PACIFIC JOURNAL OF MATHEMATICS
Vol. 177, No. 1, 1997

A BORG-LEVINSON THEOREM FOR BESSEL OPERATORS

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This paper presents a direct analog of the Borg-Levinson theorem on the recovery of a potential from the sequence of eigenvalues and norming constants for differential equations of the form

$$-y''(x) + \frac{m(m+1)y(x)}{x^2} + p(x)y(x) = \lambda y(x),$$

on the unit interval subject to various boundary conditions. This result is used to show that even zonal Schrödinger operators and Laplace operators on spheres are uniquely determined by a subsequence of their eigenvalues.

1. Introduction.

In this article we consider direct and inverse eigenvalue problems for differential equations of the form

$$-y''(x) + \frac{m(m+1)y(x)}{x^2} + p(x)y(x) = \lambda y(x), \tag{1.a}$$

on the unit interval with boundary conditions

$$\lim_{x \downarrow 0} x^{-m-1} y(x) < \infty, \quad ay(1) + by'(1) = 0, \quad a, b \in \mathbb{R}. \tag{1.b}$$

The function $p(x)$ is assumed to be real valued and square integrable. The real number m satisfies $m \ge -1/2$. When $m \ge 0$ the boundary condition at 0 can be replaced by $y(0) = 0$. Each such problem will have a sequence of eigenvalues $\lambda_n(p) \to \infty$, which are all real and simple.

Eigenvalue problems of this type arise when separation of variables is used for the study of radial Schrödinger operators $\Delta + p(r)$ on a ball in Euclidean space [**32**, pp. 160–161], zonal Schrödinger operators on spheres [**14**, **15**] or in the study of Laplace operators Δ_g for a Riemannian manifold which is a hypersurface of revolution.

When $m = 0$ there is an extensive and highly successful inverse spectral theory [**30**, **31**] for the problems (1.a,b). This theory was extended to the case $m = 1$ in [**13**], and to nonnegative integers m in [**6**, **7**]. The goal of this paper is to extend certain of the direct and inverse spectral theoretic results to the cases $m \geq -1/2$. One motivation for such an extension is that the previously analyzed cases do not include radial Schrödinger operators in even dimensions.

Our first result concerns the function taking $p \in L^2[0,1]$ to the sequence of eigenvalues.

Theorem 1.1. *The function $p \to \{\lambda_n(p) - \lambda_n(0) - \int_0^1 p(x)\, dx\}$ is continuous from $L^2_{\mathbb{R}}[0,1]$ to l^2.*

One of the principal results of inverse spectral theory is the Borg-Levinson theorem [**4**, **23**]. When $m = 0$, the right boundary condition is $y(1) = 0$, and p is an even function, this theorem says that the set of eigenvalues $\{\lambda_n\}$ uniquely determines p. Without restrictions on p and the right boundary conditions, the eigenvalues must be supplemented by a sequence of norming constants [**31**].

Our main result extends the Borg-Levinson theorem to all $m \geq -1/2$. Denote by $y_2(x, \lambda, p)$ the solution of (1.a) satisfying

$$\lim_{x \downarrow 0} x^{-m-1} y_2(x, \lambda, p) = 1.$$

Define the norming constants

$$\kappa_n(p) = y_2'(1, \lambda_n, p), \quad a \neq 0,$$

$$\kappa_n(p) = y_2(1, \lambda_n, p), \quad b \neq 0.$$

Theorem 1.2. *Suppose that for all $n \geq 1$ we have $\lambda_n(p, a, b) = \lambda_n(q, \tilde{a}, \tilde{b})$ and $\kappa_n(p) = \kappa_n(q)$. Then $p = q$ and $a/b = \tilde{a}/\tilde{b}$.*

As a corollary we have

Theorem 1.3. *Suppose that for all $n \geq 1$ we have $\lambda_n(p, a_j, b_j) = \lambda_n(q, a_j, b_j)$ for $j = 1, 2$ and for linearly independent vectors (a_1, b_1) and (a_2, b_2). Then $p = q$.*

Theorem 1.3 provides information on the problem of determining a zonal Schrödinger operator, or the Laplace operator for a hypersurface of revolution, from the spectrum. In these cases the eigenfunctions may be written as products of eigenfunctions of the Laplacian Δ_S on the N sphere and eigenfunctions of problems (1.a,b). For each eigenvalue β_k of Δ_S the eigenvalues of the corresponding problem (1.a,b) yield a subsequence $\{\lambda_{n,k}\}$ of the total spectrum. Several authors [**14**, **15**] have considered the problem of determining a zonal Schrödinger operator from its full spectrum. The case of an even zonal metric was previously considered in [5]. In the final section of this work, where we define our class of zonal potentials and zonal metrics precisely, the following result is derived from Theorem 1.3.

Theorem 1.4. *An even zonal potential on the N sphere, $N > 1$, is uniquely determined by that subsequence $\{\lambda_{n,k}\}$ of the spectrum coming from a single eigenvalue β_k of Δ_S. An even zonal metric is uniquely determined by the same data, together with the radius at $z = 0$.*

The usual treatment of inverse problems for (1.a,b) when $m = 0$ depends heavily on the fact that the solutions of (1.a) are trigonometric functions when $p = 0$. While the general case of (1.a) may still be viewed as a perturbation of the $p = 0$ case, our unperturbed equation has the form

$$-y''(x) + \frac{m(m+1)y(x)}{x^2} = \lambda y(x). \tag{1.c}$$

The second section of this work will analyze the solutions of (1.c), which are essentially Bessel functions. The estimates of the second section are extended to the case $p \neq 0$ in the third section.

The fourth section is devoted to proving Theorem 1.1. Here we employ Hardy's inequality to complete the execution of a strategy presented in [30]. In the fifth section we prove Theorems 1.2–1.4.

It will be notationally convenient to write $\omega = \sqrt{\lambda}$, and to write the square roots of eigenvalues for (1.a,b) as $\omega_n = \sqrt{\lambda_n}$. For the case $p = 0$ the eigenvalues are denoted as λ_n^0 and their square roots ω_n^0. Here the square root is chosen so that if $\lambda = r^2 e^{i\theta}$, $r \geq 0$, $-\pi < \theta \leq \pi$, then $\omega = re^{i\theta/2}$.

Several of our estimates take a distinct form when $m = -1/2$. For notational simplicity introduce the function

$$R(m,x) = \begin{cases} 1, & m > -1/2, \\ 1 + \Big|\log|x|\Big|, & m = -1/2 \end{cases}.$$

Finally, $\|p\|$ will denote the L^2 norm.

2. Solutions of the unperturbed Equation (1.c).

This section presents part of the theory of Bessel's equation in a convenient form. Bessel's equation

$$x^2 w''(x) + xw'(x) + (x^2 - \nu^2)w(x) = 0$$

is closely related to equation (1.c). In fact, taking $\nu = m + \frac{1}{2}$, $m \geq -\frac{1}{2}$ we find that if $w(x)$ is a solution to Bessel's equation, then $y(x,\lambda) = (\omega x)^{1/2} w(\omega x)$ is a solution of (1.c). The function $\phi(x) = x^{1/2}w(x)$ itself satisfies the equation

$$-\phi'' + \frac{m(m+1)\phi}{x^2} = \phi. \tag{2.a}$$

Estimates for solutions of (2.a) are the basis for understanding the eigenvalues and eigenfunctions of (1.a,b). These estimates will break into two parts: estimates for $\phi(x)$ near $x = 0$, and estimates for large values of x. It will also be helpful to extend the the estimates to complex values $z = x + it$.

To select a convenient basis for the solutions of (1.c), begin by defining

$$\phi_2(z) = (\pi z/2)^{1/2} J_{m+1/2}(z),$$

where ([**35**, p. 359])

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{\nu+2k}}{2^{\nu+2k} k! \Gamma(\nu + k + 1)}$$

is Bessel's function of the first kind of order ν. The normalization is chosen for later convenience. The subscript 2 is used to maintain notation consistent with previous work [**6**, **13**, **30**].

