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ON FIELDS WITH FINITE BRAUER GROUPS

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Let K be a field of characteristic $\neq 2$, let $\mathrm{Br}(K)_2$ be the 2-primary part of its Brauer group, and let $G_K(2) = \mathrm{Gal}(K(2)/K)$ be the maximal pro-2 Galois group of K. We show that $\mathrm{Br}(k)_2$ is a finite elementary abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^r$, $r \in \mathbb{N}$, if and only if $G_K(2)$ is a free pro-2 product of a closed subgroup H which is generated by involutions and of a free pro-2 group. Thus, the fixed field of H in $K(2)$ is pythagorean. The rank r is in this case determined by the behaviour of the orderings of K. E.g., it is shown that if $r \leq 6$ then K has precisely r orderings, and if $r < \infty$ then K has only finitely many orderings.

Introduction.

It is an open problem in the theory of algebras to characterize the fields K over which there are only finitely many K-central (finite dimensional) division algebras. Equivalently, the Brauer group $\mathrm{Br}(K)$ of K should be finite. As an abelian torsion group, $\mathrm{Br}(K)$ is the direct sum of its p-primary components $\mathrm{Br}(K)_p$, p prime, so one has in fact to know when is $\mathrm{Br}(K)_p$ finite and when is it trivial. Much light is shed on this problem by the following conjecture of Brumer and Rosen [**BR**] which states: for each p either

(i) $\mathrm{Br}(K)_p = 0$;

(ii) $\mathrm{Br}(K)_p$ contains a non-trivial divisible subgroup; or

(iii) $p = 2$ and $\mathrm{Br}(K)_2$ is an elementary abelian 2-group.

This conjecture has been proven in many cases ([**BR**], [**Mer2**], [**Wu**]) - notably, Merkurjev proved it under the assumption that char $K \neq p$ and K contains the group μ_p of roots of unity of order p (or more generally, when $(K(\mu_p) : K) \leq 2$; an alternative proof was given by Kahn [**K**]). Note that when char $K = p$, $\mathrm{Br}(K)_p$ is divisible, so the conjecture is obviously true. When $\mu_p \subseteq K$ we also have $\mathrm{Br}(K)_p = 0$ if and only if $G_K(p)$ is a free pro-p group (here $G_K(p) = \mathrm{Gal}(K(p)/K)$, where $K(p)$ is the compositum of all finite Galois p-extensions of K; cf. Lemma 1.1(a) below). Thus, an essential problem is to characterize the fields K for which $\mathrm{Br}(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^r$ for some $r \in \mathbb{N}$. In this paper we characterize these fields in terms of the group $G_K(2)$. Recall that a field K is **pythagorean** if any sum of squares in K is a square; equivalently [**B1**], $G_K(2)$ is topologically generated by involutions. We prove:

Main Theorem. *The following conditions on a field K of characteristic $\neq 2$ are equivalent:*

(a) $\mathrm{Br}(K)_2$ *is a finite elementary abelian* 2*-group;*

(b) $G_K(2)$ *is a free pro-2 product* $G_K(2) = G_L(2) *_2 \hat{F}$*, where L is a pythagorean subextension of $K(2)/K$ satisfying* $(L^\times : (L^\times)^2) < \infty$ *and* $\hat{F}$ *is a free pro-2 group.*

If one omits the finiteness requirements in these two statements, then (b) still implies (a). The converse implication may also be true. However, our approach (which yields this converse result in the finite case, as in the main theorem) relies on a "realization" property for reduced quaternionic structures (see §2). This property is conjectured to hold in general, but is presently only known to hold under certain finiteness assumptions (in [**K**] Kahn gives several other conditions which are equivalent to $\mathrm{Br}(K)_2$ being of exponent 2, but these conditions do not seem to yield a pythagorean extension as needed).

This stands in an interesting analogy with the following well known property of the character group $C(K) = \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z})$ of a field K of characteristic $\neq 2$ (with G_K denoting the absolute Galois group of K): $C(K)_2$ is an elementary abelian 2-group if and only if K is pythagorean [**DD**]. In the same spirit, the Brumer-Rosen conjecture is analogous to a result of Whaples [**W**], asserting that for any field K one of (i)-(iii) above holds also when we replace $\mathrm{Br}(K)$ by $C(K)$.

In the last two sections we study the size of $\mathrm{Coker}({}_4\mathrm{Br}(K) \xrightarrow{2} \mathrm{Br}(K)_2)$ in general and relate it to the "real arithmetic" of K - in particular, to the theory of fans. It is shown that if this cokernel is finite then K has only finitely many orderings. We also get some more precise information on K when $\mathrm{Br}(K)_2$ is small (§5).

I thank Bruno Kahn for drawing my attention to the unpublished chapter [**K**] of his thesis. I also thank Jack Sonn for several discussions and an anonymous referee for a simplification in Prop. 5.9.

1. Preliminaries.

Let p be a prime number and let K be a field of characteristic $\neq p$ containing a primitive root of unity of order p. We collect several (mostly well-known) facts relating $\mathrm{Br}(K)_p$ to $G_K(p)$. For an abelian group A and a positive integer n, let ${}_nA = \mathrm{Ker}(A \xrightarrow{n} A)$. We write μ_{p^n} for the group of all roots of unity of order dividing p^n (over the prime field of K). After fixing a generator of μ_p we may identify $\mu_p \cong \mathbb{Z}/p\mathbb{Z}$ as $G_K(p)$-modules. The free pro-p product of pro-p groups $\Gamma_1, \dots, \Gamma_m$ is denoted by $\Gamma_1 *_p \cdots *_p \Gamma_m$. We will not distinguish between a central simple K-algebra and its equivalence class in $\mathrm{Br}(K)$.

