

# *Pacific Journal of Mathematics*

**ON A THEOREM OF KOCH**

FARSHID HAJIR

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**We give a short proof of a slightly stronger version of a theorem of Koch: A complex quadratic field whose ideal class group contains a subgroup of type  $(4, 4, 4)$  possesses an infinite unramified Galois pro-2 extension.**

### 1. Koch's Theorem.

If  $K$  is a finite extension of  $\mathbb{Q}$  and  $p$  is a prime number, let  $K^{(0)} = K$  and for  $n \geq 1$  define  $K^{(n)}$  to be the maximal abelian unramified  $p$ -extension of  $K^{(n-1)}$ . The smallest  $n$  such that  $K^{(n)} = K^{(n+1)}$  is called the length of the  $p$ -class field tower of  $K$ ; if no such integer  $n$  exists, we say that  $K$  has infinite  $p$ -class field tower. By a group of type  $(m_1, \dots, m_t)$  we understand a group isomorphic to  $\mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z}$ . The purpose of this note is to give a short proof of (a slightly strengthened version of) a theorem of Koch [4]:

**Theorem 1.** *If  $K$  is a complex quadratic field whose ideal class group contains a subgroup of type  $(4, 4, 4)$ , then the 2-class field tower of  $K$  is infinite.*

Koch's proof proceeds by showing that in a minimal presentation of the Galois group of the maximal unramified 2-extension of  $K$  by a free pro-2 group  $G$ , the relations lie deep in the Zassenhaus filtration of  $G$ . We replace this key ingredient of his proof, which can be thought of as the study of the quadratic unramified extensions of the genus field of  $K$  which are central over  $K$  [3, Satz 1], with a simple result from genus theory. Moreover, Koch's proof requires a generalization of the Vinberg/Gashütz sharpening of the Golod-Shafarevich theorem on the structure of pro- $p$  groups [4, Satz 3]; for our proof, the original Vinberg/Gashütz inequality suffices (for an account of these inequalities, see, e.g., Koch's book [5]). Indeed, we will need only the following result (see Martinet [8]):

**Theorem 2.** *Suppose  $F$  is a totally real field of degree  $n$ , and  $E$  is a totally complex quadratic extension of  $F$ . Let  $t$  be the number of prime ideals of  $F$  which ramify in  $E$ . The 2-rank of the ideal class group of  $E$  is at least  $t - 1$ . If*

$$t \geq 3 + 2\sqrt{n + 1},$$

then the 2-class field tower of  $E$  is infinite.

**Corollary 3.** *Suppose  $F$  is a totally real degree 4 extension of  $\mathbb{Q}$ . If two rational primes that split completely in  $F$  ramify in a complex quadratic field  $L$ , then  $E = FL$  has an infinite 2-class field tower.*

*Proof.* With notation as in the theorem, we have  $t \geq 8 \geq 3 + 2\sqrt{4+1}$ .  $\square$

*Proof of Theorem 1.* We know that at least four primes divide the discriminant  $D$  of  $K$ . If six or more primes divide  $D$ , then an application of Theorem 2 to  $K/\mathbb{Q}$  already yields the result. Assume first that exactly four primes divide  $D$ . By the criterion of Rédei-Reichardt [9] on the 4-rank of the class group of  $K$ , one knows that  $D = -p_1 \cdot p_2 \cdot p_3 \cdot p_4$  where  $p_2, p_3, p_4$  are odd primes satisfying  $\left(\frac{p_i}{p_j}\right) = +1$  for  $i, j > 1, i \neq j$ , and one of the following is satisfied:

- (I)  $p_1 = 4; p_j \equiv 1 \pmod{8}, j = 2, 3, 4$ .
- (II)  $p_1 = 8; p_j \equiv 1 \pmod{8}, j = 2, 3, 4$ .
- (III)  $p_1 = 8; p_2 \equiv 7 \pmod{8}; p_j \equiv 1 \pmod{8}, j = 3, 4$ .
- (IV)  $p_1 \equiv 3 \pmod{4}$  is an odd prime,  $p_j \equiv 1 \pmod{4}, j = 2, 3, 4$ ,  
and  $\left(\frac{p_1}{p_j}\right) = +1$  for  $j = 2, 3, 4$ .

