

Pacific Journal of Mathematics

**PROOF OF LONGUERRE'S THEOREM AND ITS EXTENSIONS
BY THE METHOD OF POLAR COORDINATES**

ZHIHONG YU

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There are several methods to prove the well-known Longuerre's theorem and its extensions in plane geometry. We now prove them by the method of polar coordinates. Our proof is characterized by its directness, simplicity, regularity, originality, and no need for any auxiliary lines.

Longuerre's Theorem. *Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle, on which p is an arbitrary point. Let S_i denote the Simson line of point p with respect to the triangle $A_jA_kA_l$ (i, j, k, l distinct) and let D_i denote the projection of p on S_i .*

The four points D_1, D_2, D_3, D_4 are collinear.

Proof. We establish a polar coordinates system (Fig. 1) with p being the pole and the extension line of po being the polar axis. Let d be the diameter of the circle. Hence the equation of the circle is $\rho = d \cos \theta$. Let $(d \cos \theta_i, \theta_i)$ ($i = 1, 2, 3, 4$), $\theta_i \in [0, 2\pi]$ be the coordinates of A_1, A_2, A_3, A_4 . Hence the two-point form equation of A_1A_2 is

$$\begin{aligned} \frac{\sin(\theta_2 - \theta_1)}{\rho} &= \frac{\sin(\theta_2 - \theta)}{d \cos \theta_1} + \frac{\sin(\theta - \theta_1)}{d \cos \theta_2}, \\ \therefore \rho[\sin(\theta_2 - \theta) \cos \theta_2 + \sin(\theta - \theta_1) \cos \theta_1] \\ &= d \sin(\theta_2 - \theta_1) \cos \theta_1 \cos \theta_2. \\ \therefore \frac{1}{2} \rho[\sin(2\theta_2 - \theta) + \sin(\theta - 2\theta_1)] \\ &= d \sin(\theta_2 - \theta_1) \cos \theta_1 \cos \theta_2, \\ \therefore \rho \sin(\theta_2 - \theta_1) \cos(\theta - \theta_1 - \theta_2) &= d \sin(\theta_2 - \theta_1) \cos \theta_1 \cos \theta_2, \\ \therefore \sin(\theta_2 - \theta_1) &\neq 0, \\ \therefore \rho \cos(\theta - \theta_1 - \theta_2) &= d \cos \theta_1 \cos \theta_2. \end{aligned}$$

This is exactly the normal form equation of A_1A_2 . Hence we have the coordinates of the foot B_1 at which pB_1 is normal to A_1A_2 : $B_1(d \cos \theta_1 \cos \theta_2, \theta_1 + \theta_2)$. By means of cyclic permutation of indices we get $B_2(d \cos \theta_2 \cos \theta_3,$

$\theta_2 + \theta_3$), $B_3(d \cos \theta_3 \cos \theta_1, \theta_3 + \theta_1)$. Obviously the coordinates of the three feet B_i satisfy the normal form equation

$$\rho \cos(\theta - \theta_1 - \theta_2 - \theta_3) = d \cos \theta_1 \cos \theta_2 \cos \theta_3.$$

Hence we get the normal form equation of the Simson line S_1 of point p with respect to $\Delta A_1 A_2 A_3$:

$$S_1 : \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3) = d \cos \theta_1 \cos \theta_2 \cos \theta_3.$$