Lemma 1.1. *Let K be as above.*
(a) *For each n, ${}_{p^n}\mathrm{Br}(K) \cong H^2(G_K(p), \mu_{p^n})$;*
(b) *Let $L_1, \dots, L_m$ be subextensions of $K(p)/K$ such that $G_K(p) = G_{L_1}(p) *_p \cdots *_p G_{L_m}(p)$. Then $\mathrm{Br}(K)_p \cong \mathrm{Br}(L_1)_p \oplus \cdots \oplus \mathrm{Br}(L_m)_p$ via restriction.*

Proof. (a) This is a consequence of the Merkurjev-Suslin theorem ([**MerS**]; cf. [**JW**, 1.7]).
(b) For any finite discrete $G_K(p)$-module A one has a natural isomorphism

$$H^2(G_K(p), A) \xrightarrow{\cong} \bigoplus_{i=1}^{m} H^2(G_{L_i}(p), A)$$

[**N**, Satz 4.1]. Taking $A = \mu_{p^n}$, the assertion follows from (a) by passing to direct limits. □

We abbreviate $H^i(K) = H^i(G_K(p), \mathbb{Z}/p\mathbb{Z})$. Let X_K be the set of orderings of K.

Lemma 1.2. *Let K be as above, let $K \subseteq L \subseteq K(p)$ be a field, and suppose that $G_K(p) = G_L(p) *_p \hat{F}$, with $\hat{F}$ a free pro-p group. Then:*
(a) *$\mathrm{Br}(K)_p \cong \mathrm{Br}(L)_p$ via restriction.*
(b) *If $p = 2$ then $\mathrm{Res}\colon X_L \to X_K$ is bijective.*

Proof. (a) The fixed field M of $\hat{F}$ in $K(p)$ satisfies ${}_p\mathrm{Br}(M) \cong H^2(M) = 0$ (Lemma 1.1(a)). Hence $\mathrm{Br}(M)_p = 0$, so we are done by Lemma 1.1(b).
(b) [**B1**] yields a natural bijection between X_K and the conjugacy classes of the involutions in $G_K(2)$, and likewise for L. As $\hat{F}$ is torsion-free, the assertion follows from the following general group-theoretic results of Herfort and Ribes [**HR**, Th. A and Th. B'] and Melnikov [**Mel**, Prop. 4.9] (independently; see also [**EH**, Lemma 5.4]): If Γ_1, Γ_2 are closed subgroups of a pro-p group G such that $G = \Gamma_1 *_p \Gamma_2$, then:
(i) every element of finite order in G is conjugate to an element of either Γ_1 or Γ_2;
(ii) elements of Γ_1 which are conjugate in G are already conjugate in Γ_1.
□

Finally, let v be a (Krull) valuation on K (and keep assuming that $\mathrm{char}\, K \neq p$ and $\mu_p \subseteq K$). We say that (K, v) is ***p*-henselian** if v has a unique extension to $K(p)$. Denote the residue field of (K, v) by k and its value group by Γ, and suppose that $\mathrm{char}\, k \neq p$. Let $\{\pi_j\}_{j\in J}$ be elements of

$K^\times$ such that the cosets of $v(\pi_j)$, $j \in J$, form a $\mathbb{Z}/p\mathbb{Z}$-linear base of $\Gamma/p\Gamma$. Denote the collection of all subsets of J with precisely $m \in \mathbb{N}$ elements by J_m. Also, let (x) be the image of $x \in K^\times$ in $H^1(K)$ under the Kummer isomorphism $K^\times/(K^\times)^p \cong H^1(K)$, and let $\langle (x) \rangle$ be the generated subgroup. There is a natural epimorphism $G_K(p) \to G_k(p)$ [**EH**, Lemma 1.1], giving rise to inflation maps $\mathrm{Inf}\colon H^i(k) \to H^i(K)$. Parts (a) and (b) of the following lemma are proved by Wadsworth [**W**, Th. 3.6 and Remark 3.14]. Part (c) follows using Lemma 1.1(a) and the Kummer isomorphism.

Lemma 1.3. *In the setup as above and for all $n \in \mathbb{N}$ we have:*

(a) $H^n(K) = \bigoplus_{m=0}^{n} \bigoplus_{\{j_1,\dots,j_m\}\in J_m} (\mathrm{Inf}(H^{n-m}(k)) \cup \langle(\pi_{j_1})\rangle \cup \cdots \cup \langle(\pi_{j_m})\rangle).$

(b) *For each $0 \le m \le n$ and each set $\{j_1,\dots,j_m\} \in J_m$, the map $H^{n-m}(k) \longrightarrow H^n(K)$, given by $\varphi \mapsto \mathrm{Inf}(\varphi) \cup (\pi_{j_1}) \cup \cdots \cup (\pi_{j_m})$, is injective.*

(c) ${}_p\mathrm{Br}(K) \cong {}_p\mathrm{Br}(k) \oplus (k^\times/(k^\times)^p)^J \oplus (\mathbb{Z}/p\mathbb{Z})^{J_2}$.

2. Quaternionic structures.

All fields considered from now on will be assumed to have characteristic $\neq 2$. We keep the cohomological notation of §1, but with $p = 2$.