By using variation of parameters, any solution of (2.a) may be represented as a solution of the integral equation

$$\phi(x) = A\sin(x) + B\cos(x) - \int_x^\infty \sin(x-t)\frac{m(m+1)}{t^2}\phi(t)\ dt.$$

By comparing this representation with the known asymptotics [**35**, p. 368] for Bessel's function J_ν we find that

$$\phi_2(x,m) = \sin\left(x - \frac{m\pi}{2}\right) - \int_x^\infty \sin(x-t)\frac{m(m+1)}{t^2}\phi_2(t,m)\ dt. \tag{2.b}$$

To complete a basis of solutions of (2.a), define

$$\phi_1(x,m) = \cos\left(x - \frac{m\pi}{2}\right) - \int_x^\infty \sin(x-t)\frac{m(m+1)}{t^2}\phi_1(t,m)\ dt. \tag{2.c}$$

Comparing with the asymptotic expansion in [**35**, p. 371] we see that

$$\phi_1(x) = (\pi x/2)^{1/2} Y_{m+1/2}(x),$$

where Y_ν is a Bessel function of the second kind as defined by Weber. One can check directly that the Wronskian of ϕ_1 and ϕ_2 satisfies $\lim_{x\to\infty} W(\phi_1, \phi_2) = 1$, and since the Wronskian is constant it must be equal to 1 for all x.

The functions ϕ_1 and ϕ_2 have analytic extensions in the complex plane. Their behaviour near 0 is described in the following lemma.

Lemma 2.1. *The equation*

$$-\frac{d^2\phi(z)}{dz^2} + \frac{m(m+1)\phi(z)}{z^2} = \phi(z) \tag{2.d}$$

has a basis $\phi_1(z), \phi_2(z)$ *which is analytic for* $\mathrm{Re}(z) \geq 0$, $z \neq 0$*. For* z *sufficiently small*

$$\phi_2(z) = z^{m+1} f(z), \quad m \geq -1/2,$$
$$\phi_1(z) = \begin{cases} z^{-m} g(z), & m > -1/2, \\ z^{1/2} \log(z) g(z), & m = -1/2, \end{cases}$$

where $f(z)$ *and* $g(z)$ *are analytic and nonzero in a neighborhood of* $z = 0$.

Proof. It follows from the expression for Bessel's function J_ν that $\phi_2(z)$ is z^{m+1} times an entire function not vanishing at 0.

The behaviour of ϕ_1 can be analyzed by using the elementary technique of reduction of order [**9**, p. 84]. Any solution of (2.d) has the form

$$\phi(z) = C\phi_2(z) + \phi_2(z) \int^z 1/\phi_2^2(\zeta)\, d\zeta.$$

Since $\phi_2(z)$ has an isolated root at $z = 0$, there will be no loss of generality if we take the specific form

$$\phi(z) = \phi_2(z) \int_a^z 1/\phi_2^2(\zeta)\, d\zeta,$$

where $a > 0$ is sufficiently small, $\mathrm{Re}(z) \geq 0$ and $|z| \leq a$.

Write $\phi_2(z) = z^{m+1} f(z)$, where $1/f$ is analytic in a neighborhood of $z = 0$. Then

$$\phi(z) = z^{m+1} f(z) \int_a^z \zeta^{-2m-2} f^{-2}(\zeta)\, d\zeta.$$

Now expand in power series to conclude that for $g(z)$ analytic and not vanishing at 0,

$$\phi_1(z) = z^{-m} g(z), \qquad m > -1/2,$$
$$\phi_1(z) = z^{1/2} \log(z) g(z), \quad m = -1/2.$$

By our previous observation, the Wronskian of ϕ_1 and ϕ_2 is 1, so they are linearly independent. □

For $(z,w) \in \mathbb{C}^2$, $zw \ne 0$, $\mathrm{Re}(z) \ge 0$, and $\mathrm{Re}(w) \ge 0$, let

$$\Phi(z,w) = \phi_1(z)\phi_2(w) - \phi_1(w)\phi_2(z)$$

and

$$\Psi(z,w) = \partial \Phi / \partial w = \phi_1(z)\phi_2'(w) - \phi_1'(w)\phi_2(z).$$

For each fixed w, $\Phi(z,w)$ and $\Psi(z,w)$ are solutions of (2.d) satisfying the initial conditions

$$\Phi(w,w) = 0, \quad \frac{\partial \Phi}{\partial z}(w,w) = 1,$$
$$\Psi(w,w) = 1, \quad \frac{\partial \Psi}{\partial z}(w,w) = 0.$$

Using variation of parameters, we can write these solutions as the unique solutions of the integral equations

$$\Phi(z,w) = \sin(z-w) + \int_w^z \sin(z-\zeta)\frac{m(m+1)}{\zeta^2}\Phi(\zeta,w)\ d\zeta, \tag{2.e}$$

and

$$\Psi(z,w) = \cos(z-w) + \int_w^z \sin(z-\zeta)\frac{m(m+1)}{\zeta^2}\Psi(\zeta,w)\ d\zeta.$$

The next lemma provides estimates on the growth of Φ and Ψ. To avoid a complicated proof we have imposed a hypothesis on the locations of z and w which does not appear to be necessary, but is adequate for our needs. Say that $(z,w) \in \mathbb{C}^2$ satisfies condition α if all points in the triangle with vertices z, w, $\mathrm{Re}(z) + i\,\mathrm{Im}(w)$ have nonnegative real part and magnitude at least 1.

Lemma 2.2. *There is a constant C such that for all $(w,z) \in \mathbb{C}^2$ satisfying condition α,*

$$|\Phi(z,w)| \le C\exp(|\,\mathrm{Im}(z-w)|), \qquad |\Psi(z,w)| \le C\exp(|\,\mathrm{Im}(z-w)|),$$
$$|\partial_z\Phi(z,w)| \le C\exp(|\,\mathrm{Im}(z-w)|), \quad |\partial_z\Psi(z,w)| \le C\exp(|\,\mathrm{Im}(z-w)|).$$

Proof. Since the two cases are quite similar, only the estimates for Φ are considered. Look first at the case when $z = w \pm t$, with $t \ge 0$. If $\zeta = w \pm s$ for $0 \le s \le t$ then

$$\Phi(w \pm t, w) = \sin(\pm t) + \int_0^t \sin(\pm[t-s])\frac{m(m+1)}{[w \pm s]^2}\Phi(w \pm s, w)\ ds.$$

If $\Xi(t) = |\Phi(w \pm t, w)|$, then Gronwall's inequality [19, p. 24] implies that

$$|\Xi(t)| \le \exp\left(\int_0^t \frac{|m(m+1)|}{|w \pm s|^2}\ ds\right).$$

If the sign is $+$ then since $\mathrm{Re}(w)^2 + \mathrm{Im}(w)^2 \geq 1$ and $\mathrm{Re}(w) > 0$ we have $|w+s|^2 \geq 1 + s^2$, and

$$\int_0^t \frac{1}{|w+s|^2}\, ds \leq \tan^{-1}(t) \leq \pi/2.$$

If the sign is $-$ we take $u = t - s$ and use

$$\int_0^t \frac{1}{|w-s|^2}\, ds = \int_0^t \frac{1}{|z+u|^2}\, du$$

to get the same estimate.

Differentiation of (2.e) leads to a similar result for $\partial_z \Phi$.

