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TWO GENERALIZATIONS OF THE GLEASON-KAHANE-ŻELAZKO THEOREM

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In this article we obtain 2 generalizations of the well known Gleason-Kahane-Żelazko Theorem. We consider a unital Banach algebra $\mathfrak{A}$, and a continuous unital linear mapping φ of $\mathfrak{A}$ into $M_n(\mathbb{C})$ – the $n \times n$ matrices over $\mathbb{C}$. The first generalization states that if φ sends invertible elements to invertible elements, then the kernel of φ is contained in a proper two sided closed ideal of finite codimension. The second result characterizes this property for φ in saying that $\varphi(\mathfrak{A}_{\text{inv}})$ is contained in $\text{GL}_n(\mathbb{C})$ if and only if for each a in $\mathfrak{A}$ and each natural number k:

$$\text{trace}(\varphi(a^k)) = \text{trace}(\varphi(a)^k)\ .$$

1. Introduction.

The results are based on Aupetits work [1], where he proves that if φ is surjective and $\varphi(\mathfrak{A}_{\text{inv}})$ is contained in $\text{GL}_n(\mathbb{C})$ then φ is a Jordan homomorphism. Some of the key ingredients in Aupetits proof are some relations which involve φ and the trace. We quote [1, Rel (3), p. 15]

$$\text{trace}(\varphi(xy)) = \text{trace}(\varphi(x)\varphi(y)).$$

The results in this paper are based on some elaboration of the identity above.

Aupetits result is proved via a Liouville Theorem for harmonic functions. Since we could not get the reference quoted in [1] for this result, we have included a slightly different argument, which on the other hand is an application of the Hadamard Factorization Theorem, very much in the same way as used in [3, 4].

2. Notation and basic results.

We will consider a unital Banach algebra $\mathfrak{A}$ over the complex numbers, and let $\mathfrak{A}_{\text{inv}}$ denote the set of invertible elements in $\mathfrak{A}$. For a natural number n we let $M_n(\mathbb{C})$ denote the $n \times n$ matrices over $\mathbb{C}$ and we will let $\text{tr}(\cdot)$ denote the usual trace on $M_n(\mathbb{C})$, which satisfies $\text{tr}(I) = n$. We remind the reader that a functional f on an algebra $\mathfrak{B}$ is called a trace if $f(ab) = f(ba)$ for all

a, b from $\mathfrak{B}$. Moreover – up to scalar multiples – there exists only one trace on $M_n(\mathbb{C})$.

The following lemma is well known, but we do not have an exact reference.

Lemma 2.1. *There exists a positive real* $r(0 < r < 1)$ *such that for all* x *in* $M_n(\mathbb{C})$ *with* $\|x - I\| < r$ *we have*

$$|\det(x) - 1| < 1 \quad \textit{and} \quad \mathrm{Log}(\det(x)) = \mathrm{tr}(\mathrm{Log}(x)).$$

Proof. The existence of $r(0 < r < 1)$ such that $\|x - I\| < r \Rightarrow |\det(x) - 1| < 1$ follows from the continuity of the determinant. Suppose now x in $M_n(\mathbb{C})$ is chosen such that $\|x - I\| < r < 1$ then the power series for $\mathrm{Log}(1 + u)$ in the circle $\{u \in \mathbb{C} \mid |u| < 1\}$ converges for both $(\det(x) - 1)$ and $(x - I)$ so the expressions make sense. The equality is easily obtained when x is represented in a Jordan normal form. □

3. Main results.

We start by recapturing the basic results from [1] in Theorem 3.1, and then we present our extensions.

Theorem 3.1. *Let* $\mathfrak{A}$ *be a unital Banach algebra* φ *a continuous unital linear mapping of* $\mathfrak{A}$ *into* $M_n(\mathbb{C})$.

If $\varphi(\mathfrak{A}_{\mathrm{inv}}) \subseteq \mathrm{GL}_n(\mathbb{C})$ *then:*

(i) $\forall\, a, b \in \mathfrak{A} : \det(\varphi(e^a e^b)) = \det(e^{\varphi(a)} e^{\varphi(b)})$,

(ii) $\forall\, a, b \in \mathfrak{A} : \mathrm{tr}(\varphi(ab)) = \mathrm{tr}(\varphi(a)\varphi(b)) = \mathrm{tr}(\varphi(ba))$,

(iii) $\forall\, a, b \in \mathfrak{A} : \det(\varphi(ab)) = \det(\varphi(a)\varphi(b))$.