Our proofs make an essential use of the notion of quaternionic structures, as in [**M**]. We recall that a **quaternionic structure** is a triple $\langle G, Q, q\rangle$, consisting of an elementary abelian 2-group G (written multiplicatively) with a distinguished element -1, a set Q with a distinguished element 0, and a surjection $q\colon G \times G \to Q$, such that for all $x, x', y, y' \in G$:

(1) $q(x,y) = q(y,x)$;

(2) $q(x,-x) = 0$ (where $-x := (-1)x$);

(3) $q(x,y) = q(x',y)$ if and only if $q(xx',y) = 0$;

(4) if $q(x,y) = q(x',y')$ then there exists $z \in G$ such that $q(x,y) = q(x,z)$ and $q(x',y') = q(x',z)$.

Morphisms and direct products of quaternionic structures are defined in the obvious way (see [**M**] for more details and background). Given subgroups G_1, G_2 of G, let $\mathcal{Q}_i = \langle G_i, Q_i, q_i\rangle$ be the induced structures, where $Q_i = q(G_i \times G_i)$ and $q_i = q|_{G_i\times G_i}$, $i = 1,2$. A decomposition $G = G_1 \times G_2$ extends to a decomposition $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$ precisely when the following holds: for any $x_1, y_1 \in G_1$, $x_2, y_2 \in G_2$ we have $q(x_1y_1, x_2y_2) = 0$ if and only if $q(x_i, y_i) = 0$, $i = 1,2$ [**M**, Th. 5.8].

To a field K one associates a quaternionic structure

$$\mathcal{Q}(K) = \langle K^\times/(K^\times)^2, Q_K, q_K\rangle,$$

where -1 is $-(K^\times)^2$, Q_K is the set of all quaternion algebras in ${}_2\operatorname{Br}(K)$, and q_K is the usual quaternionic pairing. To a field extension L/K one associates in a natural way a morphism $\operatorname{Res}\colon \mathcal{Q}(K) \to \mathcal{Q}(L)$.

The category of quaternionic structures is naturally equivalent to the category of abstract Witt rings [**M**, Th. 4.5]. We will not make here any real use of the latter notion; however, in considerations involving quaternionic structures we will freely apply some known results which are formulated in the literature in terms of abstract Witt rings. Under the above-mentioned equivalence, $\mathcal{Q}(K)$ corresponds to the Witt ring $W(K)$ of K.

A quaternionic structure $\mathcal{Q} = \langle G, Q, q\rangle$ is called:

(i) **completely degenerate** if $Q = \{0\}$;

(ii) **non-degenerate** if q is non-degenerate;

(iii) **reduced** if for all $x \in G$, $q(x,x) = 0$ implies $x = 1$ (equivalently, the map $x \mapsto q(x,-1)$ is injective);

(iv) **euclidean** if $G = \{1,-1\}$ and $|Q| = 2$;

(v) **non-basic** if $G \neq \{1,-1\}$ and there exists $x \in G$ such that $q(x,y) \neq 0$ for all $y \in G \setminus \{1,-x\}$ and such that $q(-x,y) \neq 0$ for all $y \in G \setminus \{1,x\}$;

(vi) **finitely generated** if $|G| < \infty$.

Remark 2.1. Let K be a field. Then $\mathcal{Q}(K)$ is completely degenerate precisely when $H^2(K) = 0$ (by [**Mer1**]), i.e., when $G_K(2)$ is a free pro-2 group [**S1**, I-32, Prop. 21.2 and I-37, Cor. 2]. Also, $\mathcal{Q}(K)$ is reduced if and only if K is pythagorean [**M**, pp. 89-90]. Finally, $\mathcal{Q}(K)$ is euclidean if and only if $G_K(2)$ has (profinite) rank 1 but is not $\mathbb{Z}_2$. By the results of [**B1**], this means that K is a euclidean field (i.e., $G_K(2) \cong \mathbb{Z}/2\mathbb{Z}$).

The following classification theorem [**M**, Th. 6.23] is of fundamental importance:

Theorem 2.2 (Marshall)**.** *Every finitely generated reduced quaternionic structure is a direct product of finitely many quaternionic structures which are either euclidean or both reduced and non-basic.*

Following Arason, Elman and Jacob [**AEJ**], we call a quaternionic structure $\mathcal{Q}$ **realizable** if for every field K and a decomposition $\mathcal{Q}(K) = \mathcal{Q} \times \mathcal{Q}'$ of quaternionic structures there exists a subextension $K \subseteq L \subseteq K(2)$ such that $\operatorname{Res}\colon \mathcal{Q}(K) \to \mathcal{Q}(L)$ coincides with the projection $\mathcal{Q} \times \mathcal{Q}' \to \mathcal{Q}$. It is unknown whether every quaternionic structure is realizable. Yet, one has:

Proposition 2.3. *A quaternionic structure $\mathcal{Q}$ is realizable in each of the following cases:*