For the general case let $v = \mathrm{Re}(z) + i\,\mathrm{Im}(w)$, and $t = \mathrm{Im}(z-w)$. If

$$a = \partial_z \Phi(v,w), \quad b = \Phi(v,w),$$

then

$$\Phi(z,w) = a\sin(z-v) + b\cos(z-v) + \int_v^z \sin(z-\zeta)\frac{m(m+1)}{\zeta^2}\Phi(\zeta,w)\, d\zeta.$$

Since $\mathrm{Im}(v) = \mathrm{Im}(w)$ there is a constant C_2 such that

$$|a\sin(z-v) + b\cos(z-v)| \leq C_2 e^{|t|}.$$

Writing $z = v \pm it$ and $\zeta = v \pm is$ with $0 \leq s \leq t$, we have

$$\Phi(v+it,w) = a\sin(it) + b\cos(it) + \int_0^t \sin(\pm i[t-s])\frac{m(m+1)}{[v \pm is]^2}\Phi(v \pm is, w) i\, ds.$$

With $\Xi(t) = |\Phi(v+it)|$,

$$\Xi(t) \leq C_2 e^t + \int_0^t e^{t-s}\frac{|m(m+1)|}{|v \pm is|^2}\Xi(s)\, ds.$$

Rewriting this as

$$e^{-t}\Xi(t) \leq C_2 + \int_0^t \frac{|m(m+1)|}{|v \pm is|^2} e^{-s}\Xi(s)\, ds,$$

Gronwall's inequality gives the estimate

$$e^{-t}\Xi(t) \leq C_2 \exp\left(\int_0^t \frac{|m(m+1)|}{|v \pm is|^2}\, ds\right).$$

The rest of the argument parallels that of the case $\mathrm{Im}(z) = \mathrm{Im}(w)$. $\square$

Using the solutions $\phi_1(z), \phi_2(z)$ of (2.a), a pair solutions of (1.c) are given by

$$u_1(x,\lambda) = \lambda^{m/2}\phi_1(\omega x)$$
$$u_2(x,\lambda) = \lambda^{-(m+1)/2}\phi_2(\omega x).$$

Observe that $W(u_1,u_2) = 1$, so these solutions are linearly independent. Recall that $R(m,x) = 1 + \Big|\log|x|\Big|$ if $m = -1/2$; otherwise it is 1.

Lemma 2.3. *For $x > 0$ and $\mathrm{Re}(\omega) \ge 0$ there is a constant K such that the solutions u_1 and u_2 of (1.c) satisfy the estimates*

$$|u_2(x,\lambda)| \le K\left(\frac{x}{1+|\omega x|}\right)^{m+1} \exp(|\mathrm{Im}(\omega)|x),$$
$$|u_2'(x,\lambda)| \le K\left(\frac{x}{1+|\omega x|}\right)^{m} \exp(|\mathrm{Im}(\omega)|x),$$
$$|u_1(x,\lambda)| \le KR(m,x)\left(\frac{1+|\omega x|}{x}\right)^{m} \exp(|\mathrm{Im}(\omega)|x),$$
$$|u_1'(x,\lambda)| \le KR(m,x)\left(\frac{1+|\omega x|}{x}\right)^{m+1} \exp(|\mathrm{Im}(\omega)|x).$$

Proof. The proof reduces to verifying that u_1 and u_2 have the indicated bounds in the two cases $|\omega x| \to 0$ and $|\omega x| \to \infty$. For small values of $|\omega x|$ Lemma 2.1 gives the desired estimates.

For values of $z = \sqrt{\lambda}x$ with magnitude at least 1 the estimates of Lemma 2.2 can be employed. First observe that

$$\begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix} = \begin{pmatrix} \phi_1'(w) & \phi_1(w) \\ \phi_2'(w) & \phi_2(w) \end{pmatrix} \begin{pmatrix} \Phi(z,w) \\ \Psi(z,w) \end{pmatrix}. \tag{2.f}$$

Let $w = \sqrt{\lambda}x/|\sqrt{\lambda}x|$. Then the hypotheses of Lemma 2.2 are satisfied. The continuity of the 2×2 matrix in (2.f) on the compact set $|w| = 1$, $\mathrm{Re}(w) \ge 0$ is sufficient for the estimates. $\square$

The estimates of Lemma 2.2 allow us to extend (2.b) to $\mathrm{Re}(z) \ge 0$, where we have

$$\phi_2(z) = \sin(z - m\pi/2) - \int_z^\infty \sin(z-\zeta)\frac{m(m+1)}{\zeta^2}\phi_2(\zeta)\, d\zeta, \tag{2.g}$$

$$\phi_2'(z) = \cos(z - m\pi/2) - \int_z^\infty \cos(z-\zeta)\frac{m(m+1)}{\zeta^2}\phi_2(\zeta)\, d\zeta.$$

The contour is chosen so that $\zeta = t + i\,\mathrm{Im}(z)$. It follows that

$$|\phi_2(z) - \sin(z - m\pi/2)| \le C\exp(|\mathrm{Im}(z)|)\int_{\mathrm{Re}(z)}^{\infty} \frac{1}{\mathrm{Im}(z)^2 + t^2}\,dt$$
$$= C\frac{\exp(|\mathrm{Im}(z)|)}{|\mathrm{Im}(z)|}\left[\pi/2 - \tan^{-1}\left(\frac{\mathrm{Re}(z)}{|\mathrm{Im}(z)|}\right)\right]$$

and

$$|\phi_2'(z) - \cos(z - m\pi/2)| \le C\frac{\exp(|\mathrm{Im}(z)|)}{|\mathrm{Im}(z)|}\left[\pi/2 - \tan^{-1}\left(\frac{\mathrm{Re}(z)}{|\mathrm{Im}(z)|}\right)\right].$$

The next lemma expresses these estimates in terms of u_2.

Lemma 2.4. *For $x > 0$ and $\mathrm{Re}(\omega) \ge 0$ there is a constant C such that*

$$\left|u_2(x,\lambda) - \omega^{-(m+1)}\sin(\omega x - m\pi/2)\right| \le C\left|\omega^{-(m+1)}\right|\frac{\exp(|\mathrm{Im}(\omega x)|)}{|\omega x|},$$

and

$$\left|u_2'(x,\lambda) - \omega^{-m}\cos(\omega x - m\pi/2)\right| \le C\left|\omega^{-m}\right|\frac{\exp(|\mathrm{Im}(\omega x)|)}{|\omega x|}.$$

3. Solutions of the perturbed Equation (1.a).

The behaviour of solutions of (1.a) near $x = 0$ is described in [**6**]. Lemmas 2.1 and 2.3 of that paper show that (1.a) has a unique solution, which will be denoted $y_2(x,\lambda,p)$, satisfying

$$\lim_{x\downarrow 0} x^{-m-1}y_2(x,\lambda,p) = 1. \tag{3.a}$$

These lemmas use the assumption that $m \ge 0$. The same proofs actually work if $m > -1/2$. In case $m = -1/2$ the proofs are modified slightly, with the function x^{-m} replaced by $x^{1/2}\log(x)$.

Estimates for the solutions of (1.a) can be obtained via variation of parameters integral representations. The solutions of interest to us satisfy

$$y_2(x,\lambda,p) = u_2(x,\lambda) - \int_0^x G(x,t,\lambda)p(t)y_2(t,\lambda,p)\,dt. \tag{3.b}$$

The kernel is

$$\begin{aligned} G(x,t,\lambda) &= u_1(t,\lambda)u_2(x,\lambda) - u_1(x,\lambda)u_2(t,\lambda) \\ &= \omega^{-1}[\phi_1(\omega t)\phi_2(\omega x) - \phi_1(\omega x)\phi_2(\omega t)] \\ &= \omega^{-1}\Phi(\omega t,\omega x). \end{aligned}$$

Since $G(x,x,\lambda)=0$ we also have the representation

$$y_2'(x,\lambda,p) = u_2'(x,\lambda) - \int_0^x H(x,t,\lambda)p(t)y_2(t,\lambda,p)\ dt \tag{3.c}$$

where

$$\begin{aligned} H(x,t,\lambda) = \partial G/\partial x &= u_1(t,\lambda)u_2'(x,\lambda) - u_1'(x,\lambda)u_2(t,\lambda) \\ &= \phi_1(\omega t)\phi_2'(\omega x) - \phi_1'(\omega x)\phi_2(\omega t) \\ &= \Psi(\omega t, \omega x). \end{aligned}$$

Lemma 3.1. *Suppose that $x>0$, $t>0$ and $\mathrm{Re}(\omega)\geq 0$. Then there is a constant K such that the function $G(x,t,\lambda)$ satisfies the estimates*

$$|G(x,t,\lambda)| \leq K\left(\frac{x}{1+|\omega x|}\right)^{m+1} R(m,t)\left(\frac{1+|\omega t|}{t}\right)^m \exp(|\mathrm{Im}(\omega)|(x-t))$$

for $x\geq t$. The function $H(x,t,\lambda)$ satisfies the estimates

$$|H(x,t,\lambda)| \leq K\left(\frac{x}{1+|\omega x|}\right)^{m} R(m,t)\left(\frac{1+|\omega t|}{t}\right)^m \exp(|\mathrm{Im}(\omega)|(x-t))$$

for $x\geq t$.

