PACIFIC JOURNAL OF MATHEMATICS
Vol. 177, No. 1, 1997

ON SYZYGIES OF PROJECTIVE VARIETIES

S.P. Inamdar

In this article, we give a criterion for an embedding of a projective variety to be defined by quadratic equations and for it to have linear syzygies. Our criterion is intrinsic in nature and implies that embedding corresponding to a sufficiently high power of any ample line bundle will have linear syzygies up to a given order.

Introduction.

Let X be a projective variety. The conditions that guarantee projective normality and quadratic generations of the ideal defining the embedding of X were studied classically. These results of Mumford et al. were considered by Mark Green as the first step towards understanding the higher syzygies. We say that a line bundle L on X satisfies Property N_p if the ideal defining the embedding of X in the complete linear system of L is generated by quadratic equations and has linear syzygies till p^{th} stage (see [2]). In [3], Green introduced the notion of Koszul cohomology and proved that vanishing of certain higher Koszul cohomology is equivalent to the Property N_p.

In [4], it was proved that the required higher Koszul cohomology groups vanish for sufficiently ample line bundles ([4, Theorem 3.2]). However, the proof given in [4] makes a tacit assumption that the ideal sheaf of the diagonal embedding of X in its two-fold product X^2 is locally free which is not true.

If X is a smooth projective variety, using different methods, L. Ein and R. Lazarsfeld obtained an effective bound on the power of L required for satisfying Property N_p (see [2]).

In this article, we give a criterion for a line bundle L to have Property N_p in terms of vanishing of certain first cohomology. Our method does not assume the smoothness of X. Using this criterion, we see that sufficiently high power of L will have property N_p for fixed p. Therefore, we also see that higher Koszul cohomology of a sufficiently high power of an ample line bundle vanish. This answers the question 5.13 of [3]. Here, we would like to mention that when X is a smooth variety, a vanishing theorem of M. Nori can also be used to answer this question ([8, Proposition 3.4]).

Our method does not give an effective bound on the required power of an ample line bundle to have certain linear syzygies. However, the vanishing conditions of our criterion are explicit in nature and hopefully will lead us to effective bounds.

Acknowledgements. The author would like to thank Wilberd van der Kallen and the mathematics department of Universiteit Utrecht for their hospitality when this work was done. This article was written when the author was visiting the University of North Carolina at Chapel Hill.

1. A Criterion for Regularity.

Let X be a projective variety over an algebraically closed field of characteristic zero. Let L be a line bundle on X. We assume that $H^0(X, L) \neq 0$. Let $V = H^0(X, L)$ and let $S(V)$ denote the symmetric algebra generated by V. We want to study the resolution of the ring $R = \oplus H^0(X, L^n)$ as a $S(V)$-module.

Assume now that L is generated by its global sections. Also we assume that $H^1(X, L^b) = 0$ for all $b \geq 1$. Consider the exact sequence:

$(*)$ $$0 \to M_L \to V \otimes \mathcal{O}_X \to L \to 0.$$

Property N_p is equivalent to following vanishing condition ([2, Lemma 1.6])

$$H^1(X, \wedge^a M_L \otimes L^b) = 0 \qquad \text{for } \ p+1 \geq a \geq 0; \ b \geq 1.$$

In characteristic zero case, the wedge product is a direct summand of the tensor product. Therefore, R (or L) will have Property N_p if following holds:

$$H^1(X, \otimes^a M_L \otimes L^b) = 0 \qquad \text{for } \ p+1 \geq a \geq 0; \ b \geq 1.$$

We would like to give a geometric interpretation to this vanishing condition and show that if L is ample and if we fix p_0, there exists a number n_0 such that L^n has property N_{p_0} for every $n \geq n_0$.

We proceed by induction.

Lemma 1.1. *Let $I_{D(X)}$ denote the ideal sheaf of the diagonal embedding of X in X^2.*

1. $H^0(X, M_L \otimes L^{b+1}) = H^0(X^2, I_{D(X)} \otimes L \times L^{b+1})$.
2. $H^1(X^2, I_{D(X)} \otimes L \times L^{b+1}) = 0 \Rightarrow H^1(X, M_L \otimes L^{b+1}) = 0$.

Proof. We tensor $(*)$ by L^{b+1} and write the associated long exact sequence of cohomology:

$$0 \to H^0(M_L \otimes L^{b+1}) \to V \otimes H^0(L^{b+1}) \to H^0(L^{b+2}) \\ \to H^1(M_L \otimes L^{b+1}) \to 0.$$

Now, we note that the multiplication map is the restriction map to the diagonal. Therefore, $V \otimes H^0\left(L^{b+1}\right) \to H^0\left(L^{b+2}\right)$ will be surjective if $H^1\left(X^2, I_{D(X)} \otimes L \times L^{b+1}\right) = 0$. From this the assertions of the Lemma follows. □

The following lemma appears as Lemma 1.5 in [6].

