symmetric space G/H has discrete series if

$$\text{rank of } G/H = \text{ rank of } K/K\cap H. \tag{1.1}$$

Later on Oshima and Matsuki ([**OM**]) proved if G/H has discrete series then it must satisfy the rank condition (1.1). That is to say, the semisimple symmetric space G/H has discrete series if and only if it has a compact Cartan subspace. Othima and Matsuki also gave a description of all discrete series for semisimple symmetric spaces. Since the group G can be regarded as symmetric space $G\times G/\mathrm{d}(G)$, the results of Flensted-Jensen, Othima and Matsuki are the generalizations of Harish-Chandra's work.

If G is a semisimple Lie group and H a closed reductive subgroup, it is very natural to ask the following question: when does the semisimple homogeneous space G/H has discrete series? Based on the observation of the results mentioned above, we would guess G/H has discrete series if and and if it has a compact Cartan subspace. Then we face the following fundamental question: how can we define a Cartan subspace for a semisimple homogeneous space?

It is the purpose of this paper to undertake the task to define a proper Cartan subspace for a class of semisimple homogeneous spaces. This new class of homogeneous spaces are called polar spaces in this paper and they are natural generalizations of semisimple symmetric spaces. We actually generalize all of the harmonic analysis results on compact symmetric spaces to compact spaces. We also prove that a noncompact polar space G/H has discrete series if and only if it has a compact Cartan subspace, which generalizes the results of Harish-Chandra, Flensted-Jensen, Oshima and Matsuki. The complete classification of discrete series and the Plancherel formula discrete series for nonsymmetric polar spaces will be given in a another paper [**Hu**].

Now suppose G is a connected semisimple compact Lie group and H a closed subgroup. In order for the homogeneous space G/H to have the same kind of properties of a compact symmetric space, the first thing we expect is that G/H has a Cartan subspace. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively and $\mathfrak{q}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. The tangent space of G/H at the identity eH can be identified with the vector space $\mathfrak{q}$. We want that all maximal abelian subalgebras contained in $\mathfrak{q}$ are conjugated by the action of H. This action of H is the adjoint action of G restricted to H and will be denoted by $\mathrm{Ad}_G(H)$. Hence it is very natural to require $\mathrm{Ad}_G(H)$ on $\mathfrak{q}$ to be a polar representation (cf. Section 2 for the definition). This justifies the name polar space we choose for these homogeneous spaces. It was Dadok who introduced the notion of polar representations and classified irreducible polar representations [**Da**]. The polar representations are also closed related to the work of Palais and Terng [**PT**] and the work of Heintze, Palais, Terng and Thorbergsson [**HPTT**].

More precisely, we define the homogeneous space G/H to be a simple polar space if the action of $\mathrm{Ad}_G(H)$ on $\mathfrak{q}$ is an irreducible polar representation. A compact polar space is just the direct product of simple compact polar spaces. Another equivalent definition of polar space will be given in Section 3. If G/H is a polar space, we call (G, H) a polar pair. We also call the associated Lie algebras $(\mathfrak{g}, \mathfrak{h})$ a polar pair. It is clear that given a pair of polar Lie algebras $(\mathfrak{g}, \mathfrak{h})$ all pairs of Lie groups associated with $(\mathfrak{g}, \mathfrak{h})$ are polar pairs.

An irreducible symmetric space is a simple polar space and a symmetric space is a polar space. Moreover, any symmetric pair $(\mathfrak{g}, \mathfrak{h})$ is a polar pair, therefore all pairs (G, H) associated with it are polar pairs. It turns out besides symmetric pairs there are not very many simple polar pairs. Actually up to automorphisms of $\mathfrak{g}$ there are only two simple polar pairs of Lie algebras which are not symmetric. They are $(B_3, \mathfrak{g}_2)$ and $(\mathfrak{g}_2, A_2)$.

From representation-theoretic point of view the polar spaces are more natural subjects to study than the symmetric spaces. The Cartan subspace $\mathfrak{a}$ and Weyl group W of a polar space can be defined by the corresponding notions of polar representations. All the important results for compact symmetric spaces can be extended to compact polar spaces. For instance we can show that there exists an algebra isomorphism from the set of invariant differential operators on a polar space onto the W-invariant elements $S(\mathfrak{a}_{\mathbb{C}})^W$ of symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$ over $\mathfrak{a}_{\mathbb{C}}$. This result is even true for noncompact polar spaces since it does not depends on the various real forms. If (G, H) is a non-symmetric compact simple polar pair with G simply connected and H connected, we prove in Section 4 that irreducible representations π of G has nontrivial H-fixed vectors if and only if the highest weight of π is a integral multiple of the fundamental weight with respect to the short simple root. This result is fundamental in our study of discrete series on noncompact polar spaces in [**Hu**].

The paper is organized as follows. In Section 2 we recall the definition of polar representations. Two equivalent definitions of polar spaces are given in Section 3. We also classify the simple polar pairs up to automorphisms of conjugation In Section 4 we describe the irreducible spherical representations of a polar pair (G, H). In Section 5 we describe the invariant differential operators on polar spaces. Section 6 is devoted to the analysis on polar spaces. In Section 7 we give the definition and classification of noncompact polar spaces and the proof of the existence of discrete series for nonsymmetric polar spaces. In Section 8 we define a larger class of homogeneous spaces (called generalized polar spaces) which have Cartan subspace and compare them with spherical spaces. It is worth noting that neither all generalized polar spaces nor all spherical spaces have abelian Cartan subspaces. The investigation of discrete series on them is conceivably much more difficult than that on polar spaces. However, the effort to understand the discrete series on spherical spaces or generalized polar spaces will certainly open a new direction of interesting research.

It is a pleasure to thank David Vogan and Nolan Wallach for helpful discussions.

2. Polar representations.

Let H be a compact Lie group and $\mathfrak{h}$ be its Lie algebra. Consider a representation of H on a real vector space V. Let $\langle\ ,\ \rangle$ be a H-invariant inner product on V. For a vector $v \in V$ let a_v be the subspace of V defined by

$$a_v = \{w \in V | \langle w, \mathfrak{h} \cdot v \rangle = 0\}.$$

In other words, a_v is the normal space to the H-orbit $H \cdot v$ at v. An easy fact about a_v is that it meets every H-orbit. A vector $v \in V$ is called regular if $\mathfrak{h} \cdot v$ is of maximal possible dimension.