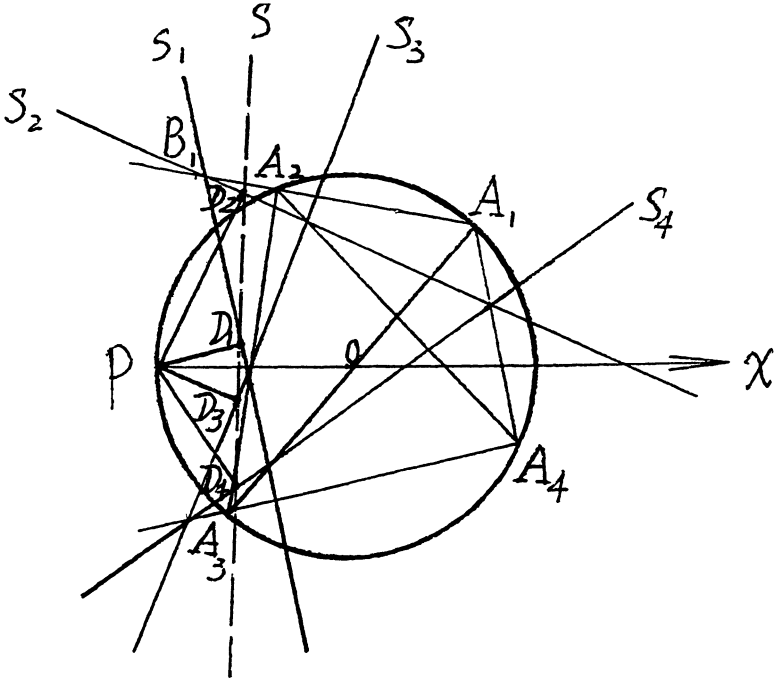


Figure 1.

Similarly, by the same means we can obtain the normal form equations of the other three Simson lines with respect to $\Delta A_j A_k A_l$. They are

$$S_2 : \rho \cos(\theta - \theta_1 - \theta_2 - \theta_4) = d \cos \theta_1 \cos \theta_2 \cos \theta_4,$$

$$S_3 : \rho \cos(\theta - \theta_2 - \theta_3 - \theta_4) = d \cos \theta_2 \cos \theta_3 \cos \theta_4,$$

$$S_4 : \rho \cos(\theta - \theta_3 - \theta_4 - \theta_1) = d \cos \theta_3 \cos \theta_4 \cos \theta_1.$$

Hence the coordinates of the four projections D_i are:

$$D_1(d \cos \theta_1 \cos \theta_2 \cos \theta_3, \theta_1 + \theta_2 + \theta_3),$$

$$\begin{aligned} D_2(d \cos \theta_1 \cos \theta_2 \cos \theta_4, \theta_1 + \theta_2 + \theta_4), \\ D_3(d \cos \theta_2 \cos \theta_3 \cos \theta_4, \theta_2 + \theta_3 + \theta_4), \\ D_4(d \cos \theta_3 \cos \theta_4 \cos \theta_1, \theta_3 + \theta_4 + \theta_1). \end{aligned}$$

It is obvious that the above-mentioned coordinates satisfy the normal form equation of the line

$$S: \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3 - \theta_4) = d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4.$$

Thus the four points D_i are collinear. □

The above equation of S represents a straight line containing the points D_i and this straight line is named the Simson line of a point p with respect to four concyclic points.

Extension I. *Let A_1, A_2, A_3, A_4, A_5 be five points on a circle and let p be an arbitrary point on this circle. Let S_i denote the Simson line of p with respect to the 4-tuple $A_j A_k A_l A_m$ (i, j, k, l, m distinct) and let D_i denote the projection of p on the line S_i ($i = 1, 2, 3, 4, 5$). Then the five points D_i are collinear.*

Proof. We establish a polar coordinates system (Fig. 2) with p being the pole and the extension line of po being the polar axis. Let d be the diameter of the circle. The equation of the circle is $\rho = d \cos \theta$. Let $(d \cos \theta_i, \theta_i)$ ($\theta_i \in [0, 2\pi]$) be the coordinates of A_i . According to the above Longuerre's theorem and its proof we can get the normal form equations of S_i . They are:

$$\begin{aligned} S_1: \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3 - \theta_4) &= d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4, \\ S_2: \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3 - \theta_5) &= d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_5, \\ S_3: \rho \cos(\theta - \theta_2 - \theta_3 - \theta_4 - \theta_5) &= d \cos \theta_2 \cos \theta_3 \cos \theta_4 \cos \theta_5, \\ S_4: \rho \cos(\theta - \theta_3 - \theta_4 - \theta_5 - \theta_1) &= d \cos \theta_3 \cos \theta_4 \cos \theta_5 \cos \theta_1, \\ S_5: \rho \cos(\theta - \theta_4 - \theta_5 - \theta_1 - \theta_2) &= d \cos \theta_4 \cos \theta_5 \cos \theta_1 \cos \theta_2. \end{aligned}$$

Hence the coordinates of the five projections D_i are:

$$\begin{aligned} D_1(d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4, \theta_1 + \theta_2 + \theta_3 + \theta_4), \\ D_2(d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_5, \theta_1 + \theta_2 + \theta_3 + \theta_5), \\ D_3(d \cos \theta_2 \cos \theta_3 \cos \theta_4 \cos \theta_5, \theta_2 + \theta_3 + \theta_4 + \theta_5), \\ D_4(d \cos \theta_3 \cos \theta_4 \cos \theta_5 \cos \theta_1, \theta_3 + \theta_4 + \theta_5 + \theta_1), \\ D_5(d \cos \theta_4 \cos \theta_5 \cos \theta_1 \cos \theta_2, \theta_4 + \theta_5 + \theta_1 + \theta_2). \end{aligned}$$