Lemma 1.2. *Let X be a projective variety. Let X_1, X_2 be sub-varieties and $X_3 = X_1 \cap X_2$. Let L be an line bundle on X such that*

1. *The natural restriction map $H^0(X, L) \to H^0(X_2, L)$ is surjective, and,*
2. *$H^1(X, I_{X_1 \cup X_2} \otimes L) = 0$ where $I_{X_1 \cup X_2}$ denotes the ideal sheaf of the subvariety $X_1 \cup X_2$ of X*

then, the natural restriction map $H^0(X, I_{X_1} \otimes L) \to H^0(X_2, I_{X_3} \otimes L)$ is surjective.

Proof. Since $H^0(X, L) \to H^0(X_2, L)$ is onto, we have the following exact sequence.

$$0 \to H^0(X, I_{X_2} \otimes L) \to H^0(X, I_{X_3} \otimes L) \to H^0(X_2, I_{X_3} \otimes L) \to 0.$$

Now we consider the Mayor Vietoris sequence:

$$0 \to I_{X_1 \cup X_2} \to I_{X_1} \oplus I_{X_2} \to I_{X_3} \to 0.$$

Tensoring with L and writing the associated long exact sequence of cohomology, we get that

$$H^0(X, I_{X_1} \otimes L) \oplus H^0(X, I_{X_2} \otimes L) \to H^0(X, I_{X_3} \otimes L)$$

is surjective if

$$H^1(X, I_{X_1 \cup X_2} \otimes L) = 0.$$

Therefore we get that the composite map:

$$H^0(X, I_{X_2} \otimes L) \oplus H^0(X, I_{X_1} \otimes L) \to H^0(X, I_{X_3} \otimes L) \to \frac{H^0(X, I_{X_3} \otimes L)}{H^0(X, I_{X_2} \otimes L)}$$

is surjective.

Thus $H^0(X, I_{X_1} \otimes L) \to H^0(X_2, I_{X_3} \otimes L)$ is surjective. □

Lemma 1.3. *Assume that $H^1\left(X, M_L \otimes L^{b+1}\right) = 0$. Then,*

1. $H^0\left(X, M_L^{\otimes 2} \otimes L^{b+1}\right) = H^0\left(X^3, I_{D_{1,3} \cup D_{2,3}} \otimes L \times L \times L^{b+1}\right)$.
2. $H^1\left(X^3, I_{D_{1,3} \cup D_{2,3}} \otimes L \times L \times L^{b+1}\right) = 0 \Rightarrow H^1\left(X, M_L^{\otimes 2} \otimes L^{b+1}\right) = 0$.

Proof. We tensor (∗) by $M_L \otimes L^{b+1}$ and write the associated long exact sequence of cohomology:

$$0 \to H^0\left(M_L^2 \otimes L^{b+1}\right) \to V \otimes H^0\left(M_L \otimes L^{b+1}\right) \to H^0\left(M_L \otimes L^{b+2}\right) \to H^1\left(M_L^2 \otimes L^{b+1}\right) \to 0.$$

Therefore, if $V \otimes H^0\left(M_L \otimes L^{b+1}\right) \to H^0\left(M_L \otimes L^{b+2}\right)$ is surjective, then $H^1\left(M_L^2 \otimes L^{b+1}\right) = 0$. Now, by Lemma 1.1, we have:

1. $H^0\left(X, M_L \otimes L^{b+1}\right) = H^0\left(X^2, I_{D(X)} \otimes L \times L^{b+1}\right)$.
2. $H^0\left(X, M_L \otimes L^{b+2}\right) = H^0\left(X^2, I_{D(X)} \otimes L \times L^{b+2}\right)$.

Therefore, $V \otimes H^0\left(M_L \otimes L^{b+1}\right) = H^0\left(X^3, I_{D_{2,3}} \otimes L \times L \times L^{b+1}\right)$.

Also, $H^0\left(M_L \otimes L^{b+2}\right) = H^0\left(D_{1,3}, I_{D(X)} \otimes L \times L \times L^{b+2}\right)$.

Here, the notation is obvious ($D_{i,j}$ denotes the partial diagonal embedding and the line bundle is also denoted by same symbol upon restriction to subvarieties). Therefore, we need that the natural restriction map between $H^0\left(X^3, I_{D_{2,3}} \otimes L \times L \times L^{b+1}\right)$ and $H^0\left(D_{1,3}, I_{D(X)} \otimes L \times L \times L^{b+1}\right)$ should be surjective.

Now we apply the Lemma 1.2, to the following varieties.

X^3 and subvarieties $D_{1,3}$ and $D_{2,3}$ with the line bundle $L \times L \times L^{b+1}$ and get the proof of the Lemma. □

Remark 1.4. Now we will prove the general statement. Note that to use the inductive argument, we not only need the vanishing of the H^1 at the previous level, but also a "geometric description" of the H^0 of the previous level.