Proof. For an a in $\mathfrak{A}$ we define an entire function $f(z)$ by

$$f(z) = \det(\varphi(e^{za})e^{-z\varphi(a)}).$$

As usual $f(z) \neq 0$ for all z and the order p of f satisfies $p \leq 1$ since

$$|f(z)| \leq \|\varphi\|^n \exp(|z| n(\|a\| + \|\varphi(a)\|)).$$

By Hadamards Factorization Theorem [2, p. 291; 5 p. 250] we have $f(z) = e^{\alpha + \beta z}$, but $f(0) = 1$ so $f(z) = e^{\beta z}$.

Following Lemma 2.1 we get that for some positive real r we have

$$\begin{aligned} \forall\, z\,,\ |z| < r : \beta z &= \mathrm{tr}(\mathrm{Log}(I + z\varphi(a) + O(z^2))) - z\,\mathrm{tr}(\varphi(a)) \\ &= O(z^2). \end{aligned}$$

Hence $\beta = 0$ and $\det(\varphi(e^a)) = \det(e^{\varphi(a)})$.

In the general case define an entire function $f(w,z)$ in 2 variables by $f(w,z) = \det(\varphi(e^{wa}e^{zb})e^{-w\varphi(a)}e^{-z\varphi(b)})$. Let w be fixed then the function $g(z) = f(w,z)$ is entire, never vanishing, of order 1 and by the previous result $g(0) = 1$, hence there exists a complex function $\alpha(w)$ such that $f(w,z) = g(z) = e^{\alpha(w)z}$. By analogy we find a complex function $\beta(z)$ such that $f(w,z) = e^{w\beta(z)}$. Hence there exists a constant γ such that $f(w,z) = e^{\gamma wz}$. On the other hand the function $k(z) = f(z,z)$ is easily seen to be of order less than or equal to 1 so $\gamma = 0$ and (i) follows. ☐

Applying Lemma 2.1 to both sides in the following identity

$$\det\left(e^{w\varphi(a)}\right)\,\det\left(e^{z\varphi(b)}\right) = \det\left(\varphi\left(e^{wa}e^{zb}\right)\right)$$

shows that there exists a positive real r such that for all z, w in $\mathbb{C}$ with $|z| < r\,,\ |w| < r$

$$\begin{aligned}
&w\,\mathrm{tr}\left(\varphi\left(a\right)\right) + z\,\mathrm{tr}\left(\varphi\left(b\right)\right) = \mathrm{tr}\left(\mathrm{Log}\left(\varphi\left(e^{wa}e^{zb}\right)\right)\right)\\
&\quad = \mathrm{tr}\left(\mathrm{Log}\left(I + w\varphi\left(a\right) + z\varphi\left(b\right) + wz\varphi\left(ab\right) + w^2 p_1\left(w,z\right) + z^2 p_2\left(w,z\right)\right)\right)\\
&\quad = w\,\mathrm{tr}\left(\varphi\left(a\right)\right) + z\,\mathrm{tr}\left(\varphi\left(b\right)\right) + wz\,\mathrm{tr}\left(\varphi\left(ab\right)\right)\\
&\quad - \frac{1}{2} wz\,\mathrm{tr}\left(\varphi\left(a\right)\varphi\left(b\right) + \varphi\left(b\right)\varphi\left(a\right)\right) + w^2 p_3\left(w,z\right) + z^2 p_4\left(w,z\right),
\end{aligned}$$

where $p_i(w,z)$ are power series. Hence (ii) follows from the properties of the trace. The relation (iii) is a consequence of (i) since for $|z| > \|a\| + \|b\|$ we have $(z-a) = z(1-\frac{a}{z}) = z\exp\left(\mathrm{Log}(I - \frac{a}{z})\right)$ and a similar expression for b and hence for $|z| > \|a\| + \|b\|$ we have

$$\det(\varphi((a-z)(b-z))) = \det(\varphi(a-z))\det(\varphi(b-z)).$$

Since the functions involved are entire, we get (iii) for $z = 0$.

The relation (ii) is the basis for the following result.

Theorem 3.2. *Let $\mathfrak{A}$ be a unital Banach algebra and φ a continuous, unital linear mapping of $\mathfrak{A}$ into $M_n(\mathbb{C})$.*

If $\varphi(\mathfrak{A}_{\mathrm{inv}}) \subseteq \mathrm{GL}_n(\mathbb{C})$ then $\mathfrak{A}$ has a proper closed two sided ideal J – of finite codimension – which contains the kernel of φ.