(a) *$\mathcal{Q}$ is non-basic;*

(b) *$\mathcal{Q}$ is completely degenerate;*

(c) *$\mathcal{Q}$ is euclidean.*

Proof. (a) This is [**AEJ**, Th. 4.8].
(b) Consider a decomposition $\mathcal{Q}(K) = \mathcal{Q} \times \mathcal{Q}'$ with $\mathcal{Q} = \langle G, Q, q\rangle$ completely degenerate and $\mathcal{Q}' = \langle G', Q', q'\rangle$. Let A, A' be $\mathbb{F}_2$-linear bases of G, G', respectively. Denoting the Frattini subgroup by Φ, we have a perfect duality

$$G_K(2)/\Phi(G_K(2)) \ \times \ K^\times/(K^\times)^2 \longrightarrow \{\pm 1\},$$

given by $(\bar\sigma, \bar a) \mapsto \langle \bar\sigma, \bar a\rangle = \sigma(\sqrt{a})/\sqrt{a}$ for $\sigma \in G_K(2)$ and $a \in K^\times$ (with $\bar\sigma$, $\bar a$ denoting the corresponding cosets, and with $\sqrt{a}$ being a fixed square root of a). Choose subsets Σ, Σ' of $G_K(2)$ such that the cosets of $\Sigma \cup \Sigma'$ form an $\mathbb{F}_2$-linear basis of $G_K(2)/\Phi(G_K(2))$ dual to $A \cup A'$. Let L be the fixed field of Σ in $K(2)$. For any $b \in L^\times$ the duality yields $a \in K^\times$ such that $\langle\bar\sigma, \bar a\rangle = \langle \bar\sigma, \bar b\rangle$ for all $\sigma \in \Sigma$ and such that $\langle \bar\sigma', \bar a\rangle = 1$ for all $\sigma' \in \Sigma'$. Then $a \equiv b \bmod (L^\times)^2$ and $(a) \in G$. As $\mathcal{Q}$ is completely degenerate, we conclude that the quaternionic pairing q_L is trivial and $\mathrm{Res}\colon \mathcal{Q}(K) \to \mathcal{Q}(L)$ coincides with the projection $\mathcal{Q}(K) \to \mathcal{Q}$.
(c) In this case the projection $\mathcal{Q}(K) = \mathcal{Q} \times \mathcal{Q}' \to \mathcal{Q}$ corresponds to a signature on K, hence also to an ordering $P \in X_K$ [**M**, pp. 74-75]. We take L to be a euclidean closure of K with respect to P [**B1**] (alternatively, one can argue along the lines of (b)). □

The following is the immediate quaternionic structure analog of [**JWr**, Th. 3.4 and Remark 3.5]:

Lemma 2.4 (Jacob, Ware). *The following conditions on a field K and fields $K \subseteq L_1, \dots, L_n \subseteq K(2)$ are equivalent:*
(a) *$\mathcal{Q}(K) = \mathcal{Q}(L_1) \times \cdots \times \mathcal{Q}(L_n)$ via restriction;*
(b) *$G_K(2) = G_{L_1}(2) *_2 \cdots *_2 G_{L_n}(2)$.*

Lemma 2.5. *Every quaternionic structure $\mathcal{Q}$ decomposes as $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$, with $\mathcal{Q}_1$ non-degenerate and $\mathcal{Q}_2$ completely degenerate.*

Proof. Write $\mathcal{Q} = \langle G, Q, q\rangle$ and decompose $G = G_1 \times G_2$, with $G_2 = \{x \in G \mid \forall y \in G: \ q(x,y) = 0\}$. For any $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$ we have $q(x_1x_2, y_1y_2) = 0$ precisely when $q(x_1, y_1) = 0$. Therefore, by the criterion mentioned earlier, this decomposition of G gives rise in a natural way to a decomposition $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$ of quaternionic structures with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ as in the lemma. □

Lemma 2.6. *The following conditions on a quaternionic structure $\mathcal{Q} = \langle G, Q, q\rangle$ are equivalent:*
(a) *If $x \in G$ satisfies $q(x,x) = 0$ then $q(x,y) = 0$ for all $y \in G$;*

(b) $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$ *with* $\mathcal{Q}_1$ *(resp.,* $\mathcal{Q}_2$*) a reduced (resp., completely degenerate) quaternionic structure.*

Proof. (a)$\Rightarrow$(b): First decompose $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$ as in Lemma 2.5 and write $\mathcal{Q}_i = \langle G_i, Q_i, q_i \rangle$, $i = 1, 2$. Suppose that $x \in G_1$ satisfies $q_1(x, x) = 0$. Then also $q(x, x) = 0$, so by assumption $q(x, y) = 0$ for all $y \in G$. In particular, $q_1(x, y) = 0$ for all $y \in G_1$, whence $x = 1$. This shows that $\mathcal{Q}_1$ is reduced.
(b)$\Rightarrow$(a): Denote again $\mathcal{Q}_i = \langle G_i, Q_i, q_i \rangle$, $i = 1, 2$. Let $x, y \in G$ and suppose that $q(x, x) = 0$. Decompose $x = x_1 x_2$, $y = y_1 y_2$, with $x_1, y_1 \in G_1$, $x_2, y_2 \in G_2$. By the decomposition criterion mentioned above, $q_1(x_1, x_1) = 0$. Since $\mathcal{Q}_1$ is reduced, $x_1 = 1$. As $\mathcal{Q}_2$ is completely degenerate, $q_2(x_2, y_2) = 0$. By the decomposition criterion again, $q(x_2, y_1) = 0$. Therefore $q(x, y) = q(x_2, y_1 y_2) = 0$. □

3. Elementary abelian 2-primary Brauer groups.

We view the cyclic algebra defined by $\chi \in C(K)$ and $b \in K^\times$ as a bilinear pairing $C(K) \otimes K^\times \to \mathrm{Br}(K)$, $\chi \otimes b \mapsto (\chi, b)$ [**S2**, Ch. XIV, §1]. In particular, if $\chi = \chi_a$ is the character in ${}_2C(K)$ with kernel $G_{K(\sqrt{a})}$, $a \in K^\times$, then (χ_a, b) is the quaternion algebra (a, b) in ${}_2\mathrm{Br}(K)$.