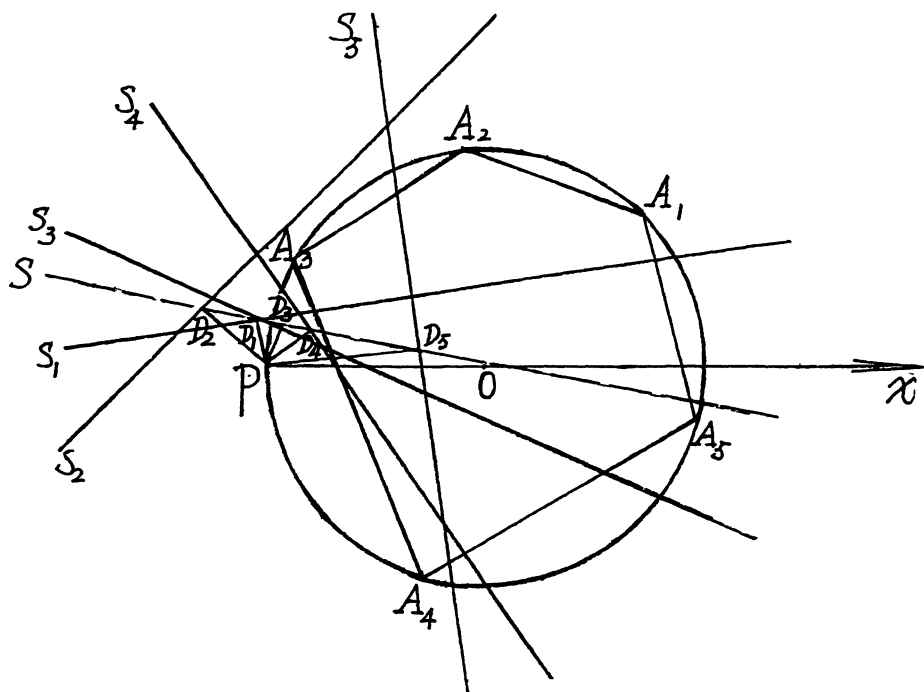


Figure 2.

Clearly the coordinates of D_i satisfy the normal form equation of the line

$$S: \rho \cos \left(\theta - \sum_{i=1}^5 \theta_i \right) = d \prod_{i=1}^5 \cos \theta_i.$$

Thus the five points D_i are collinear. □

Extension II. Let A_1, A_2, \dots, A_n be n points on a circle and let p be an arbitrary point on this circle. Let S_i denote the Simson line of p with respect to the $(n-1)$ -tuple $(n-1)$ -gonal polygon $A_j A_k \cdots A_x$ (i, j, k, \dots, x distinct) and let D_i denote the projection of p on the line S_i ($i = 1, 2, \dots, n$).

Then the n points D_i are collinear.

Proof. We again establish a polar coordinates system with p being the pole and the extension line of po being the polar axis. One can immediately verify that

$$\rho \cos \left(\theta - \sum_{i=1}^n \theta_i \right) = d \prod_{i=1}^n \cos \theta_i$$

represents a straight line containing the n points D_i . Hence the n points D_i are collinear. \square

Received March 9, 1995 and revised July 19, 1995. Translation by Tschüh Jatschiang.

TAIZHOU GENERAL RUBBER FACTORY
162 YANGZHOU ROAD
TAIZHOU CITY, JIANGSU PROVINCE
PEOPLE'S REPUBLIC OF CHINA

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by

E. F. Beckenbach (1906-1982)

F. Wolf (1904-1989)

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The *Pacific Journal of Mathematics* (ISSN 0030-8730) is published monthly except for July and August. Regular subscription rate: \$245.00 a year (10 issues). Special rate: \$123.00 a year to individual members of supporting institutions.

Subscriptions, back issues published within the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at the University of California, c/o Department of Mathematics, 981 Evans Hall, Berkeley, CA 94720 (ISSN 0030-8730) is published monthly except for July and August. Second-class postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 6143, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS at University of California,
Berkeley, CA 94720, A NON-PROFIT CORPORATION

This publication was typeset using AMS-LATEX,
the American Mathematical Society's TEX macro system.
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