Proposition 2.1 ([**D**]). *Fix a regular vector $v_0 \in V$. The following conditions are equivalent:*

(i) *For any regular vector $v \in V$, $\mathfrak{h} \cdot v = h \cdot (\mathfrak{h} \cdot v_0)$ for some $h \in H$.*

(ii) *For any regular vector $v \in V$, $a_v = h \cdot a_{v_0}$ for some $h \in H$.*

(iii) *For any $w \in a_{v_0}$, $\langle \mathfrak{h} \cdot w, a_{v_0} \rangle = 0$.*

Definition 2.2. A representation of H on V is called polar if it satisfies one of the three equivalent conditions in Proposition 2.1. If $v \in V$ is regular then the subspace a_v is called a Cartan subspace.

Here are a few examples of polar representations:

(a) The 1-dimensional trivial representations.

(b) The natural representations of the classical compact Lie groups.

(c) The nontrivial representations of the exceptional compact Lie groups of minimal possible dimension.

(d) The adjoint representations.

(e) The action of $\mathrm{Ad}_G(H)$ on $\mathfrak{g}/\mathfrak{h}$ provided G/H is a symmetric space.

Remark 2.3. All irreducible polar representations are classified by Dadok [**D**]. It turns out that there is only one irreducible polar representations which does not belong to above list. It is the 8-dimensional irreducible representation of Spin(7).

The polar representations play an important role in studying H-invariant differential equations by separation of variables and in analyzing the H-orbit structure in V. In the rest of this paper we will use it to study harmonic analysis on certain homogeneous spaces, which are to be defined as polar spaces.

Definition 2.4. Let (π, V) be a polar representation of H. Let a be a Cartan subspace. The Weyl group W of the polar representation (π, V) is defined as $N_H(a)/Z_H(a)$.

The Weyl group defined above is a finite group. The Weyl groups of the representation in Remark 2.3 and the 7-dimensional irreducible representation of G_2 are $\mathbb{Z}_2$. (Cf. Lemma 2.7 and Table 1 of [**DK**].) We need a Chevalley-type restriction theorem late in Section 5:

Theorem 2.5 ([DK]). *Let (π, V) be a polar representation of H. Let a be a Cartan subspace. Then the ring of H-invariant $\mathbb{C}[V]^H$ is isomorphic via restriction to the ring of Weyl group W-invariant $\mathbb{C}[a]^W$.*

3. Compact polar pairs.

In this section we will give two equivalent definitions of compact polar pairs and classify simple polar pairs. Let $\mathfrak{g}$ be a semisimple Lie algebra and σ an involution of $\mathfrak{g}$. Let $\mathfrak{h}$ and $\mathfrak{q}$ be the eigenspaces of eigenvalue 1 and -1 respectively. We have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and

$$(3.1) \qquad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}, \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}.$$

Then $\mathfrak{h}$ is actually a subalgebra of $\mathfrak{g}$ and $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. Conversely, if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and there is a subspace $\mathfrak{q}$ complement to $\mathfrak{h}$ such that the condition (3.1) is satisfied, then we can define an involution σ with $\sigma(x) = x$ for $x \in \mathfrak{h}$ and $\sigma(x) = -x$ for $x \in \mathfrak{q}$. That makes $(\mathfrak{g}, \mathfrak{h})$ a symmetric pair. Now we want to loosen the condition of existing involutions to define polar pairs. We only define polar pairs of compact type in this section. Let $\langle\ ,\ \rangle$ be the Killing form of $\mathfrak{g}$. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$. It follows from the fact that Killing form is invariant under the action of $\mathfrak{g}$ (hence under $\mathfrak{h}$) that we have $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$. For $x \in \mathfrak{q}$ denote by $\mathfrak{q}^x$ the centralizer of x in $\mathfrak{q}$. An element $x \in \mathfrak{q}$ is called regular if the tangent space $\mathrm{ad}(\mathfrak{h})x = [\mathfrak{h}, x]$ of the H-orbit $H \cdot x$ in $\mathfrak{q}$ is of maximal possible dimension.

Definition 3.1. Let $\mathfrak{g}$ be a compact semisimple Lie algebra with $\mathfrak{h}$ a subalgebra and $\mathfrak{q}$ the orthogonal complement of $\mathfrak{h}$. We say $(\mathfrak{g}, \mathfrak{h})$ is a polar pair if the following condition holds: For any regular $x \in \mathfrak{q}$, one has

(i) $\mathfrak{q}^x$ is an abelian Lie algebra.

(ii) $\mathfrak{q} = \mathfrak{q}^x \oplus [\mathfrak{h}, x]$.

(iii) $\mathfrak{g}^x = \mathfrak{h}^x \oplus \mathfrak{q}^x$.

Suppose $(\mathfrak{g}, \mathfrak{h})$ is a polar pair and (G, H) is a pair of Lie groups associated with $(\mathfrak{g}, \mathfrak{h})$, i.e. G is a connected Lie group with Lie algebra $\mathfrak{g}$ and H a closed subgroup with Lie algebra $\mathfrak{h}$. Then we say (G, H) a polar pair of Lie groups. The corresponding homogeneous space G/H is called a polar space. The space $\mathfrak{q}^x$ is called a Cartan subspace if x is regular. The rank of G/H is defined to be the dimension of a Cartan subspace.