Proof. Since the verifications of the various cases are quite similar, the details are only provided for G in case $m>-1/2$.

The estimates for G follow from estimates for $\Phi(z,w)$. Suppose that the real parts of z and w are nonnegative. If $|z|\leq|w|\leq 1$, then Lemma 2.1 gives

$$|\Phi(z,w)| \leq K\left|\frac{w^{m+1}}{z^m}\right|.$$

If $|z|\leq 1\leq |w|$ then $\phi_1(z)$ and $\phi_2(z)$ are estimated using Lemma 2.1, while $\phi_1(w)$ and $\phi_2(w)$ are estimated using (2.f) and Lemma 2.2. In this case we find

$$\begin{aligned} |\Phi(z,w)| &\leq K_1\exp(|\mathrm{Im}(w)|)|z^{-m}| \\ &\leq K_1\exp(|\mathrm{Im}(w-z)+\mathrm{Im}(z)|)|z^{-m}| \\ &\leq K|z^{-m}|\exp(|\mathrm{Im}(w-z)|) \end{aligned}$$

Notice that estimates for $G(x,t,\lambda)$ only require consideration of (z,w) such that z is a positive multiple of w. By Lemma 2.2, if $|w|,|z|\geq 1$, then

$$|\Phi(z,w)| \leq K\exp(|\mathrm{Im}(z-w)|).$$

For $x \ge t$ these estimates for Φ yield the following estimates for $G(x,t,\lambda)$:

$$\begin{aligned}
&|G(x,t,\lambda)| \le K\omega^{-1}\exp(|\operatorname{Im}(\omega)|(x-t)), && |\omega t| \ge 1,\\
&|G(x,t,\lambda)| \le Kx\left(\frac{x}{t}\right)^m, && |\omega x| \le 1,\\
&|G(x,t,\lambda)| \le K|\omega^{-1}||\omega t|^{-m}\exp(|\operatorname{Im}(\omega)|(x-t)), && |\omega t| \le 1,\ |\omega x| \ge 1.
\end{aligned}$$

Consolidating these estimates gives the result for G. □

Estimates for solutions of (1.a) may now be obtained by using Lemma 3.1 with (3.b,c). In case $m = 1$ similar estimates were developed in [**13**]. (They have a typographical error in (1.12) and (1.13).) For notational convenience define

$$E(x,\lambda) = \exp\left(\int_0^x \frac{R(m,t)t|p(t)|}{1+|\omega t|}\,dt\right) - 1.$$

Lemma 3.2. *The solution y_2 of (1.a) satisfies the estimates*

$$|y_2(x,\lambda,p) - u_2(x,\lambda)| \le C\left(\frac{x}{1+|\omega x|}\right)^{m+1}\exp(|\operatorname{Im}(\omega)|x)E(x,\lambda),$$

$$|y_2'(x,\lambda,p) - u_2'(x,\lambda)| \le C\left(\frac{x}{1+|\omega x|}\right)^{m}\exp(|\operatorname{Im}(\omega)|x)E(x,\lambda).$$

Proof. The proof of the first inequality uses Gronwall's inequality ([**9**, p. 37]).

From (3.b), Lemma 2.3 and Lemma 3.1 we have

$$\begin{aligned}
|y_2(x,\lambda,p)| \le{}& K\frac{x^{m+1}\exp(|\operatorname{Im}(\omega)|x)}{(1+|\omega x|)^{m+1}}\\
&+\int_0^x K\left(\frac{x}{1+|\omega x|}\right)^{m+1}R(m,t)\left(\frac{1+|\omega t|}{t}\right)^m\\
&\qquad\exp(|\operatorname{Im}(\omega)|(x-t))|p(t)||y_2(t,\lambda,p)|\,dt.
\end{aligned}$$

Rewrite this inequality as

$$|y_2(x,\lambda,p)|\frac{(1+|\omega x|)^{m+1}}{x^{m+1}}\exp(-|\operatorname{Im}(\omega)|x)$$

$$\le K + K\int_0^x \frac{R(m,t)t|p(t)|}{1+|\omega t|}\left(\frac{1+|\omega t|}{t}\right)^{m+1}\exp(-|\operatorname{Im}(\omega)|t)|y_2(t,\lambda,p)|\,dt.$$

The division by x^{m+1} still leaves us with functions continuous on $[0,1]$ by Lemma 2.1 of [**6**]. Letting

$$\xi(x) = |y_2(x,\lambda,p)|\frac{(1+|\omega x|)^{m+1}}{x^{m+1}}\exp(-|\operatorname{Im}(\omega)|x),$$

Gronwall's inequality gives

$$\xi(x) \le K \exp\left(\int_0^x \frac{R(m,t)t|p(t)|}{1+|\omega t|}\, dt\right).$$

Inserting this estimate for y_2 back into the integral equation (3.b) gives the first estimate for y_2.

The difference $|y_2'(x,\lambda,p) - u_2'(x,\lambda)|$ is estimated using (3.c) and the initial estimate for y_2. Thus

$$\begin{aligned}
&|y_2'(x,\lambda,p) - u_2'(x,\lambda)| \\
&\qquad \le K\left(\frac{x}{1+|\omega x|}\right)^m \exp\left(|\operatorname{Im}(\omega)|x\right) \int_0^x \frac{R(m,t)t|p(t)|}{1+|\omega t|} \\
&\qquad\qquad\qquad \exp\left(\int_0^t \frac{R(m,s)s|p(s)|}{1+|\omega s|}\, ds\right)\, dt,
\end{aligned}$$

yielding the second estimate. □

In addition to the solution y_2, we will be interested in solutions whose initial data is specified at $x = 1$. Let $z_2(x,\lambda,p)$ be the solution of (1.a) satisfying

$$z_2(1,\lambda,p) = -b, \quad z_2'(1,\lambda,p) = a, \quad a,b \in \mathbb{R}. \tag{3.d}$$

To estimate this solution it will suffice to quote Theorem 1.3 of [**30**].

Lemma 3.3. *The solution z_2 of (1.a) satisfies*

$$\begin{aligned}
&|z_2(x,\lambda,p) + b\cos(\omega[1-x]) + a\sin(\omega[1-x])/\omega| \\
&\qquad\qquad \le \frac{K(x)}{|\omega|} \exp(|\operatorname{Im}(\omega)|[1-x]), \quad |\omega| \ge 1,
\end{aligned}$$

where

$$K(x) \le \exp\left(\int_x^1 \left[\frac{|m(m+1)|}{t^2} + |p(t)|\right]\, dt\right).$$

4. The potential to eigenvalue map.

The main goal for this section is to prove Theorem 1.1. The approach follows [**30**, pp. 35–37], although more delicate estimates arise here. These are handled with Hardy's inequality.

It will be helpful to start with initial estimates on the eigenvalues λ_n^0 for the case $p = 0$. Equation (1.c) with boundary conditions (1.b) is formally self adjoint, and integration by parts shows that all eigenvalues are real. In order to satisfy the boundary condition at 0, any eigenfunction for the $p = 0$ problem must be a multiple of

$$u_2(x, \lambda) = \lambda^{-(m+1)/2}\phi_2(\omega x).$$

For $\lambda = 0$ the power series for Bessel's function $J_{m+1/2}$ shows that 0 is an eigenvalue if and only if $a + b(m+1) = 0$. For $\lambda \neq 0$ the boundary condition at $x = 1$ becomes

$$a\phi_2(\omega) + b\omega\phi_2'(\omega) = 0,$$

or $aJ_{m+1/2}(\omega) + b\omega J_{m+1/2}'(\omega) = 0$ in terms of the Bessel functions. In this form the study is classical [**36**, p. 482, p. 506]; a brief analysis is presented here.