Theorem 3.1. *The following conditions on a field K are equivalent:*
(a) $\mathrm{Br}(K)_2$ *is an elementary abelian 2-group.*
(b) *If $a \in K^\times$ satisfies $(a, a) = 0$ then $(a, b) = 0$ for all $b \in K^\times$.*
(c) $\mathcal{Q}(K) = \mathcal{Q}_1 \times \mathcal{Q}_2$ *with* $\mathcal{Q}_1$ *(resp.,* $\mathcal{Q}_2$*) a reduced (resp., completely degenerate) quaternionic structure.*
(d) $\cup(-1)\colon H^2(K) \to H^3(K)$ *is injective.*

Proof. (a)$\Leftrightarrow$(b): By the results of Merkurjev and Suslin [**MerS**, §16], the cyclic algebra pairing ${}_4C(K) \otimes K^\times \to {}_4\mathrm{Br}(K)$ is surjective. Thus, (a) means that $(2\chi, b) = 0$ for all $\chi \in {}_4C(K)$ and $b \in K^\times$. Furthermore, there is an exact sequence

$$ {}_4C(K) \xrightarrow{2} {}_2C(K) \xrightarrow{\delta} {}_2\mathrm{Br}(K), $$

where $\delta(\chi_a) = (a, a)$ [**S3**, p. 4]. Thus, as χ ranges over ${}_4C(K)$, the character 2χ ranges over all $\chi_a \in {}_2C(K)$ with $a \in K^\times$ satisfying $(a, a) = 0$.
(b)$\Leftrightarrow$(c): Use Lemma 2.6.
(a)$\Leftrightarrow$(d): [**LLT**, Cor. A4] (and [**Mer1**]) yields an exact sequence

$$ {}_4\mathrm{Br}(K) \xrightarrow{2} {}_2\mathrm{Br}(K) \cong H^2(K) \xrightarrow{\cup(-1)} H^3(K). $$

□

Other equivalent conditions are given in [**K**, Th. 2]. In light of Remark 2.1 we get:

Corollary 3.2. *A field K is pythagorean if and only if:*

(i) $\mathrm{Br}(K)_2$ *is an elementary abelian* 2*-group; and*

(ii) *the quaternionic pairing* $q_K\colon K^\times/(K^\times)^2 \times K^\times/(K^\times)^2 \to {}_2\mathrm{Br}(K)$ *is non-degenerate.*

Corollary 3.3. *Let K be a field such that $G_K(2) = G_L(2) *_2 \hat{F}$, with $K \subseteq L \subseteq K(2)$ a pythagorean field and $\hat{F}$ a free pro-*2 *group. Then* $\mathrm{Br}(K)_2$ *is an elementary abelian* 2*-group.*

Proof. Let M be the fixed field of $\hat{F}$ in $K(2)$. By Lemma 2.4, $\mathcal{Q}(K) \cong \mathcal{Q}(L) \times \mathcal{Q}(M)$ via restriction. By Remark 2.1, $\mathcal{Q}(L)$ is reduced and $\mathcal{Q}(M)$ is completely degenerate, so we are done by Theorem 3.1. $\square$

Proof of the Main Theorem. Assume (a). Theorem 3.1 yields a decomposition $\mathcal{Q}(K) = \mathcal{Q}_1 \times \mathcal{Q}_2$ with $\mathcal{Q}_1$ reduced and $\mathcal{Q}_2$ completely degenerate. Write $\mathcal{Q}_1 = \langle G_1, Q_1, q_1\rangle$. Since the map $G_1 \to Q_1$ given by $x \mapsto q(x,-1)$ is injective, and since Q_1 embeds into the finite group ${}_2\mathrm{Br}(K)$, the structure $\mathcal{Q}_1$ is finitely generated. By Theorem 2.2 and Proposition 2.3, $\mathcal{Q}_1$ is the product of finitely many quaternionic structures which are reduced and realizable. Moreover, $\mathcal{Q}_2$ is also realizable. Using Lemma 2.4 we get $K \subseteq L_1, \ldots, L_n, L_{n+1} \subseteq K(2)$ such that $G_K(2) = G_{L_1}(2) *_2 \cdots *_2 G_{L_{n+1}}(2)$, such that $\mathcal{Q}(L_1), \ldots, \mathcal{Q}(L_n)$ are reduced and $\mathcal{Q}(L_{n+1})$ is completely degenerate, and such that $\mathcal{Q}_1 \cong \mathcal{Q}(L_1) \times \cdots \times \mathcal{Q}(L_n)$. By Remark 2.1, $L_1, \ldots, L_n$, hence also $L = L_1 \cap \cdots \cap L_n$, are pythagorean, and $G_{L_{n+1}}(2)$ is a free pro-2 group. Clearly, $G_K(2) = G_L(2) *_2 G_{L_{n+1}}(2)$. Finally, $L^\times/(L^\times)^2 \cong L_1^\times/(L_1^\times)^2 \times \cdots \times L_n^\times/(L_n^\times)^2 \cong G_1$, whence $(L^\times : (L^\times)^2) < \infty$.