Proposition 3.2. *If $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair, then it is a polar pair.*

Proof. Denote by σ the corresponding involution associated with $(\mathfrak{g}, \mathfrak{h})$. The Killing form $\langle\ ,\ \rangle$ is invariant under the automorphism σ. Let $\mathfrak{q}$ denote the eigenspace of σ of eigenvalue -1. For $x \in \mathfrak{q}$ regular the space $\mathfrak{q}^x$ is a Cartan subspace for the symmetric pair and hence a maximal abelian subalgebra contained in $\mathfrak{q}$ (cf. the proof of Theorem 1 and Remark 3 in [**KR**]). This shows the condition (i) of Definition 3.1 is satisfied. The condition (iii) is obviously true for symmetric pairs. Proposition 5 of [**KR**] shows that for any $x \in \mathfrak{q}$ one has $\dim \mathfrak{h}/\mathfrak{h}^x = \dim \mathfrak{q}/\mathfrak{q}^x$. Then we have $\dim \mathfrak{q} = \dim \mathfrak{q}^x + \dim [\mathfrak{h}, x]$. In order to show that the condition (ii) is satisfied, we only have to check $\mathfrak{q}^x \cap [\mathfrak{h}, x] = 0$. This follows from the identity $\langle [\mathfrak{h}, x], \mathfrak{q}^x\rangle = \langle \mathfrak{h}, [x, \mathfrak{q}^x]\rangle = 0$. □

From representation-theoretic point of view it is more natural to consider a polar space. The notions of Cartan subspaces and Weyl groups can be defined by the corresponding notions for polar representations. At the Lie algebra level there are not very many polar pairs other than symmetric pairs. We now classify all of the simple polar pairs. Given a real vector space V denote by $V_{\mathbb{C}}$ its complexification. For a regular $x \in \mathfrak{q}$ denote by $\mathfrak{a}$ the Cartan space $\mathfrak{q}^x$. Let $\Sigma = \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ be the set of roots of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Fix a choice of the set of positive roots Σ^+. Let $\mathfrak{n}_{\mathbb{C}}$ be the subspace of $\mathfrak{g}_{\mathbb{C}}$ spanned by the positive root vectors. The subspace $\mathfrak{n}_{\mathbb{C}}$ is a nilpotent subalgebra. Condition (ii) of Definition 3.4 implies the equality

$$\dim \mathfrak{q}/\mathfrak{q}^x = \dim \mathfrak{h}/\mathfrak{h}^x. \tag{3.2}$$

It is clear the centralizer $\mathfrak{g}^{\mathfrak{a}}$ of $\mathfrak{a}$ in $\mathfrak{g}$ is contained in $\mathfrak{g}^x$. Actually Condition (iii) of Definition 3.4 implies $\mathfrak{g}^{\mathfrak{a}} = \mathfrak{g}^x$. Since $\mathfrak{g}^x$ is a subalgebra, we have $[\mathfrak{h}^x, \mathfrak{q}^x] \subset \mathfrak{q}^x$. Then the identity $\langle [\mathfrak{h}^x, \mathfrak{q}^x], \mathfrak{q}^x\rangle = \langle \mathfrak{h}^x, [\mathfrak{q}^x, \mathfrak{q}^x]\rangle = 0$ implies $[\mathfrak{h}^x, \mathfrak{q}^x] = 0$. Therefore we have the equality

$$\dim_{\mathbb{C}} \mathfrak{n}_{\mathbb{C}} = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{h}^x - \dim \mathfrak{q}^x). \tag{3.3}$$

We can obtain an Iwasawa-type decomposition theorem for $\mathfrak{g}_{\mathbb{C}}$:

Lemma 3.3. *Let $(\mathfrak{g}, \mathfrak{h})$ be a polar pair defined in Definition* 3.1, *then one has*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}.$$

Proof. It follows from (3.2) and (3.3)

$$\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = \dim_{\mathbb{C}} \mathfrak{h}_{\mathbb{C}} + \dim_{\mathbb{C}} \mathfrak{a}_{\mathbb{C}} + \dim_{\mathbb{C}} \mathfrak{n}_{\mathbb{C}}.$$

We need to show $\mathfrak{h}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$ is a direct sum. Since $\mathfrak{a} \subset \mathfrak{q}$ and $[\mathfrak{a}, \mathfrak{h}] \subset \mathfrak{q}$, we have $\mathfrak{h}_{\mathbb{C}} \cap \mathfrak{a}_{\mathbb{C}} = 0$ and $\mathfrak{h}_{\mathbb{C}} \cap \mathfrak{n}_{\mathbb{C}} = 0$. The direct sum $\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ is a solvable Lie algebra. We need only to show $\mathfrak{h}_{\mathbb{C}} \cap (\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}) = 0$. Suppose it is not. There exists a nonzero element $y = y_1 + y_2 \in \mathfrak{h}_{\mathbb{C}}$ with $y_1 \in \mathfrak{a}_{\mathbb{C}}$ and $y_2 \in \mathfrak{n}_{\mathbb{C}}$. Then neither y_1 nor y_2 can be zero. Denote by $(\, , \,)$ the Killing form of $\mathfrak{g}_{\mathbb{C}}$. The spaces $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}$ are orthogonal to each other with respect to $(\, , \,)$. Hence we have $(y, y_1) = (y_1, y_1) + (y_1, y_2) = 0$. Since y_2 is the linear combination of positive root vectors, we have $(y_1, y_2) = 0$. It follows $(y_1, y_1) = 0$. This is a contradiction. □

Corollary 3.4. *If (G, H) is a polar pair as defined in Definition* 3.1, *then it is also a spherical pair. That is to say the H-fixed vectors of an irreducible G-module is at most* 1*-dimensional.*

Proof. The solvable algebra $\mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ is contained in some Borel subalgebra $\mathfrak{b}_{\mathbb{C}}$. Let $B_{\mathbb{C}}$ be a Borel subgroup of $G_{\mathbb{C}}$ correspond to $\mathfrak{b}_{\mathbb{C}}$. Then $B_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$. □

The spherical pairs (G, H) are classified by Krämer [**K2**] in the case G is simple and by Brion [**B**] in general. In order to get the complete list of all polar pairs we need only to inspect the list of the spherical pairs. We say a polar pair is simple if it cannot be decomposed into direct product of two polar pairs. The list of simple symmetric pairs is given in [**Hel**]. By examination of the lists, we have the following classification theorem:

Theorem 3.5. *Besides symmetric pairs there are only two simple polar pairs $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras up to automorphisms of $\mathfrak{g}$. They are $(B_3, \mathfrak{g}_2)$ and $(\mathfrak{g}_2, A_2)$.*

Note that if $\mathfrak{h}$ is simple and $\mathfrak{g} = \mathfrak{h} \times \mathfrak{h}$ with $\mathfrak{h}$ embedded diagonally in $\mathfrak{g}$ then $(\mathfrak{g}, \mathfrak{h})$ is also a simple symmetric pair. It is worth remarking that both of the non-symmetric polar pairs involve $\mathfrak{g}_2$ and have been interesting to theoretical physicists (cf. [**G**]).