Observe that the integral formulas (2.g) imply that there are at most finitely many roots of

$$a\phi_2(\omega) + b\omega\phi_2'(\omega) = 0$$

on the imaginary axis. It will be convenient to consider both the function

$$f(\omega) = a\phi_2(\omega) + b\omega\phi_2'(\omega)$$

whose zeroes we want, and its derivative. From the integral equation (2.b)

$$\begin{aligned}(4.\text{a})\quad f(\omega) = {}& a\sin\left(\omega - \frac{m\pi}{2}\right) - a\int_\omega^\infty \sin(\omega - t)\frac{m(m+1)}{t^2}\phi_2(t)\ dt \\ & + b\omega\cos\left(\omega - \frac{m\pi}{2}\right) - b\omega\int_\omega^\infty \cos(\omega - t)\frac{m(m+1)}{t^2}\phi_2(t)\ dt\end{aligned}$$

and

$$\begin{aligned}f'(\omega) = {}& a\cos\left(\omega - \frac{m\pi}{2}\right) - a\int_\omega^\infty \cos(\omega - t)\frac{m(m+1)}{t^2}\phi_2(t)\ dt \\ & - b\omega\sin\left(\omega - \frac{m\pi}{2}\right) + b\omega\int_\omega^\infty \sin(\omega - t)\frac{m(m+1)}{t^2}\phi_2(t)\ dt.\end{aligned}$$

By Lemma 2.2 $\phi_2(t)$ is bounded for $t > 1$, so

(4.b)

$$f(\omega) = b\omega\cos(\omega - m\pi/2) + \left[a + \frac{b}{2}m(m+1)\right]\sin(\omega - m\pi/2) + O(1/\omega),$$

and

$$f'(\omega) = -b\omega\sin(\omega - m\pi/2) + \left[a + \frac{b}{2}m(m+1)\right]\cos(\omega - m\pi/2) + O(1/\omega).$$

From (4.b) positive eigenvalues occur, in case $b = 0$, when a condition of the form
$$0 = \sin(\omega - m\pi/2) + O(1/\omega),$$
is satisfied. Thus for n sufficiently large there is at least one root of the form
$$\tilde{\omega}_n^0 = n\pi + m\pi/2 + O(n^{-1}), \quad n \in \mathcal{Z}^+.$$
Since
$$f'(\omega) = \cos(\omega - m\pi/2) + O(1/\omega),$$
the mean value theorem implies that for ω large there is exactly one root of this form. A similar analysis for $b \neq 0$ shows that for $|\omega|$ large there is exactly one real root of the form
$$\tilde{\omega}_n^0 = [n + 1/2]\pi + m\pi/2 + O(n^{-1}), \quad n \in \mathcal{Z}^+.$$

It will be helpful later to refine these estimates on the location of eigenvalues for $b \neq 0$. Let
$$\tilde{\omega}_n^0 = (n + 1/2)\pi + \frac{m\pi}{2} + \epsilon_n.$$
Equation (4.b) leads to
$$\tan(\epsilon_n) = \left[\frac{a}{b} + \frac{m(m+1)}{2}\right] / \tilde{\omega}_n^0 + O(n^{-2})$$
or
$$\epsilon_n = \left[\frac{a}{b} + \frac{m(m+1)}{2}\right] \left[(n + 1/2)\pi + \frac{m\pi}{2}\right]^{-1} + O(n^{-2}).$$
This gives

(4.c)
$$\tilde{\omega}_n^0 = [n + 1/2]\pi + m\pi/2 + \left[\frac{a}{b} + \frac{m(m+1)}{2}\right] n^{-1} + O(n^{-2}), \quad b \neq 0.$$

Once again we appeal to [6] Lemma 2.1, which shows that $y_2(1, \lambda)$ and $y_2'(1, \lambda)$ are analytic for all $\lambda \in \mathbb{C}$. The next lemma is modeled on Lemma 2.2 of [30].

Lemma 4.1. *For N a sufficiently large integer, the functions $ay_2(1, \lambda) + by_2'(1, \lambda)$ and $au_2(1, \lambda) + bu_2'(1, \lambda)$ have the same number of roots in the half plane*
$$\mathrm{Re}(\lambda) < [N + (m+1)/2]^2\pi^2, \quad b = 0,$$
$$\mathrm{Re}(\lambda) < [N + 1 + m/2]^2\pi^2, \quad b \neq 0.$$

For each $n > N$ the function $ay_2(1,\lambda) + by_2'(1,\lambda)$ has exactly one simple root in

$$|\omega - n\pi - m\pi/2| < \pi/2, \quad b = 0,$$
$$|\omega - n\pi - (m+1)\pi/2| < \pi/2, \quad b \neq 0.$$

Proof. The cases $b = 0$ and $b \neq 0$ are similar, so we only consider the second case. For $K > N$ consider the contours

$$|\omega| = [K+1+m/2]\pi, \quad \mathrm{Re}(\omega) = [K+1+m/2]\pi, \quad |\omega - n\pi - (m+1)\pi/2| = \pi/2.$$

By Lemma 3.2 there is a constant C such that

$$|a[y_2(1,\lambda) - u_2(1,\lambda)] + b[y_2'(1,\lambda) - u_2'(1,\lambda)]| \le C|\omega^{-(m+1)}| \exp(|\operatorname{Im}(\omega)|),$$

since by the Cauchy-Schwarz inequality

$$E(x,\lambda) = \exp\left(\int_0^x \left|\frac{tR(m,t)p(t)}{1+|\omega t|}\right| \, dt\right) - 1 \le K(\|p\|)/|\omega|.$$

By Lemma 2.4 there is a constant C such that

$$\left|a\left[u_2(1,\lambda) - \omega^{-(m+1)}\sin(\omega - m\pi/2)\right] + b\left[u_2'(1,\lambda) - \omega^{-m} cos(\omega - m\pi/2)\right]\right|$$
$$\le C\left|\omega^{-(m+1)}\right| \exp(|\operatorname{Im}(\omega)|).$$

On the contours,

$$\left|a\omega^{-(m+1)}\sin(\omega - m\pi/2) + b\omega^{-m}\cos(\omega - m\pi/2)\right| > C_1|\omega^{-m}| \exp(|\operatorname{Im}(\omega)|),$$

and the result follows from Rouche's Theorem. ∎

Define $g(x,\lambda,p) = y_2(x,\lambda,p)/\|y_2\|$ and $g_n(x,p) = g(x,\lambda_n,p)$, so that g_n is a normalized eigenfunction for (1.a) satisfying the boundary conditions (1.b). As in [**30**] Theorem 2.3, $\partial_p \lambda_n = g_n^2$. We begin by writing

$$\lambda_n - \lambda_n^0 = \int_0^1 \frac{d}{dt}\lambda_n(tp)dt = \int_0^1 \langle g_n^2(x,tp), p\rangle \, dt. \tag{4.d}$$

Lemma 4.2. *If $p \in L^2[0,1]$ then*

$$\lambda_n = \lambda_n^0 + O(1)$$

and

$$g_n(x,\lambda_n) = \sqrt{2}\phi_2(\omega_n^0 x) + O\left(\frac{\log(n)}{n}\right).$$

Proof. Begin with

$$\int_0^1 y_2^2(t,\lambda)\ dt = \int_0^1 u_2^2(t,\lambda)\ dt + 2\int_0^1 u_2(t,\lambda)[y_2 - u_2]\ dt + \int_0^1 [y_2 - u_2]^2\ dt.$$