Conversely, assume (b). Then $\mathrm{Br}(K)_2 \cong \mathrm{Br}(L)_2$ via restriction (Lemma 1.2(a)). Since the quaternionic pairing $L^\times/(L^\times)^2 \otimes L^\times/(L^\times)^2 \to {}_2\mathrm{Br}(L)$ is surjective [**Mer1**] and $(L^\times : (L^\times)^2) < \infty$, the group ${}_2\mathrm{Br}(L)$ is finite. But ${}_2\mathrm{Br}(L) = \mathrm{Br}(L)_2$ (Corollary 3.2), so $\mathrm{Br}(K)_2$ is a finite elementary abelian 2-group. $\square$

Remark 3.4. Any elementary abelian 2-group is realizable as $\mathrm{Br}(K)_2$ for some pythagorean field K. Indeed, let A be any set (considered as a discrete topological space) and let X be its one-point compactification. Since X is Boolean (i.e., Hausdorff, compact and totally disconnected), a construction of Craven [**C**] yields a pythagorean field K satisfying the strong approximation property (SAP) and for which $X_K \cong X$. By Corollary 3.2, $\mathrm{Br}(K)_2 = {}_2\mathrm{Br}(K) \cong H^2(K)$. It follows from [**Er**, Th. 3 and Lemma 2] that $\mathrm{Br}(K)_2 \cong \{\pm 1\}^A$.

4. The T-invariant.

Even when $\mathrm{Br}(K)_2$ is not an elementary abelian 2-group, one may still extract some information regarding the "reduced" structure of K by considering

$$T(K) = \mathrm{Coker}({}_4\mathrm{Br}(K) \xrightarrow{2} {}_2\mathrm{Br}(K)).$$

For $a, b \in K^\times$ let $\overline{(a,b)}$ be the image of the quaternion algebra (a,b) under the natural projection ${}_2\mathrm{Br}(K) \to T(K)$. By a result of Lam, Leep and Tignol [**LLT**, Cor. 5.14], $\overline{(a,b)} = 0$ if and only if the Pfister form $\langle\langle -a, -b, 1\rangle\rangle$ is 0 in $W(K)$, or equivalently, if and only if $(a) \cup (b) \cup (-1) = 0$ in $H^3(K)$. Moreover, by [**LLT**, Cor. A4] again, $T(K) \cong H^2(K) \cup \langle(-1)\rangle$ naturally. Note that if $0 \neq a \in K^2 + K^2$ then $\langle\langle -a, 1\rangle\rangle$ is isotropic, hence $\langle\langle -a, -b, 1\rangle\rangle = 0$ in $W(K)$ for all $b \in K^\times$. Consequently, the quaternionic pairing induces a bilinear map

$$K^\times/((K^2+K^2) \setminus \{0\}) \otimes K^\times/((K^2+K^2) \setminus \{0\}) \longrightarrow T(K),$$

which is surjective by [**Mer1**] (recall that $(K^2+K^2) \setminus \{0\}$ is a group with respect to multiplication [**L1**, Ch. X, Cor. 1.7]). Given a field extension L/K, the restriction map of the Brauer groups induces a functorial restriction homomorphism $\mathrm{Res}\colon T(K) \to T(L)$.

Lemma 4.1. (a) *Let $L_1, \dots, L_m$ be subextensions of $K(2)/K$ such that $G_K(2) = G_{L_1}(2) *_2 \cdots *_2 G_{L_m}(2)$. Then $T(K) \cong T(L_1) \oplus \cdots \oplus T(L_m)$ via restriction.*
(b) *Suppose that (K, v) is 2-henselian, let k, J, J_2 be as in §1, and assume that* $\mathrm{char}\, k \neq 2$ *and that $-1 \notin K^2$. Then:*

$$T(K) \cong T(k) \oplus (H^1(k) \cup \langle(-1)\rangle)^J \oplus (\mathbb{Z}/2\mathbb{Z})^{J_2}.$$

Proof. (a) Use Lemma 1.1(b).
(b) This follows from Lemma 1.3. □

4.2 Remarks. (1) If $\sqrt{-1} \in K$ then $T(K) = H^2(K) \cup \langle(-1)\rangle = \{0\}$.
(2) The triviality of $T(K)$ does not imply that $\mathrm{Br}(K)_2$ is divisible. For example, let $K = \mathbb{Q}(\sqrt{-1}, t)$, with t transcendental. By (1), $T(K) = \{0\}$. But by a result of Fein, Schacher and Sonn [**FSS**], $\mathrm{Br}(K)_2$ has a direct summand which is generated as an abelian group by elements $x, y_2, y_3, y_4, \dots$ subject to defining relations $2x = 0$, $x = 2^i y_i$, $i = 2, 3, \dots$. Here $2y_2 - 4y_3$ is an involution which is not divisible by 4.

5. Quantitative results.

In this section we estimate the size of $\mathrm{Br}(K)_2$ in case it is an elementary abelian 2-group. We remark however, that in the special case when K is pythagorean with finitely many square classes (or more generally, with finite chain length - see below), one has a rather complete structure theory for $G_K(2)$ (see [**J**] or [**Mi**]) which allows a case-by-case computation of $\mathrm{Br}(K)_2$ (by means of Lemma 1.1(b) and Lemma 1.3). To treat the general case, we use instead the theory of fans and the related notion of a strictly pythagorean field. Throughout this section we assume general familiarity with the theory of ordered fields, e.g., as in [**L2**].