Remark 3.6. Given a polar pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, there is a unique pair of Lie groups (G, H) associated with it providing G is connected and simply connected and H is connected. If two Lie algebra pairs $(\mathfrak{g}, \mathfrak{h})$ are conjugated by an automorphism of $\mathfrak{g}$, then their corresponding unique group pairs (G, H) are conjugated by an automorphism of G. However, if we do not require G being simply connected, then the Lie group pairs associated with the two conjugated Lie algebra pairs may not be conjugated. For example both the pair (SO(8), SO(7)) and the pair (SO(8), Spin(7)) are associated with a Lie algebra pair (so(8), so(7)) (up to an automorphism of so(8)). Another example is the pair (SO(8), SO(3)×SO(5)) and the pair (SO(8), SU(2)·Sp(2)). If the closed subgroup H is required to be connected, there is only one group pair (G_2, SU(3)) associated with the non-symmetric simple polar pair $(\mathfrak{g}_2, A_2)$ and there are two group pairs $(\mathrm{SO}(7), \mathrm{G}_2)$ and $(\mathrm{Spin}(7), \mathrm{G}_2)$ associated with the other non-symmetric simple polar pair $(B_3, \mathfrak{g}_2)$.

If (G, H) is a simple polar pair defined in Definition 3.1, then $\mathrm{Ad}_G(H)$ acting on $\mathfrak{g}/\mathfrak{h}$ is an irreducible polar representation. Actually the inverse is also true. This gives another equivalent definition of compact polar spaces.

Definition 3.7. Let G be a semisimple compact Lie group with Lie algebra $\mathfrak{g}$ and H a closed subgroup of G with Lie algebra $\mathfrak{h}$. We say $(\mathfrak{g}, \mathfrak{h})$ is a simple polar pair if the action of $\mathrm{Ad}_G(H)$ on $\mathfrak{g}/\mathfrak{h}$ is an irreducible polar representation. The corresponding homogeneous space G/H is called a simple polar space. In general, we say $(\mathfrak{g}, \mathfrak{h})$ is a polar pair if we have the decomposition of ideals $\mathfrak{g} = \oplus \mathfrak{g}_i, \mathfrak{h} = \oplus \mathfrak{h}_i$ and $\mathfrak{h}_i \subset \mathfrak{g}_i$ such that each $(\mathfrak{g}_i, \mathfrak{h}_i)$ is a simple polar pair. The corresponding homogeneous space G/H is called a polar space.

It is clear all simple symmetric pairs are simple polar pairs under this new definition. It follows from Theorem 3.5 that a polar pair defined in Definition 3.1 is a polar pair in the sense of Definition 3.7. We can classify all simple polar pairs (defined in Definition 3.7) by using the results of Krämer **[K1]**. If the action of $\mathrm{Ad}_G(H)$ on $\mathfrak{g}/\mathfrak{h}$ is irreducible, then $\mathfrak{g}$ has to be either simple or the direct product of two copies of simple Lie algebra $\mathfrak{h}$. In the case G is simple all such possible pairs (G, H) are listed in Table 1-4 in **[K1]**. Now we can inspect which pair induces an irreducible polar representation of H on $\mathfrak{g}/\mathfrak{h}$. It produces the same list we had in Theorem 3.5. Therefore we have

Proposition 3.8. *Definition* 3.1 *and Definition* 3.7 *are equivalent.*

4. Spherical representations of polar pairs.

Let (G, H) be a polar pair defined in Section 3 with the associated Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$. We say a representation π of G over a complex vector space V is spherical if π has nontrivial H-fixed vectors. In this section we characterize spherical representations of a polar pair (G, H) with G simply connected and H connected. Let $\mathfrak{a}$ be a Cartan subspace of G/H. Let $\mathfrak{b}$ be an abelian subalgebra of $\mathfrak{h}$ such that $\mathfrak{c} = \mathfrak{a} \oplus \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$. We use subscript $\mathbb{C}$ to denote a complexification of a real vector space or Lie algebra (for example $\mathfrak{a}_\mathbb{C}$ is the complexification of $\mathfrak{a}$). Let $\Sigma = \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{c}_\mathbb{C})$ denote the set of roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{c}_\mathbb{C}$. Let $\Sigma_\mathfrak{q}$ be the set of roots in Σ which do not vanish identically on $\mathfrak{a}_\mathbb{C}$. Let $\Sigma_1 = \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{a}_\mathbb{C})$ be the set of roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{a}_\mathbb{C}$. Let Σ^+, $\Sigma_\mathfrak{q}^+$ and Σ_1^+ be the compatible choices of the sets of positive roots. Here is Helgason's theorem on spherical representations of symmetric pairs (cf. **[He2]** Theorem 4.1 and Corollary 4.2 of Chapter IV).

Theorem 4.1 (Helgason). *Suppose (G, H) is a compact symmetric pair* *with G simply connected and H connected. Let $\mathfrak{a}$ be a Cartan subspace of G/H. Extend $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{c} = \mathfrak{a} \oplus \mathfrak{b}$ of $\mathfrak{g}$ with $\mathfrak{b} \subset \mathfrak{h}$. Then the following two conditions are equivalent:*

(i) *$\lambda \in \mathfrak{c}_{\mathbb{C}}^*$ is the highest weight of an irreducible spherical representation of G.*

(ii) *$\lambda(\mathfrak{b}_{\mathbb{C}}) = 0$, $\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}_+$ for any $\alpha \in \Sigma_1^+$.*

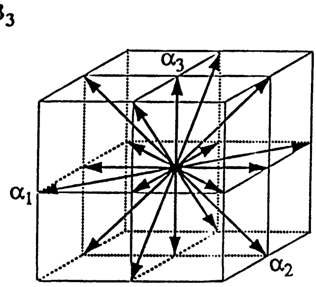

Figure 1.

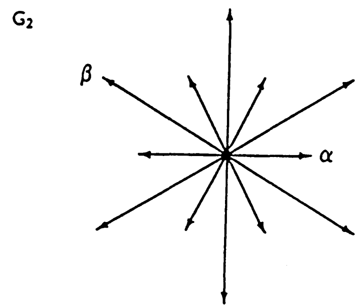

Figure 2.