Proof. Define $J = \{a \in \mathfrak{A} \mid \forall\, b \in \mathfrak{A} : \mathrm{tr}(\varphi(ab)) = 0\}$, then J is obviously a closed right ideal, but by the trace property – $\mathrm{tr}(\varphi(ab)) = \mathrm{tr}(\varphi(ba))$ – from (ii) in Theorem 3.1 we see that J is a left ideal as well. The property $\mathrm{tr}(\varphi(ab)) = \mathrm{tr}(\varphi(a)\varphi(b))$ from (ii) above shows that $\ker\varphi \subseteq J$. Hence J is of finite codimension and since $\varphi(I) = I$ we get $I \notin J$ and J is a proper ideal. □

Corollary 3.3. *Let k, n be natural numbers if φ is a unital linear mapping of $M_k(\mathbb{C})$ into $M_n(\mathbb{C})$ which satisfy $\varphi(\mathrm{GL}_k(\mathbb{C})) \subseteqq \mathrm{GL}_n(\mathbb{C})$, then φ is injective and k divides n.*

Proof. Since M_k has no nontrivial ideals, we get $\ker \varphi = \{0\}$ and φ is injective. By (ii) in Theorem 3.1 we get that $\mathrm{tr}_n \circ \varphi$ is a trace on $M_k(\mathbb{C})$ which satisfies $\mathrm{tr}_n(\varphi(I)) = \mathrm{tr}_n(I) = n$. By the uniqueness (up to scalar multiples) of the trace on $M_k(\mathbb{C})$ we have for all a in $M_k(\mathbb{C})$: $\mathrm{tr}_n(\varphi(a)) = (n/k)\,\mathrm{tr}_k(a)$. Let e be a rank 1 projection in $M_k(\mathbb{C})$ then $\sigma(\varphi(e)) \subseteq \sigma(e) = \{0, 1\}$ so $\mathrm{tr}_n(\varphi(e)) \in \mathbb{N}_0$. On the other hand $\mathrm{tr}_n(\varphi(e)) = (n/k)\,\mathrm{tr}_k(e) = n/k$ so $n/k \in \mathbb{N}$. □

Corollary 3.4. *If $\varphi(\mathfrak{A}_{\mathrm{inv}}) \subseteq \mathrm{GL}_n(\mathbb{C})$ then there exists a unital finitedimensional algebra $\mathfrak{B}$ and a linear mapping ψ of $M_n(\mathbb{C})$ into $\mathfrak{B}$ such that $\psi \circ \varphi$ is a unital homomorphism.*

Proof. Just another formulation of the result $\ker \varphi \subseteq J \neq \mathfrak{A}$, combined with elementary algebra. □

The property (ii) from Theorem 3.1 and the theorem above are generalizations of the original Gleason-Kahane-Ẑelazko Theorem. The following result yields another generalisation as well as a characterization of the mappings φ which satisfy $\varphi(\mathfrak{A}_{\mathrm{inv}}) \subseteq \mathrm{GL}_n(\mathbb{C})$.

Theorem 3.5. *Let $\mathfrak{A}$ be a unital Banach algebra and φ a unital continuous linear mapping into $M_n(\mathbb{C})$. Then $\varphi(\mathfrak{A}_{\mathrm{inv}}) \subseteq \mathrm{GL}_n(\mathbb{C})$ if and only if*

$$\forall\, k \in \mathbb{N}\, \forall\, a \in \mathfrak{A} : \mathrm{tr}\left(\varphi\left(a^k\right)\right) = \mathrm{tr}\left(\varphi(a)^k\right).$$

Proof. Suppose first that $\varphi(\mathfrak{A}_{\mathrm{inv}}) \subseteq \mathrm{GL}_n(\mathbb{C})$. Let r be the positive real comming from Lemma 2.1 and let a be in $\mathfrak{A}$ and z in $\mathbb{C}$, then there exists a positive real $r_1, 0 < r_1 < \|a\|^{-1}$ such that for $|z| < r_1$

$$\|\varphi(I + za) - I\| < r \quad \text{and} \quad \|\exp(\varphi(\mathrm{Log}(I + za))) - I\| < r.$$

By Theorem 3.1 (i) we get for $|z| < r_1$

$$\det(\varphi(\exp(\mathrm{Log}(I + za)))) = \det(\exp(\varphi(\mathrm{Log}(I + za))))$$

so by Lemma 2.1

$$\forall\, z, |z| < r_1 : \mathrm{tr}(\mathrm{Log}(I + z\varphi(a))) = \mathrm{tr}(\varphi(\mathrm{Log}(1 + za))).$$