Using Lemmas 2.3 and 3.2, and again noting that as $\lambda \to \infty$

$$\exp\left(\int_0^x \left|\frac{tR(m,t)p(t)}{1+|\omega t|}\right|\ dt\right) - 1 \le K(\|p\|)/|\omega|,$$

we have,

$$\int_0^1 u_2(t,\lambda)[y_2 - u_2]\ dt \le K(\|p\|)|\lambda|^{-m-1-1/2}$$

and

$$\int_0^1 [y_2 - u_2]^2\ dt \le K(\|p\|)|\lambda|^{-m-2}.$$

Thus

$$\int_0^1 y_2^2(t,\lambda)\ dt = \lambda^{-m-1}\left[\int_0^1 \phi_2^2(\omega t)dt + O(|\omega^{-1}|)\right].$$

To analyze the main term, let $\epsilon = 1/\omega$ and write

$$\begin{aligned}\int_0^1 \phi_2^2(\omega t)dt &= \int_0^\epsilon \phi_2^2(\omega t)\ dt + \int_\epsilon^1 \phi_2^2(\omega t)\ dt \\ &= \frac{1}{\omega}\int_0^1 \phi_2^2(x)\ dx + \frac{1}{\omega}\int_1^\omega \phi_2^2(x)\ dx.\end{aligned}$$

Now (2.b) implies that $\phi_2(x) = \sin(x - m\pi/2) + O(1/x)$, so that

$$\int_1^\omega \phi_2^2(x)\ dx = \omega/2 + O(\log\omega).$$

Consequently,

$$\int_0^1 y_2^2(t,\lambda)\ dt = \lambda^{-m-1}\left[\frac{1}{2} + O\left(\frac{\log(\omega)}{\omega}\right)\right].$$

Now

$$\begin{aligned}g(x,\lambda) &= \frac{y_2(x,\lambda)}{\|y_2\|} \\ &= \left[\frac{1}{2} + O\left(\frac{\log(\omega)}{\omega}\right)\right]^{-1/2}[\phi_2(\omega_n x) + O(\omega^{-1})] \\ &= \sqrt{2}\phi_2(\omega x) + O\left(\frac{\log(\omega)}{\omega}\right).\end{aligned}$$

By (4.d) the uniform bounds on $g_n(x, \lambda_n, p)$ for $\|p\|$ bounded imply

$$\lambda_n - \lambda_n^0 = O(1),$$

so

$$\omega_n = \omega_n^0 + O\left(\frac{1}{n}\right).$$

By Lemma 2.2 and (2.g) the function $\phi_2'(t)$ is uniformly bounded if $t \geq 1$. This gives the estimate for g_n as stated in the lemma for $\omega_n^0 x \geq 1$. Suppose that $x < 1/n$. In this case $|[\omega_n - \omega_n^0]x| \leq K/n^2$ and

$$|\phi_2(\omega_n x) - \phi_2(\omega_n^0 x)| \leq \left| \int_{\omega_n^0 x}^{\omega_n x} \phi'(t)\ dt \right| \leq K \left| \int_{\omega_n^0 x}^{\omega_n x} t^m\ dt \right| \leq K/n^2.$$

□

Now we're ready to show that the function $p \to \left\{ \lambda_n(p) - \lambda_n^0 - \int_0^1 p(x)\ dx \right\}$ is continuous from $L^2_{\mathbb{R}}[0,1]$ to l^2.

Proof of Theorem 1.1. The sequence

$$\left\{ \lambda_n(p) - \lambda_n^0 - \int_0^1 p(x)\ dx \right\}$$

can be written as a sum of terms, each of which is a continuous function from $L^2 \to l^2$, so the continuity will be established as we show that the sequence is in l^2. Start with

$$\begin{aligned} \lambda_n - \lambda_n^0 &= \int_0^1 \frac{d}{dt} \lambda_n(tp)\ dt \\ &= \int_0^1 \int_0^1 g_n^2(x, tp) p(x)\ dx\ dt \\ &= \int_0^1 \int_0^1 \left[\sqrt{2}\phi_2\left(\omega_n^0 x\right) + O\left(\frac{\log(n)}{n}\right)\right]^2 p(x)\ dx\ dt. \end{aligned} \tag{4.e}$$

This expression is considered in two parts: the part near zero, where we can use the Taylor expansion, and the part away from zero, where the asymptotic expansion for Bessel functions with large argument can be used.

From (4.e) and the fact that $\phi_2(x)$ is bounded for $x \in [0, \infty)$ if suffices to estimate

$$2 \int_0^1 \phi_2^2(\omega_n^0 x) p(x)\ dx.$$

For n large enough so that $\omega_n^0 > 0$, let $\epsilon_n = 1/\omega_n^0$ and break up the integral as

$$\int_0^1 \phi_2^2(\omega_n^0 x)p(x)\ dx = \int_0^{\epsilon_n} \phi_2^2(\omega_n^0 x)p(x)\ dx + \int_{\epsilon_n}^1 \phi_2^2(\omega_n^0 x)p(x)\ dx.$$

For $0 < \epsilon \le x \le 1$, the uniform bounds on ϕ_2 and the representation (2.b) imply

$$\phi_2(x) = \sin\left(x - \frac{m\pi}{2}\right) + O\left(\frac{1}{x}\right),$$

so that

$$\phi_2(\omega_n^0 x) = \sin\left(\omega_n^0 x - \frac{m\pi}{2}\right) + O\left(\frac{\epsilon_n}{x}\right)$$

and

$$\left|\int_{\epsilon_n}^1 \phi_2^2\left(\omega_n^0 x\right) p(x)\ dx - \int_{\epsilon_n}^1 \sin^2\left(\omega_n^0 x - \frac{m\pi}{2}\right) p(x)\ dx\right| \le K \int_{\epsilon_n}^1 \frac{\epsilon_n}{x}|p(x)|\ dx.$$

Now

$$\int_{\epsilon_n}^1 \sin^2\left(\omega_n^0 x - \frac{m\pi}{2}\right) p(x)\ dx = \frac{1}{2}\int_0^1 \left[1 - \cos\left(\omega_n^0 x - m\pi\right)\right] p(x)\ dx$$
$$- \frac{1}{2}\int_0^{\epsilon_n} \left[1 - \cos\left(2\omega_n^0 x - m\pi\right)\right] p(x)\ dx$$

and

$$\frac{1}{2}\int_0^1 \left[1 - \cos\left(2\omega_n^0 x - m\pi\right)\right] p(x)\ dx$$
$$= \frac{1}{2}\int_0^1 p(x)dx + \frac{1}{2}\int_0^1 \cos\left(2\omega_n^0 x - m\pi\right) p(x)\ dx.$$

To see that the sequence $\int_0^1 \cos(2\omega_n^0 x - m\pi)p(x)\ dx\ \in l^2$ note that there is an integer k such that

$$\int_0^1 \cos(2\omega_n^0 x - m\pi)p(x)\ dx$$
$$= \int_0^1 \cos([2n + 2k + (m+1)]\pi x - m\pi)p(x)\ dx + O(1/n).$$

Now write the cosine terms as a sum of exponentials and note that the exponentials come from a pair of orthonormal sets. Note too that

$$-\frac{1}{2}\int_0^{\epsilon_n} [1 - \cos(2\omega_n^0 x - m\pi)]p(x)\ dx$$

is bounded in absolute value by $K \int_0^{\epsilon_n} |p(x)| \, dx$.

The remaining expressions are handled with Hardy's inequality, which implies that if $p(x) \in L^2[0,1]$, then the sequences $c_n = \int_0^{\epsilon_n} |p(x)| \, dx$ and $d_n = \int_{\epsilon_n}^1 \epsilon_n |p(x)|/x \, dx$ are in l^2.

If $F(x) = \int_0^{1/x} |p(x)| \, dx$, then $c_n = F(\omega_n^0)$. Clearly $F(x)$ is nonnegative and monotonically decreasing. Thus $\sum_{n=1}^\infty c_n^2 < \infty$ if and only if $\int_1^\infty |F(x)|^2 \, dx < \infty$. Letting $u = 1/x$ we have

$$\int_1^\infty |F(x)|^2 dx = \int_0^1 |F(1/u)/u|^2 \, du.$$

Note that $F(1/u)/u = \frac{1}{u} \int_0^u |p(t)| \, dt$. The integral form of Hardy's inequality [20, p. 240] shows that if $p(x) \in L^2[0,1]$ then $F(1/u)/u \in L^2[0,1]$.