Lemma 4.2. *Suppose that $(\mathfrak{g}, \mathfrak{h})$ is a non-symmetric simple polar pair. Let γ be a long simple root in Σ^+. Then the restriction of γ to $\mathfrak{a}_{\mathbb{C}}$ is zero. In other words, the vector $H_\gamma \in \mathfrak{c}_{\mathbb{C}}$ is orthogonal to $\mathfrak{a}_{\mathbb{C}}$, where $H_\gamma \in \mathfrak{c}_{\mathbb{C}}$ is determined by $\langle H, H_\gamma \rangle = \gamma(H)$ for any $H \in \mathfrak{c}_{\mathbb{C}}$.*

Proof. By the classification of polar pairs it is sufficient to verify the lemma for the polar pairs $(\mathfrak{g}_2, A_2)$ and $(B_3, \mathfrak{g}_2)$. In the case $(\mathfrak{g}_2, A_2)$, the set of positive roots Σ^+ of $\mathfrak{g}_2$ is $\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$ (see Figure 1). Here α is the short simple root and β is the long simple root. The Cartan subspace $\mathfrak{a}_{\mathbb{C}}$ is $\mathbb{C}H_{2\alpha+\beta}$. Since the number of positive roots in Σ_1^+ is 5, they are simply the projection of the positive roots of $\Sigma_{\mathfrak{q}}^+$ to $\mathfrak{a}_{\mathbb{C}}$. The subalgebra $\mathfrak{h}_{\mathbb{C}}$ is the conjugation by $\mathrm{Ad}(\mathrm{Exp} X_{2\alpha+\beta})$ of the subalgebra generated by the root vectors $\{X_\gamma\}$ associated with all long roots.

In the other case $(B_3, \mathfrak{g}_2)$, we identify the root system as the vectors $\pm(e_i \pm e_j)$ and $\pm e_i$ in $\mathbb{R}^3$ (see Figure 2). The two long simple roots are $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. The only short simple root is $\alpha_3 = e_3$. Since the number of positive roots in Σ_1^+ is 6, the restriction of a root to $\mathfrak{a}_{\mathbb{C}}$ is simply the projection of $\Sigma_{\mathfrak{q}}^+$ to the subspace $\mathbb{C}(e_1 + e_2 + e_3)$. Hence the lemma follows. $\square$

The following theorem describes the spherical representations of non-symmetric polar pairs.

Theorem 4.3. *Suppose (G, H) is a simple non-symmetric pair with G simply connected and H connected. In other words (G, H) is either* $(\mathrm{G}_2, \mathrm{SU}(3))$

or (Spin(7), G_2). *Let* $\mathfrak{a}$ *be a Cartan subspace of* G/H. *Extend* $\mathfrak{a}$ *to a Cartan subalgebra* $\mathfrak{c} = \mathfrak{a} \oplus \mathfrak{b}$ *of* $\mathfrak{g}$ *with* $\mathfrak{b} \subset \mathfrak{h}$. *Then the following two conditions are equivalent:*

(i) $\lambda \in \mathfrak{c}_{\mathbb{C}}^*$ *is the highest weight of an irreducible spherical representation of* G.

(ii) $\lambda = n\mu$, *for some positive integer* n.

Here $\mu \in \mathfrak{c}^*$ *is the fundamental weight associated with the short simple root, i.e.* μ *is determined by*

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \begin{cases} 0, & \text{if } \alpha \text{ is a long simple root} \\ 1, & \text{if } \alpha \text{ is the short simple root.} \end{cases}$$

Proof. Suppose that (π, V) is an irreducible spherical representation of (G, H) with highest weight $\lambda \in \mathfrak{c}_{\mathbb{C}}^*$ over $\mathbb{C}$ and a highest weight vector v_λ. The representation π can be extended to a representation of $G^{\mathbb{C}}$. Then its differential is an irreducible representation of $\mathfrak{g}_{\mathbb{C}}$ with nontrivial $\mathfrak{h}_{\mathbb{C}}$-annihilated vectors. Recall that we have the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$. Then V is spanned by the vectors in the set $\pi(H)v_\lambda$. Put $P = \int_H \pi(h)dh$. Then $P\pi(h)v_\lambda = Pv_\lambda$, so $PV = \mathbb{C}Pv_\lambda$. Since (G, H) is a spherical pair, the H-invariant vectors PV is 1-dimensional. Hence Pv_λ is a nonzero H-fixed vector. If $X \in \mathfrak{b}$, then

$$\pi(\exp X)Pv_\lambda = Pe^{\lambda(X)}v_\lambda = e^{\lambda(X)}Pv_\lambda.$$

Hence $e^{\lambda(X)} = 1$ and $\lambda(\mathfrak{b}) = 0$. Hence by Lemma 4.2, we have (i) implies (ii).

Let (δ, V) be the irreducible representation of G with the highest $\lambda = \mu$ and a highest weight vector v_μ. Then it is clear that δ has nontrivial H-fixed vectors. In the case $G = \mathrm{Spin}(7)$, δ is the 8-dimensional spin representation. The restriction of δ to $H = G_2$ is decomposed into direct sum of the trivial representation and the nontrivial representation of minimal possible dimension. In the case $G = G_2$, δ is the 7-dimensional representation. The restriction of δ to $H = \mathrm{SU}(3)$ is decomposed into the direct sum of the trivial representation, the 3-dimensional natural representation and its complex conjugation. In either case Pv_μ is a nonzero H-fixed vector. The irreducible representation π with the highest weight $\lambda = n\mu$ is a subrepresentation of tensor product of V of n-copies. Hence $v_\lambda = v_\mu \otimes v_\mu \otimes \cdots \otimes v_\mu$ is a highest vector of π. Then $Pv_\lambda = Pv_\mu \otimes Pv_\mu \otimes \cdots \otimes Pv_\mu$ is a nonzero H-fixed vector. This proves that (ii) implies (i). □

Remark 4.4. The nature diffeomorphisms between simple polar spaces and symmetric spaces $G_2/\mathrm{SU}(3) \cong \mathrm{SO}(7)/\mathrm{SO}(6) \cong S^6$, $\mathrm{Spin}(7)/G_2 \cong$

$SO(8)/SO(7) \cong S^7$ suggest that spherical representations of the polar pairs $(G_2, SU(3))$ and $(Spin(7), G_2)$ are related to that of symmetric pair $(SO(7), SO(6))$ and $(SO(8), SO(7))$ respectively. Actually the irreducible spherical representations of $(G_2, SU(3))$ are exactly the irreducible spherical representations of $(SO(7), SO(6))$ restricted to G_2 and the irreducible spherical representations of $(Spin(7), G_2)$ are exactly the spherical representations of $(SO(8), SO(7))$ restricted to $Spin(7)$. The similar remarks using this geometric picture can be made in the next two sections after Theorem 5.4 and Theorem 6.3.