Let $C(X,G)$ be the group of all continuous maps from a topological space X into a discrete group G (when $X=\emptyset$ we set $C(X,G)=\{0\}$). Denote the fundamental ideal of $W(K)$ by $I(K)$. The **(reduced) stability index** $\mathrm{st}(K)$ of K is the minimal positive integer s (∞ if no such integer exists) such that the total signature sgn: $I(K)^s\to C(X_K,2^s\mathbb{Z})$ is surjective (where $I(K)^0=W(K)$) [**L2**, §13]. For $S\subseteq K$ let $X_K(S)=\{P\in X_K\mid S\subseteq P\}$ endowed with the Harrison topology [**L2**, p. 1].

Recall that a field E is **strictly pythagorean** if E^2 is a fan. Equivalently, E is formally real and $E(2)$-hereditarily pythagorean (i.e., all formally real subextensions $E(2)/E$ are pythagorean) [**B2**, p. 89, Th. 2]. In this case $G_E(2)\cong\mathbb{Z}_2^m\rtimes(\mathbb{Z}/2\mathbb{Z})$ for some cardinal number m, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}_2^m$ by inversion, and $|X_E|=2^m$ [**B2**, pp. 86-87 and p. 124]. Considering m also as an ordinal number, we may construct such a strictly pythagorean field E as follows: order $\Gamma=\mathbb{Z}^m$ lexicographically and let E be the field of formal power series $\sum_{\gamma\in\Gamma}a_\gamma t^\gamma$, with $a_\gamma\in\mathbb{R}$ and with $\{\gamma\in\Gamma\mid a_\gamma\neq 0\}$ well-ordered.

The following lemma (essentially due to Becker) allows one to estimate from below the size of $T(K)$:

Lemma 5.1. *Let m be a positive integer and let K be a field with $\mathrm{st}(K)\geq m$. There exists a strictly pythagorean field $K\subseteq E\subseteq K(2)$ such that:*
(a) $E=KE^2$;
(b) $|X_E|=2^m$;
(c) $\mathrm{Res}\colon X_E\to X_K$ *is injective;*
(d) $T(E)=\mathrm{Br}(E)_2\cong(\mathbb{Z}/2\mathbb{Z})^{(m^2+m+2)/2}$;
(e) $\mathrm{Res}\colon T(K)\to T(E)$ *is surjective.*

Proof. Since m is positive, K is formally real. As $\mathrm{st}(K)\geq m$, [**L2**, Th. 13.7] yields a fan S_0 on K with $|X_K(S_0)|\geq 2^m$. Hence $(K^\times:S_0^\times)\geq 2^{m+1}$. Take a subgroup of S of $K^\times$ containing $S_0^\times$ but not -1 such that $(K^\times:S)=2^{m+1}$. By [**L2**, Th. 5.5(2) and Remark 5.2], $S\cup\{0\}$ is a fan, whence $|X_K(S)|=$

2^m. [**B2**, p. 143: Th. 7] now yields a strictly pythagorean field $K \subseteq E \subseteq K(2)$ satisfying (a)-(c). Assertion (d) follows from [**B2**, p. 128: Th. 20]. By [**Mer1**] (or by [**B2**, p. 128: Th. 20] again), $\mathrm{Br}(E)_2$ is generated by quaternion algebras. Combined with (a), this gives (e). □

For each $P \in X_K$ fix a euclidean closure $\bar{K}_P$ of K with respect to P [**B1**]. Since $T(\bar{K}_P) = \mathrm{Br}(\bar{K}_P)_2 \cong \mathbb{Z}/2\mathbb{Z}$ (with $(-1,-1)$ as the non-trivial element), we may identify $\prod_{P\in X_K} T(\bar{K}_P)$ with the group of all mappings $X_K \to \{\pm 1\}$. Under this identification, the image of the restriction $T(K) \to \prod_{P\in X_K} T(\bar{K}_P)$ is contained in $C(X_K, \{\pm 1\})$, by [**Mer1**] again. In this manner we get an $\mathbb{F}_2$-linear map $\Lambda_K \colon T(K) \to C(X_K, \{\pm 1\})$ (which is independent of the choice of the euclidean closures).

Recall that the **chain length** $\mathrm{cl}(K)$ of a pythagorean field K is the supremum of all $n \in \mathbb{N}$ for which there exists a proper chain $H_K(a_0) \subset H_K(a_1) \subset \cdots \subset H_K(a_n)$ of (subbasic) Harrison sets, with $a_0, a_1, \ldots, a_n \in K$ ([**L2**, §8], [**EH**, §2]).

Proposition 5.2. *Let K be a field.*

(a) *If K is pythagorean and* $\mathrm{cl}(K) < \infty$ *then Λ_K is injective.*

(b) *Λ_K is surjective if and only if* $\mathrm{st}(K) \le 2$.

Proof. (a) As $T(K) = {}_2\mathrm{Br}(K)$ (by Corollary 3.2), this follows from Jacob's results in [**J**, §5]. Note that when K is pythagorean and not formally real (a case not covered in [**J**]) $\sqrt{-1} \in K$, so $T(K) = 0$ by Remark 4.2(1).