Incidentally, Koch's theorem was originally stated for case (IV) only. Let  $F = \mathbb{Q}(\sqrt{p_3}, \sqrt{p_4})$  and  $E = F(\sqrt{-p_1 \cdot p_2})$ . In all cases,  $(p_2)$  and the unique rational prime divisor of  $(p_1)$  split completely in  $F$ . Hence, by Corollary 3,  $E$  has an infinite 2-class field tower. Since  $E/K$  is an unramified 2-extension,  $K$  has an infinite 2-class field tower as well. Now suppose exactly five primes  $p_1, \dots, p_5$  divide the discriminant of  $K$ ; using the Rédei-Reichardt criterion [9], or its equivalent form due to Rédei [10], it is straightforward to check that for some  $i, 1 \leq i \leq 5$ , we have

$$p_i \equiv 1 \pmod{4}, \quad \left(\frac{p_i}{p_j}\right) = 1, j \neq i.$$

Now let  $F = \mathbb{Q}(\sqrt{p_i}), E = K(\sqrt{p_i})$ ;  $E/F$  is a CM-extension with 8 ramified primes. By Theorem 2,  $E$  has an infinite 2-class field tower, and so does  $K$ .  $\square$

## 2. Further Remarks.

Koch and Venkov [6] have proved that a complex quadratic field whose ideal class group has a subgroup of type  $(p, p, p)$  for some odd prime  $p$  has an

infinite  $p$ -class field tower. Therefore, a complex quadratic field possesses an infinite Hilbert class field tower whenever its ideal class group contains a subgroup of type  $(m, m, m)$  with  $m \geq 3$ . On the other hand, the field  $\mathbb{Q}(\sqrt{-105})$ , whose ideal class group is of type  $(2, 2, 2)$ , has a finite class field tower, since its root discriminant is just below the Odlyzko bound (see e.g. [8]). I am indebted to the referee for the above remark.

Note that the proof of Koch's theorem we have given relies only on the existence of two primes that split completely in a real biquadratic field. For instance, the primes 31, 89 split completely in  $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ , hence  $\mathbb{Q}(\sqrt{-2 \cdot 5 \cdot 31 \cdot 89})$  has an infinite 2-class field tower; its 2-ideal class group is of type  $(4, 2, 2)$ .

Taussky-Todd [12] proved that a number field with 2-ideal class group of type  $(2, 2)$  has a finite 2-class field tower of length at most 2. It is natural to ask whether there are number fields with infinite 2-class field tower whose 2-class group is of type  $(4, 2)$  or  $(2, 2, 2)$  (simplest non-cyclic 2-groups after type  $(2, 2)$ ). Using a minor variation on an idea first introduced by Schoof [11], we now show that there are complex quadratic fields with these properties. Consider, for example,  $K = \mathbb{Q}(\sqrt{-5 \cdot 7 \cdot 41 \cdot 61})$ , which has 2-ideal class group of type  $(2, 2, 2)$ . To show that this field has infinite 2-tower, let  $H_0$  be the Hilbert class field of  $K_0 = \mathbb{Q}(\sqrt{5 \cdot 41 \cdot 61})$ , a real quadratic field with class number 16. Since 7 is inert in  $K_0$ , it splits into 16 prime ideals in  $H_0$ , all of which ramify in the CM extension  $L = H_0(\sqrt{-7})$ . Theorem 2 shows that  $L$ , an unramified 2-extension of  $K$ , has an infinite 2-class field tower, proving the claim. In fact, for any prime  $q$  satisfying  $q \equiv 7 \pmod{5 \cdot 41 \cdot 61}$  (there are infinitely many such primes by Dirichlet's theorem), the same argument shows that  $K_q = \mathbb{Q}(\sqrt{-5 \cdot 41 \cdot 61 \cdot q})$  has infinite 2-class field tower; furthermore, by Rédei-Reichardt,  $K_q$  has 2-class group of type  $(2, 2, 2)$ .

For the second example, let  $K = \mathbb{Q}(\sqrt{-5 \cdot 11 \cdot 461})$ ; this field has 2-ideal class group of type  $(4, 2)$ . Observe that the rational prime ideal (11) splits into 16 prime ideals in  $H_0$ , the Hilbert class field of the real quadratic field  $K_0 = \mathbb{Q}(\sqrt{5 \cdot 461})$  with class number 16. Therefore, by the same argument as above,  $L = H_0(\sqrt{-11})$ , and thereby  $K$ , have infinite 2-class field tower. Let  $H$  be the 2-Hilbert class field of  $K$ . Benjamin [1] has shown that the 2-class field tower of a complex quadratic field  $E$  with 2-class group of type  $(4, 2)$  has length at most 2 if the 2-Hilbert class field of  $E$  has elementary 2-class group ( $E = \mathbb{Q}(\sqrt{-5 \cdot 13})$  is an example). Since  $K$  has infinite 2-tower, we conclude that  $H$  does not have elementary abelian 2-class group.

Finally, note that the 2-rank of the ideal class group of  $L$  is at least 15. Using Louboutin [7], we compute the 2-rank of the ideal class group of the biquadratic field  $E = \mathbb{Q}(\sqrt{-11}, \sqrt{5 \cdot 461})$  to be 2. The arguments of [2] then show that the 2-rank of the ideal class group of  $L$  is 15, 16 or 17.

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