5. Invariant differential operators on polar spaces.

In this section we define the Weyl group W of a compact polar space and show that the invariant differential operators on a polar space is isomorphic to $S(\mathfrak{a}_{\mathbb{C}})^W$, the W-invariant elements in the symmetric algebra $S(\mathfrak{a}_{\mathbb{C}})$ of a Cartan subspace $\mathfrak{a}_{\mathbb{C}}$.

Definition 5.1. Let (G, H) be a polar pair. The Weyl group of the polar space is defined to be the Weyl group of the associated polar representation of H on $\mathfrak{g}/\mathfrak{h}$ (cf. Definition 2.4).

Theorem 5.2. *Let $X = G/H$ be a polar space. The G-invariant differential operators $\mathbb{D}(X)$ on X is isomorphic to $U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}/(U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}_{\mathbb{C}} \cap U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}})$.*

This theorem follows from a more general theorem about invariant differential operators on so called reductive homogeneous spaces. (Cf. Theorem 4.6 of Chapt. II in [**He2**].) We will retain the notations in Section 3. Recall that we have the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$. Denote by ρ the half of the sum of all positive roots in $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$.

Lemma 5.3. *Let $(\mathfrak{g}, \mathfrak{h})$ be a polar pair. For each $D \in U(\mathfrak{g}_{\mathbb{C}})$ there is a unique element $D_{\mathfrak{a}_{\mathbb{C}}} \in U(\mathfrak{a}_{\mathbb{C}}) = S(\mathfrak{a}_{\mathbb{C}})$ such that*

$$D - D_{\mathfrak{a}_{\mathbb{C}}} \in \mathfrak{h}_{\mathbb{C}} U(\mathfrak{a}_{\mathbb{C}}) + U(\mathfrak{a}_{\mathbb{C}})\mathfrak{n}_{\mathbb{C}}.$$

Proof. The existence of $D_{\mathfrak{a}_{\mathbb{C}}}$ follows from the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ and PBW theorem. The uniqueness is due to the fact

$$U(\mathfrak{a}_{\mathbb{C}}) \cap (\mathfrak{h}_{\mathbb{C}} U(\mathfrak{a}_{\mathbb{C}}) + U(\mathfrak{a}_{\mathbb{C}})\mathfrak{n}_{\mathbb{C}}) = 0.$$

□

Theorem 5.4. *Let $X = G/H$ be a polar space of rank r. Let $\phi : U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{a}_{\mathbb{C}}) = S(\mathfrak{a}_{\mathbb{C}})$ be defined by*

$$\phi(D)(\lambda) = D_{\mathfrak{a}_{\mathbb{C}}}(\lambda - \rho) \quad \textit{for all} \ \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Then $U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}_{\mathbb{C}} \cap U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}$ is the kernel of ϕ. So ϕ defines an algebra isomorphism of $\mathbb{D}(X)$ onto $S(\mathfrak{a}_{\mathbb{C}})^W$. Hence $\mathbb{D}(X)$ is an polynomial algebra generated by r homogeneous algebraically independent elements.

For the case X is a symmetric space the proof is due to Harish-Chandra. The proof for the non-symmetric polar space follows the same line of Harish-Chandra's proof for symmetric spaces (cf. the proof of Theorem 5.17 of Chapt. II in [**He2**]). However, we do need some modifications to show the image of ϕ is actually the Weyl group invariant elements $S(\mathfrak{a}_{\mathbb{C}})^W$. This will be done by using the following lemma (Lemma 5.5) to replace Theorem 5.16 of Chapt. II in [**He2**] used in the proof mentioned above. To show the map ϕ is surjective we use Theorem 2.5.

Denote by $\langle\ ,\ \rangle$ the Killing form of $\mathfrak{g}_{\mathbb{C}}$. Consider $\mathfrak{g}_{\mathbb{C}}$ with the Hilbert space inner product $(X,Y) = -\langle X, \tau Y\rangle$, where τ is the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{h}_{\mathbb{C}}$, the map $(N, X, H) \mapsto \exp H \exp X \exp N$ is a holomorphic diffeomorphism of a neighbourhood of $(0,0,0)$ onto a neighbourhood $G_0^{\mathbb{C}}$ of e in $G^{\mathbb{C}}$. Here $G^{\mathbb{C}}$ is the complexification of the compact Lie group G. Hence the map A defined by

$$\exp N \exp X \exp H \mapsto X$$

is a well-defined holomorphic mapping of $G_0^{\mathbb{C}}$ into $\mathfrak{a}_{\mathbb{C}}$. We may take $G_0^{\mathbb{C}}$ as the diffeomorphic image of an open ball $B(0) \subset \mathfrak{g}_{\mathbb{C}}$ with center 0. Then $G_0^{\mathbb{C}}$ is invariant under the conjugations by elements $h \in H$ and so is $G_0 = G_0^{\mathbb{C}} \cap G$.

Lemma 5.5. *In the setting of Theorem 4.3 we have for $\lambda = n\mu \in \mathfrak{a}_{\mathbb{C}}^*$*

$$f_\lambda(g) = \int_H e^{-\lambda(A(h^{-1}gh))} dh, \quad g \in G_0$$

is a H-bi-invariant eigenfunction of each $D \in U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}$ with eigenvalue $D_{\mathfrak{a}_{\mathbb{C}}}(\lambda)$ and $f_\lambda = f_{s\lambda}$ for each $s \in W$.