□

By expanding in power series and comparing terms we get

$$\forall\, k \in \mathbb{N} : \quad (-1)^{k-1}k^{-1}\,\mathrm{tr}\left(\varphi(a)^k\right) = (-1)^{k-1}k^{-1}\,\mathrm{tr}\left(\varphi(a^k)\right),$$

and the first part of the proof is complete. Let us now suppose, that for each k in $\mathbb{N}$ and any a in $\mathfrak{A}$ $\mathrm{tr}(\varphi(a^k)) = \mathrm{tr}(\varphi(a)^k)$. Let b be an element in $\mathfrak{A}$ and let $\mathfrak{C}$ denote the abelian unital Banach algebra generated by b and all its resolvents in $\mathfrak{A}$. Since $\mathfrak{C}$ is abelian we get for c, d in $\mathfrak{C}$ $cd = \frac{1}{4}((c+d)^2 - (c-d)^2)$ so

$$(*)\ \mathrm{tr}(\varphi(cd)) = \frac{1}{4}\left(\mathrm{tr}\left((\varphi(c) + \varphi(d))^2 - (\varphi(c) - \varphi(d))^2\right)\right) = \mathrm{tr}(\varphi(c)\varphi(d)).$$

Having this identity we may as in the proof of Theorem 3.2 define a two sided ideal $J_{\mathfrak{C}}$ in $\mathfrak{C}$ by

$$J_{\mathfrak{C}} = \{c \in \mathfrak{C} \mid \forall\, d \in \mathfrak{C} : \mathrm{tr}(\varphi(cd)) = 0\}.$$

Again $J_{\mathfrak{C}} \neq \mathfrak{C}$ and

$$(\ker \varphi \cap \mathfrak{C}) \subseteq J_{\mathfrak{C}}.$$

Since $J_{\mathfrak{C}}$ is a proper two sided ideal in $\mathfrak{C}$ and b is invertible in $\mathfrak{C}$, $b \notin J_{\mathfrak{C}}$, and $\varphi(b) \neq 0$. Let $p(x)$ be a monic polynomial which satisfies $p(\varphi(b)) = 0$. The roots for p are divided into two groups $L = \{\lambda_1, \ldots, \lambda_k\}$ and $M = \{\mu_1, \ldots, \mu_l\}$ corresponding to the criteria: for each λ_i we have $(b - \lambda_i)$ is not invertible in $\mathfrak{C}$ and for each μ_j, $(b - \mu_j)$ is invertible in $\mathfrak{C}$. Finally, there exist exponents $r_1, \ldots, r_k$ and $s_1, \ldots, s_l$ such that

$$p(x) = \left(\prod_{i=1}^{k}(x - \lambda_i)^{r_i}\right)\left(\prod_{j=1}^{l}(x - \mu_j)^{s_j}\right).$$

In order to link properties of $p(b)$ to properties of $p(\varphi(b)) = 0$ we state and prove that for any polynomial q and any c, d in $\mathfrak{C}$ we have

$$(**) \qquad \mathrm{tr}(\varphi(q(c)d)) = \mathrm{tr}(\varphi(q(c))\varphi(d)) = \mathrm{tr}(q(\varphi(c))\varphi(d)).$$

The proof of $(**)$ follows from the proof of the special case where $q(x) = x^s$, $s \in \mathbb{N}$. Let $z \in \mathbb{C}$ then by assumption

$$\forall\, z \in \mathbb{C} : \quad \mathrm{tr}\left(\varphi\left((c + zd)^{s+1}\right)\right) = \mathrm{tr}\left((\varphi(c) + z\varphi(d))^{s+1}\right)$$

and then by comparing terms

$$\operatorname{tr}(\varphi(c^s d)) = \operatorname{tr}\left(\varphi(c)^s \varphi(d)\right).$$

Having $(**)$ we get $p(b) \in J_{\mathfrak{C}}$. Since the elements $(b - \mu_j)$ are invertible in $\mathfrak{C}$ we get for the polynomial $q(x) = \prod_{i=1}^{k}(x - \lambda_i)^{r_i}$ that $q(b) \in J_{\mathfrak{C}}$. By definition of $J_{\mathfrak{C}}$ and by $(**)$ we then have

$$\forall\, s \in \mathbb{N}: \quad \operatorname{tr}\left((q(\varphi(b)))^s\right) = \operatorname{tr}\left(\varphi\left(q(b)^s\right)\right) = 0.$$

The matrix $q(\varphi(b))$ is then nilpotent and then for each λ in $\sigma(\varphi(b))$ there exists an $i \in \{1, \dots k\}$ such that $\lambda = \lambda_i \in \sigma(b)$, and we have proved that $\sigma(\varphi(b)) \subseteq \sigma(b)$, so $\varphi(\mathfrak{A}_{\text{inv}}) \subseteq \mathrm{GL}_n(\mathbb{C})$.

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