For the second case write

$$d_n = \int_{\epsilon_n}^1 \epsilon_n |p(x)|/x \, dx = \frac{1}{n} n\epsilon_n \left[\int_{\epsilon_1}^1 |p(x)|/x \, dx + \sum_{k=1}^{n-1} \int_{\epsilon_{k+1}}^{\epsilon_k} |p(x)|/x \, dx \right].$$

The term $n\epsilon_n$ does not affect convergence. Let $a_0 = \int_{\epsilon_1}^1 |p(x)|/x dx$ and

$$a_k = \int_{\epsilon_{k+1}}^{\epsilon_k} |p(x)|/x \, dx, \quad k > 0.$$

Let $p_0^2 = \int_{\epsilon_1}^1 |p(x)|^2 dx$ and

$$p_k^2 = \int_{\epsilon_{k+1}}^{\epsilon_k} |p(x)|^2 \, dx, \quad k > 0.$$

Notice that $\sum_{k=1}^\infty p_k^2 = \int_0^1 |p(x)|^2 dx < \infty$. Now for $k > 0$,

$$|a_k|^2 \le \left[\int_{\epsilon_{k+1}}^{\epsilon_k} 1/x^2 dx \right] p_k^2 \le [\pi + O(1/k)] p_k^2.$$

Apply Hardy's inequality for sums [20, p. 239] to conclude that $\{d_n\} \in l^2$. $\square$

5. The main theorem and zonal problems on spheres.

Based on the estimates developed in the previous three sections, the proof of the Borg-Levinson Theorem from [31] Theorem 1.5 may be adopted more or less intact. The proof is sketched for completeness.

Recall that norming constants may be defined by

$$\kappa_n(p) = y_2'(1, \lambda_n, p), \quad a \neq 0,$$
$$\kappa_n(p) = y_2(1, \lambda_n, p), \quad b \neq 0.$$

Proof of Theorem 1.2. The functions $z_2(x, \lambda, p)$ defined in (3.d) are solutions of (1.a) which satisfy the boundary conditions (1.b) at $x = 1$. If $\lambda = \lambda_n(p)$ then y_2 is a nonzero multiple of z_2. In fact

$$y_2(x, \lambda, p) = \begin{cases} -y_2(1, \lambda_n, p) z_2(x, \lambda_n, p)/b, & b \neq 0 \\ y_2'(1, \lambda_n, p) z_2(x, \lambda_n, p)/a, & a \neq 0 \end{cases}.$$

Starting with the equation

$$(-D^2 + p - \lambda)\partial_\lambda y_2 = y_2,$$

we multiply by y_2 and integrate to get

$$\int_0^1 y_2^2(x, \lambda)\ dx = \left(-y_2 \partial_\lambda y_2' + y_2' \partial_\lambda y_2\right)(1, \lambda). \tag{5.a}$$

Evaluation at $\lambda = \lambda_n(p)$ leads to

(5.b)
$$\int_0^1 y_2^2(x, \lambda_n(p))\ dx = \begin{cases} \left(-y_2(1, \lambda_n)/b\right)\left(a\partial_\lambda y_2 + b\partial_\lambda y_2'\right)(1, \lambda_n), & b \neq 0 \\ \left(y_2'(1, \lambda_n)/a\right)\left(a\partial_\lambda y_2 + b\partial_\lambda y_2'\right)(1, \lambda_n), & a \neq 0 \end{cases}.$$

This last computation requires a comment if $m < 0$. To verify that

$$\lim_{x \downarrow 0} -y_2(x, \lambda)\frac{\partial y_2'(x, \lambda)}{\partial \lambda} + y_2'(x, \lambda)\frac{\partial y_2(x, \lambda)}{\partial \lambda} = 0,$$

one can use the integral equation

$$y_2(x, \lambda) = x^{m+1} + \frac{1}{2m+1}\int_0^x (t^{-m}x^{m+1} - x^{-m}t^{m+1})(p(t) - \lambda) y_2(t, \lambda)\ dt$$

for $m > -1/2$ as in Lemma 2.1 of [6], and its analog when $m = -1/2$.

Consider the function

$$f(\lambda) = \frac{[y_2(x, \lambda, p) - y_2(x, \lambda, q)][z_2(x, \lambda, p) - z_2(x, \lambda, q)]}{a y_2(1, \lambda, p) + b y_2'(1, \lambda, p)}.$$

For each $x \in (0, 1]$ the numerator is entire in λ, and the denominator is entire in λ with zeroes exactly at the eigenvalues λ_n. By (5.b) the roots of the denominator are simple. The residues are

$$R_n = \left[\int_0^1 y_2^2(x, \lambda_n)\ dx\right]^{-1} [y_2(x, \lambda_n, p) - y_2(x, \lambda_n, q)]^2 \geq 0.$$

The estimates from Lemmas 2.4, 3.2, 3.3 and 4.2 show that if $c_n = \omega_n^0 + \pi/2$, then

$$\lim_{|\lambda|=c_n} \lambda f(\lambda) = 0.$$

This implies that the sum of the nonnegative residues R_n is zero in the sense that

$$\lim_{N\to\infty} \sum_{n=1}^{N} R_n = 0.$$

Since the eigenfunctions for p and q agree, these functions agree almost everywhere.

Finally, by Theorem 1.1 the sequence $\{\lambda_n^0\}$ is uniquely determined up to an l^2 correction. The cases $b = 0$ and $b \neq 0$ are readily distinguished by the coarse asymptotics of the eigenvalues, while (4.c) shows that a/b is uniquely determined when $b \neq 0$. □

As in the case of regular Sturm-Liouville problems, p is also uniquely determined if a pair of spectra is known for distinct sets of boundary conditions.

Proof of Theorem 1.3. Since $a_j y(1,\lambda) + b_j y'(1,\lambda)$ are entire functions with order of growth 1/2, Hadamard's product theorem [**1**] implies they are determined up to a nonzero constant factor by their zeroes, which are the eigenvalues. Assuming for notational simplicity that 0 is not an eigenvalue,

$$a_j y(1,\lambda) + b_j y'(1,\lambda) = C_j \prod (1 - \lambda/\lambda_{n,j}).$$

Once the constants C_j are identified, then, since the vectors (a_1, b_1) and (a_2, b_2) are linearly independent, both functions $y(1,\lambda)$ and $y'(1,\lambda)$ are known for all λ, and thus the norming constants $y(1, \lambda_n(1))$ and $y'(1, \lambda_n(1))$ are known.

There are similar techniques for identifying C_j in the cases $b = 0$ or $b \neq 0$. We consider $b \neq 0$. Let $\mu_k = [2k\pi + m/2]^2$. By Lemmas 2.4 and 3.2

$$\lim_{k\to\infty} \mu_k^{m/2} [a_j y(1,\mu_k) + b_j y'(1,\mu_k)] = b_j = C_j \lim_{k\to\infty} \mu_k^{m/2} \prod_n (1 - \mu_k/\lambda_{n,j}).$$

Thus C_j is determined from the eigenvalue sequence. □

Theorems 1.1-1.3 are applicable to the study of direct and inverse spectral theory of Schrödinger operators $\Delta + p(r)$ on the ball in R^N with spherically symmetric potentials, and for Schrödinger operators with zonal potentials on the $N+1$ sphere. They also provide some information about the spectral determination of even zonal metrics for the $N+1$ sphere. Since there is a discussion of problems on the ball in [7], the discussion in this work will be limited to the zonal operators on spheres. We take as our starting point the Laplacian for a zonal metric.