Proof. Put $f_1(g) = e^{-\lambda(A(g))}$ for $g \in G_0^{\mathbb{C}}$. It is clear for any $H \in \mathfrak{h}$ we have $Hf_1 \equiv 0$, so that $DHf_1 \equiv 0$ for any $D \in U(\mathfrak{g}_{\mathbb{C}})$. We also have $NDf_1 \equiv 0$ for any $N \in \mathfrak{n}_{\mathbb{C}}$. Hence $Df_1 = D_{\mathfrak{a}_{\mathbb{C}}}(\lambda) f_1$, and

$$D\left(\int_H f_1(h^{-1}gh)dh\right) = \int_H (Df_1)(h^{-1}gh)dh = D_{\mathfrak{a}_{\mathbb{C}}}(\lambda) \int_H f_1(h^{-1}gh)dh.$$

□

6. Analysis on compact polar spaces.

In this section we define and determine the spherical functions related to the spherical representations in Section 4. We also describe the joint eigenfunctions of $\mathbb{D}(X)$ and eigenspace representations.

Definition 6.1. Let (G, H) be a compact polar pair and $X = G/H$ be the corresponding polar space. A complex valued smooth function ϕ on G with $\phi(e) = 1$ is called a spherical function if

(i) ϕ is H-bi-invariant,

(ii) $D\phi = \lambda_D \phi$, for each $D \in \mathbb{D}(X)$, where λ_D is a complex number.

As in the case of symmetric pair, the spherical functions and spherical representations are closely related. Since a polar pair (G, H) is a spherical pair, the H-bi-invariant smooth functions $C^\natural(G)$ on G is a commutative algebra with respect to the convolution. Therefore the proof of Theorem 3.4 and Theorem 4.2 of Chapt. II in [**He2**] gives the following theorem.

Theorem 6.2. *Let (G, H) be a compact polar pair. We have*

(i) *If ϕ is a spherical function, then the G representation π associated with ϕ is irreducible and spherical.*

(ii) *If π is an irreducible spherical representation with the unit H-fixed vector v_H. Then the function $\phi(g) = (v_H, \pi(g)v_H)$ is a spherical function on G. Moreover, we have*

$$\phi(g) = \int_H \chi(g^{-1}h)dh,$$

where χ is the character of π.

In the rest of this section we assume (G, H) is a polar pair with G simply connected and H connected. Let $\widehat{G}_H$ be the set of equivalent classes of spherical representations of G. For each $\delta \in \widehat{G}_H$ let V_δ be a representation space. The H-fixed vectors V_δ^H of V_δ is spanned by a single unit vector v_H. Let $C_\delta(G/H)$ denote the space consisting of functions ϕ defined by $\phi(g) = \langle v, \delta(g)v_H\rangle$. Each $C_\delta(G/H)$ has a unique spherical function $\phi_\delta(g) = \langle v_H, \delta(g)v_H\rangle$. Denote by α_δ the homomorphism $\mathbb{D}(G/H) \to \mathbb{C}$ determined by $D\phi_\delta = \alpha_\delta(D)\phi_\delta$ for each $D \in \mathbb{D}(G/H)$. The joint eigenspace is defined to be

$$E_\delta(G/H) = \{f \in C^\infty(G/H) \mid Df = \alpha(D)f, \quad \text{for all } D \in \mathbb{D}(G/H)\}.$$

We can use Theorem 3.5 of Chapt. V in [**He2**] to obtain the following theorem.

Theorem 6.3. *We have the following Hilbert space decomposition*

$$L^2(G/H) = \bigoplus_{\delta \in \widehat{G}_H} C_\delta(G/H).$$

Moreover, $C_\delta(G/H) = E_\delta(G/H)$ *and the natural representation on* $C_\delta(G/H)$ *is of class* δ *and it coincides with the eigenspace representation on* $E_\delta(G/H)$.

7. Noncompact polar spaces and discrete series.

This section is devoted to noncompact polar spaces. From now on we denote by $\mathfrak{g}$ a noncompact semisimple Lie algebra with a Cartan involution θ. Let $\mathfrak{h}$ be a θ-stable reductive subalgebra of $\mathfrak{g}$. We can define $(\mathfrak{g}, \mathfrak{h})$ to be a polar pair if its compact form $(\mathfrak{g}^c, \mathfrak{h}^c)$ is a compact polar pair. We may also define the noncompact polar pairs in the same fashion as we did for compact polar pairs in Definition 3.1. Let $\langle\ ,\ \rangle$ be the Killing form on $\mathfrak{g}$. Define an inner product on $\mathfrak{g}$ by $(X,Y) = -\langle X, \theta Y\rangle$, $X, Y \in \mathfrak{g}$. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. It is clear that $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$. A element $x \in \mathfrak{q}$ is called regular if $\mathrm{Ad}(\mathfrak{h})x = [\mathfrak{h}, x]$ is of maximal possible dimension.

Definition 7.1. We say $(\mathfrak{g}, \mathfrak{h})$ is a polar pair, if the following condition is satisfied: for any regular semisimple element $x \in \mathfrak{q}$ one has

(i) $\mathfrak{q}^x$ is an abelian Lie algebra.

(ii) $\mathfrak{q} = \mathfrak{q}^x \oplus [\mathfrak{h}, x]$.

(iii) $\mathfrak{g}^x = \mathfrak{h}^x \oplus \mathfrak{q}^x$.

Remark 7.2. It is clear that if $(\mathfrak{g}, \mathfrak{h})$ is a noncompact polar pair then its compact form $(\mathfrak{g}^c, \mathfrak{h}^c)$ is a compact polar pair defined in Section 3. Therefore for a regular semisimple $x \in \mathfrak{q}$ the space $\mathfrak{q}^x$ consists of only semisimple elements and will be called a Cartan subspace. If (G, H) is a Lie group pair associated with a noncompact polar pair $(\mathfrak{g}, \mathfrak{h})$, then we say (G, H) is a noncompact polar pair of Lie groups. The corresponding homogeneous space G/H is called a noncompact polar space. Note that a nonlinear semisimple Lie group may not have a compact form. However, if G is linear then (G, H) is a noncompact polar pair if and only if its compact form (G^c, H^c) is a compact polar pair.