(b) [**Mer1**] gives an epimorphism $I(K)^2 \to {}_2\mathrm{Br}(K)$ mapping the Pfister form $\langle\langle -a, -b\rangle\rangle$ to (a, b). We get a natural commutative square:

$$\begin{tikzcd} I(K)^2 \arrow[r] \arrow[d, "\mathrm{sgn}"'] & {}_2\mathrm{Br}(K) \arrow[r] & T(K) \arrow[d, "\Lambda_K"'] \\ C(X_K, 4\mathbb{Z}) \arrow[r] & C(X_K, 4\mathbb{Z}/8\mathbb{Z}) \arrow[r, "\cong"] & C(X_K, \{\pm 1\}) \end{tikzcd}$$

If $\mathrm{st}(K) \le 2$ then the left vertical map is surjective, whence so is Λ_K.

Conversely, suppose that $\mathrm{st}(K) \ge 3$. Let E be as in Lemma 5.1 with $m = 3$. Identifying X_E with its image under the injection $\mathrm{Res}\colon X_E \to X_K$, we get a commutative square:

$$\begin{tikzcd} T(E) \arrow[r, "\Lambda_E"] & C(X_E, \{\pm 1\}) \\ T(K) \arrow[u, "\mathrm{Res}"] \arrow[r, "\Lambda_K"] & C(X_K, \{\pm 1\}). \arrow[u, "\mathrm{Res}"'] \end{tikzcd}$$

Since X_E is finite, the right vertical map is surjective. As $T(E) \cong (\mathbb{Z}/2\mathbb{Z})^7$ and $C(X_E, \{\pm 1\}) \cong (\mathbb{Z}/2\mathbb{Z})^8$, the map Λ_E is not surjective. Therefore Λ_K is not surjective. □

Corollary 5.3. *Let K be a field such that either $|X_K| \leq 7$ or $\dim_{\mathbb{F}_2} T(K) \leq 6$. Then $|X_K| \leq \dim_{\mathbb{F}_2} T(K)$.*

Proof. In both situations, Lemma 5.1 implies that $\mathrm{st}(K) \leq 2$. By Proposition 5.2(b), Λ_K is surjective. □

Remark 5.4. In general, Λ_K need not be injective. For example, let K be a field in which -1 is a sum of 8, but not less, squares (such fields exist by a result of Pfister [**L1**, Ch. XI, Th. 2.8]). Then $X_K = \emptyset$, but by the observations in §4, $\overline{(-1,-1)} \neq 0$ in $T(K)$.

Proposition 5.5. *Let K be a field with $\mathrm{Br}(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^r$ and $0 \leq r \leq 6$. Then $|X_K| = r$.*

Proof. In light of the main theorem and Lemma 1.2, we may assume that K is pythagorean. By Lemma 5.1 and the assumption, $\mathrm{st}(K) \leq 2$. By Corollary 5.3, $|X_K| < \infty$, hence also $\mathrm{cl}(K) < \infty$. The assertion therefore follows from Proposition 5.2. □

Similarly we obtain:

Proposition 5.6. *Let K be a pythagorean field with $|X_K| \leq 7$. Then $\mathrm{Br}(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{|X_K|}$.*

Proposition 5.7. *Let K be a field with $\mathrm{Br}(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^r$ and $0 \leq r \leq 3$. Then $G_K(2)$ is the free pro-2 product of r copies of $\mathbb{Z}/2\mathbb{Z}$ and of a free pro-2 group.*

Proof. In light of the main theorem and Lemma 1.2(a), we may assume again that K is pythagorean. Lemma 5.1 implies that $\mathrm{st}(K) \leq 1$. Thus K is an SAP field [**L2**, Cor. 17.11]. By a result of Eršov [**Er**, Th. 3], $G_K(2)$ is then a free pro-2 product of finitely many copies of $\mathbb{Z}/2\mathbb{Z}$. The fixed fields in $K(2)$ of these copies are euclidean, hence have $\mathbb{Z}/2\mathbb{Z}$ as their 2-primary Brauer group. Conclude from Lemma 1.1(b) that the number of these copies is r, as required. □

Remark 5.8. The bounds in Propositions 5.5-5.7 are the best possible. Indeed, let K be a strictly pythagorean field with 8 orderings (see the remarks before Lemma 5.1). Then $\mathrm{Br}(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^7$. Thus, Proposition 5.5 is false for $r = 7$ and Proposition 5.6 is false for $|X_K| = 8$. When K is a strictly pythagorean field with 4 orderings, $\mathrm{Br}(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^4$ and $G_K(2) \cong \mathbb{Z}_2^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$, with the generator of $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{Z}_2^2$ by inversion. This Galois group is not a free pro-2 product of 4 copies of $\mathbb{Z}/2\mathbb{Z}$ and of a free pro-2 group (e.g., since the chain length of the former group is 2 and that of the latter is 4; cf. [**EH**, Lemma 2.1]). Consequently, in Proposition 5.7 one cannot take $r = 4$.

Proposition 5.9. *Let K be a field with $T(K)$ finite. Then X_K is finite.*

Proof. Let D be the kernel of the homomorphism $K^\times \to H^3(K)$ given by $a \mapsto (a) \cup (-1) \cup (-1)$. Equivalently, D is the set of all $a \in K^\times$ which are sums of four squares in K (cf. §4). The signature map therefore induces an embedding $X_K \hookrightarrow \mathrm{Hom}(K^\times/D, \{\pm 1\})$. But

$$K^\times/D \cong H^1(K) \cup \langle(-1)\rangle \cup \langle(-1)\rangle \subseteq H^2(K) \cup \langle(-1)\rangle \cong T(K).$$

Therefore $K^\times/D$, whence also X_K, are finite. □

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