Let $(x_1, \dots, x_{N+1}, z)$ denote the standard coordinates on $\mathbb{R}^{N+2}$, and consider the hypersurface defined by the equations

$$\sum_{n=1}^{N+1} x_n^2 = r^2(z), \quad -1 \le z \le 1.$$

Assume that r is an even function of z, that $r(-1) = 0 = r(1)$, and that $0 < r(z) < \infty$ for $|z| < 1$.

Let $s(z)$ denote the arc length

$$s(z) = \int_{-1}^{z} \sqrt{1 + [dr(t)/dt]^2} \ dt,$$

and let $L = s(1)$. A metric on this hypersurface is induced from the standard metric on $\mathbb{R}^{N+2}$. Calculations similar to those in [33, pp. 157–162] show that the Laplacian for functions in these coordinates takes the form

$$\Delta_g = r^{-N}(s)\partial_s r^N(s)\partial_s + r^{-2}(s)\Delta_S$$

where Δ_S denotes the Laplacian for the N-sphere.

This operator can be put in the Liouville form with respect to s by the similarity transformation

$$r^{N/2}\Delta_g r^{-N/2} = \partial_s^2 - \frac{N^2}{4}\left(\frac{r'}{r}\right)^2 - \frac{N}{2}\left(\frac{r'}{r}\right)' + r^{-2}(s)\Delta_S$$

where the function

$$-\frac{N^2}{4}\left(\frac{r'}{r}\right)^2 - \frac{N}{2}\left(\frac{r'}{r}\right)'$$

is even about the midpoint of $[0, L]$. For the standard $N+1$ sphere $r(s) = \sin(s)$. Assume that the hypersurface looks like an $N+1$ sphere near $z = -1$ in the sense that $r(s) \in C^2[0, L]$, and for some $p_0 \in C^2[0, L/2]$

$$r(s) = s[1 + p_0(s)], \quad \lim_{s \downarrow 0} p_0(s) = 0, \quad \lim_{s \downarrow 0} p_0'(s)/s \ \text{ exists.}$$

Then for $0 < s < L/2$,

$$r^{-2}(s) = s^{-2} + p_1(s), \quad p_1(s) \in C[0, L/2],$$

and

$$-\frac{N^2}{4}\left(\frac{r'}{r}\right)^2 - \frac{N}{2}\left(\frac{r'}{r}\right)' = \frac{-N(N-2)}{4s^2} + p_2(s), \quad p_2(s) \in C[0, L/2].$$

Say that a metric on the $N+1$ sphere is an even zonal metric if satisfies these conditions.

Suppose now that for some eigenvalue β_k of Δ_S the corresponding eigenvalues of Δ_g, which are eigenvalues of

$$\partial_s^2 y + \left[-\frac{N^2}{4}\left(\frac{r'}{r}\right)^2 - \frac{N}{2}\left(\frac{r'}{r}\right)' + r^{-2}(s)\beta_k \right] y = \lambda y, \tag{6.a}$$

are known. Letting

$$m(m+1) = \frac{N^2 - 2N - 4\beta_k}{4},$$

regularity of the eigenfunctions for Δ_g, requires ([**11**, p. 328] or [**34**, p. 137]) that the boundary conditions are

$$\lim_{s \downarrow 0} s^{-m-1} y(s) < \infty, \quad \lim_{s \uparrow L} L - s^{-m-1} y(s) < \infty.$$

In the case of a zonal Schrödinger operator the analogous expression is

(6.b)

$$\partial_s^2 y + \left[-\frac{N^2}{4}\left(\frac{r'}{r}\right)^2 - \frac{N}{2}\left(\frac{r'}{r}\right)' + r^{-2}(s)\beta_k \right] y + p(s)y = \lambda y, \quad p \in L^2[0, L],$$

where $r(s) = \sin(s)$.

Since the coefficients of (6.a) and (6.b) are even, the eigenvalues $\{\lambda_{n,k}\}$ for β_k fixed are the union of the two sets of eigenvalues $\{\mu_n\}$ and $\{\zeta_n\}$ for the problems on $[0, L/2]$ with boundary conditions $y(L/2) = 0$, and $y'(L/2) = 0$ respectively. To prove this equivalence, first note that since p is even, the odd and even extensions respectively of these eigenfunctions satisfy (6.a,b). Conversely, suppose we have an eigenfunction for (6.a) or (6.b). Then reflection about $L/2$ gives another eigenfunction with the same eigenvalue. Since the eigenspaces are one dimensional the reflection must give a real multiple of the original, and since the norm is the same the multiplier must be ± 1.

Next we consider how to split the eigenvalues $\{\lambda_{n,k}\}$, with β_k given, into the disjoint sequences $\{\mu_n\}$ and $\{\zeta_n\}$ without explicit information about the eigenfunctions. Since the case of a zonal Schrödinger operator is more general, only the case (6.b) needs to be considered. The boundary condition at $s = 0$ already restricts the solutions of (6.b) to a one dimensional space, implying that the sequences $\{\mu_n(p)\}$ and $\{\zeta_n(p)\}$, which come from the boundary conditions $y(L/2) = 0$ and $y'(L/2) = 0$ respectively, have no elements in common.

Now for p fixed and $0 \le t \le 1$ consider the sequences $\{\mu_n(tp)\}$ and $\{\zeta_n(tp)\}$. Since these eigenvalues are continuous functions of t, and the eigenvalues are simple, and the sequences $\{\mu_n(tp)\}$ and $\{\zeta_n(tp)\}$ have no elements in common, the ordering of these sequences is independent of t. Consequently, the splitting of the sequence $\{\lambda_{n,k}\}$ is determined by the splitting for $p = 0$. In fact a similar argument using the explicitly solvable case $-y'' = \lambda y$ shows that the ordering is

$$\zeta_1 < \mu_1 < \zeta_2 < \mu_2 < \dots .$$

For both problems (6.a) and (6.b) the gross asymptotics of the subsequence $\lambda_{n,k}$ of eigenvalues determines L by scaling. In the case of an even zonal Schrödinger operator $p(s)$ is determined by Theorem 1.3.

For the metric problem the subsequence $\lambda_{n,k}$ of eigenvalues determines $p(s) \in C[0, L]$ where

$$-\frac{N^2}{4}\left(\frac{r'}{r}\right)^2 - \frac{N}{2}\left(\frac{r'}{r}\right)' + r^{-2}(s)\beta_k = \frac{\beta_k - N(N-2)}{4s^2} + p(s), \quad 0 < s < L/2.$$

Since $r(s)$ is even, the solution will satisfy the initial condition $r'(L/2) = 0$. Suppose that the radius $r(L/2)$ is known. This data specifies initial conditions for the differential equation for $r(s)$. By assumption the solution is bounded and positive for $|s - L/2| < L/2$, so by Theorem 4.1 of [**9**] the solution may be continued for all such s, so $r(s)$ is uniquely determined by solving the differential equation.

This completes the proof of Theorem 1.4.

Knowledge of the value of r at $z = 0$ may not be needed for uniqueness. This is the case if $\beta_k = 0$. The equation for r then becomes

$$-\frac{N^2}{4}\left(\frac{r'}{r}\right)^2 - \frac{N}{2}\left(\frac{r'}{r}\right)' = \frac{-N(N-2)}{4s^2} + p(s), \quad 0 < s < L/2.$$

Letting $r'/r = \log(r)' = u$ we have a first order equation for u, with the initial condition $u(L/2) = 0$. The desired function $r(s)$ will be among the solutions

$$r(s) = \exp\left(C + \int^s u(t)\, dt\right).$$

All of these functions are positive constant multiples of a single positive function, which we write as

$$r(s) = KU(s), \quad K > 0.$$

Now the parametrization by arc length means that

$$(dz/ds)^2 + (dr/ds)^2 = 1.$$

Thus a candidate for the function $r(s)$ must satisfy

$$K^2(dU/ds)^2 < 1$$

and

$$(dz/ds)^2 + K^2(dU/ds)^2 = 1.$$

This implies that $z(0)$ is monotonically increasing in K, and so only one value of K can yield the required value $z(0) = -1$.

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Received May 24, 1995 and revised October 3, 1995.
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