By the same argument used in the proof of Proposition 3.2 we know that any noncompact symmetric pair is a noncompact polar pair. Denote by $G_{2(2)}$ the noncompact (split) simple Lie group of type $\mathfrak{g}_2$. The following are a few examples of noncompact polar pair of Lie groups which do not associated with symmetric pair of Lie algebras: $(G_{2(2)}, SU(2,1))$, $(G_{2(2)}, SL(3,\mathbb{R}))$, $(Spin_e(3,4), G_{2(2)})$ and $(SO_e(3,4), G_{2(2)})$. Note that there are also noncompact polar pairs of groups which are associated with noncompact symmetric pair of algebras such as ($SO_e(4,4)$, $Spin_e(3,4)$). By Remark 7.2 and Theorem 3.5 we have

Theorem 7.3. *Besides noncompact symmetric pairs there are only three noncompact simple polar pairs* $(\mathfrak{g}, \mathfrak{h})$ *up to automorphisms of* $\mathfrak{g}$. *They are* $(\mathfrak{g}_{2(2)}, \mathrm{su}(2,1))$, $(\mathfrak{g}_{2(2)}, \mathrm{sl}(3,\mathbb{R}))$ *and* $(\mathrm{so}(3,4), \mathfrak{g}_{2(2)})$.

It is well known that a symmetric space has a discrete series if and only if it has a compact Cartan subspace. This is due to Harish-Chandra for the group case and Flensted-Jensen, Sekiguchi and Oshima for the symmetric space case. We would like to extend this result to polar spaces.

Theorem 7.4. *Let U/K be a compact polar space with U connected and simple connected and K connected. Let G/H be a noncompact real form of U/K (which means $G_{\mathbb{C}} = U_{\mathbb{C}}$ and $K_{\mathbb{C}} = H_{\mathbb{C}}$). Then G/H has discrete series if and only if it has a compact Cartan subspace.*

Here is the idea of the proof of the theorem. By Theorem 7.3 and the known results on symmetric spaces it suffices to verify the theorem for the three nonsymmetric simple polar spaces $G_{2(2)}/SU(2,1)$, $G_{2(2)}/SL(3,\mathbb{R})$ and $Spin(3,4)/G_{2(2)}$. Since all of the three polar spaces have compact Cartan subspaces we need to show that they all have discrete series. We actually classify all discrete series and obtain the Plancherel formulae on these three noncompact polar spaces in [**Hu**]. Here we only prove the existence of discrete series for the three polar spaces.

The three non-symmetric polar spaces $X = G/H$, $G_{2(2)}/SU(2,1)$, $G_{2(2)}/SL(3,\mathbb{R})$ and $Spin_e(3,4)/G_{2(2)}$, are isomorphic to real hyperbolic spaces in a special way. Now we describe the isomorphisms. Consider the following three quadruplets:

(1) Let G=$Spin_e(3,4)$ be the subgroup of G'=$SO_e(4,4)$. Let H'=$SO_e(4,3)$ be the standard subgroup of G'. Then $H = G \cap H'$ is isomorphic to $G_{2(2)}$.

(2) Let G=$G_{2(2)}$ be the subgroup of G'=$SO_e(4,3)$. Let H'=$SO_e(4,2)$ be the standard subgroup of G'. Then $H = G \cap H'$ is isomorphic to $SU(2,1)$.

(3) Let G=$G_{2(2)}$ be the subgroup of G'=$SO_e(3,4)$. Let H'=$SO_e(3,3)$ be the standard subgroup of G'. Then $H = G \cap H'$ is isomorphic to $SL(3,\mathbb{R})$.

List of the Three Quadruplets				
	G	G′	H′	H
(1)	$\mathrm{Spin}_e(3,4)$	$\mathrm{SO}_e(4,4)$	$\mathrm{SO}_e(4,3)$	$\mathrm{G}_{2(2)}$
(2)	$\mathrm{G}_{2(2)}$	$\mathrm{SO}_e(4,3)$	$\mathrm{SO}_e(4,2)$	SU(2,1)
(3)	$\mathrm{G}_{2(2)}$	$\mathrm{SO}_e(3,4)$	$\mathrm{SO}_e(3,3)$	SL(3,$\mathbb{R}$)

In all of the three cases the inclusion of $G \hookrightarrow G'$ induces a diffeomorphism of G/H onto G'/H'. In other words, the natural embedding $G/H \hookrightarrow G'/H'$ is surjective. The very significance of these diffeomorphisms is that we can make use the known results [**R**] on real hyperbolic spaces to obtain the desired results on polar spaces. A discrete series representation of G'/H' restricted to G is an irreducible representation of G, which is a discrete series for G/H.

8. Generalized polar spaces and spherical spaces.

Let G be a semisimple compact Lie group with Lie algebra $\mathfrak{g}$. Let and H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. The homogeneous space G/H is called a simple polar space if the action $\mathrm{Ad}_G(H)$ on $\mathfrak{g}/\mathfrak{h}$ is an irreducible polar representation. This was defined in Section 3. We may loosen the condition that the representation $\mathrm{Ad}_G(H)$ is irreducible to define generalized polar spaces.

Definition 8.1. Let G be a semisimple compact Lie group with Lie algebra $\mathfrak{g}$ and H a closed subgroup of G with Lie algebra $\mathfrak{h}$. We say (G, H) is a generalized simple polar pair if the action of $\mathrm{Ad}_G(H)$ on $\mathfrak{g}/\mathfrak{h}$ is a polar representation (cf. Section 2). The corresponding homogeneous space G/H is called a generalized simple polar space. A generalized compact polar space is defined to be the direct product of generalized simple polar spaces.

One can define the Cartan subspace and Weyl group for generalized polar spaces accordingly. It is proved in Section 3 that all polar spaces are spherical spaces. One might hope that all generalized polar spaces are also spherical spaces, which is unfortunately false. One might hope that all spherical space are generalized polar spaces, which is not true either.

Remark 8.2. There are spherical spaces which are not generalized polar spaces. Consider $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $H = \mathrm{SU}(2)$ embedded in G diagonally. The homogeneous space G/H is a spherical space. However, the representation $\mathrm{Ad}_g(H)$ on $\mathfrak{g}/\mathfrak{h}$ is not polar.

It is conceivable that harmonic analysis on either generalized polar spaces or spherical spaces are quite different from that on the polar spaces. This is because neither the generalized polar spaces nor the spherical spaces have abelian Cartan subspaces. The investigation of discrete series on the noncompact form of these homogeneous spaces will be substantially more difficult than that on polar spaces. However, we hope that the effort to understand the harmonic analysis on generalized polar spaces and spherical spaces will open the new direction of interesting research.

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