Inductive Hypothesis: Let $\Sigma^{(p)} = D_{1,p} \cup \cdots \cup D_{p-1,p}$ be the subvariety of union of partial diagonals in X^p. Let $I_{\Sigma^{(p)}}$ denote its ideal sheaf. Then,

1. $H^0\left(X, M_L^{\otimes p-1} \otimes L^{b+1}\right) = H^0\left(X^p, I_{\Sigma^{(p)}} \otimes L \times \cdots \times L \times L^{b+1}\right)$
2. $H^1\left(X^p, I_{\Sigma^{(p)}} \otimes L \times \cdots \times L \times L^{b+1}\right) = 0 \Rightarrow H^1\left(X, M_L^{\otimes} \otimes L^{b+1}\right) = 0$

Now we can state the general lemma as follows:

Lemma 1.5. *Assume that L has property N_{p-2}. Then,*

1. $H^0\left(X, M_L^{\otimes p} \otimes L^{b+1}\right) = H^0\left(X^{p+1}, I_{\Sigma^{(p+1)}} \otimes L \times \cdots \times L \times L^{b+1}\right)$
2. $H^1\left(X^{p+1}, I_{\Sigma^{(p+1)}} \otimes L \times \cdots \times L \times L^{b+1}\right) = 0 \Rightarrow H^1\left(X, M_L^{\otimes p} \otimes L^{b+1}\right) = 0$

Proof. Tensor (∗) by $M_L^{p-1} \otimes L^{b+1}$ and write the long exact sequence of cohomology. We get:

$$0 \to H^0\left(M_L^p \otimes L^{b+1}\right) \to V \otimes H^0\left(M_L^{p-1} \otimes L^{b+1}\right) \to H^0\left(M_L^{p-1} \otimes L^{b+2}\right) \to H^1\left(M_L^p \otimes L^{b+1}\right) \to 0$$

Therefore, if $V \otimes H^0\left(M_L^{p-1} \otimes L^{b+1}\right) \to H^0\left(M_L^{p-1} \otimes L^{b+2}\right)$ is surjective, then $H^1\left(M_L^p \otimes L^{b+1}\right) = 0$. Now, by the inductive hypothesis, we have:

1. $H^0\left(X, M_L^{p-1} \otimes L^{b+1}\right) = H^0\left(X^p, I_{\Sigma^{(p)}} \otimes L \times \cdots L \times L^{b+1}\right)$
2. $H^0\left(X, M_L^{p-1} \otimes L^{b+2}\right) = H^0\left(X^p, I_{\Sigma^{(p)}} \otimes L \times \cdots \times L \times L^{b+2}\right)$

Therefore, $V \otimes H^0\left(M_L^{p-1} \otimes L^{b+1}\right) = H^0\left(X^{p+1}, I_{\Sigma_1^{(p)}} \otimes L \times \cdots \times L \times L^{b+1}\right)$
Also, $H^0\left(M_L^{p-1} \otimes L^{b+2}\right) = H^0\left(D_{1,p+1}, I_{\sigma^{(p)}} \otimes L \times \cdots \times L \times L^{b+1}\right)$
Here, the notation is as before, and further, $\Sigma_1^{(p)}$ denotes the pull-back of $\Sigma^{(p)}$ under the projection onto the last p factors. Therefore, we need that the natural restriction map between $H^0\left(X^{p+1}, I_{\Sigma_1^{(p)}} \otimes L \times \cdots \times L \times L^{b+1}\right)$ and $H^0\left(D_{1,p+1}, I_{\sigma^{(p)}} \otimes L \times \cdots \times L \times L^{b+1}\right)$ should be surjective.
Now we apply the Lemma 1.2, to the following varieties.
X^{p+1} and subvarieties $D_{1,p+1}$ and $\Sigma_1^{(p)}$ with the line bundle $L \times \cdots \times L \times L^{b+1}$ and get the proof of the Lemma. □

Theorem 1.6. *Let X be a projective variety and let L be an ample line bundle on X. Then for every positive integer p_0, there exists a number n_0 such that L^n has property N_{p_0} for every $n \geq n_0$.*

Proof. We take sufficiently high power of the line bundle L so that the vanishing conditions required in all the lemmas regarding Property N_p are satisfied for every $p \leq p_0$. □

References

[1] A. Bertram, L. Ein and R. Lazarsfeld, *Vanishing Theorems, a theorem of Severi, and the equations defining projective varieties*, J. Amer. Math. Soc., **4** (1991), 587-602.

[2] L. Ein and R. Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), 51-67.

[3] M. Green, *Koszul cohomology and geometry of projective varieties*, I, J. Diff. Geom., **19** (1984), 125-171.

[4] ———, *Koszul cohomology and geometry of projective varieties*, II, J. Diff. Geom., **20** (1984), 279-289.

[5] S.P. Inamdar, *A note on Frobenius splitting of Schubert varieties and linear syzygies*, Amer. Jour. Math., **116** (1994), 1587-1590.

[6] S.P. Inamdar and V.B. Mehta, *Frobenius splitting of Schubert varieties and linear syzygies*, Amer. Jour. Math., **116** (1994), 1569-1586.

[7] L. Manivel, *On the syzygies of flag manifolds*, Preprint, August 1994.

[8] M. Nori, *Algebraic cycles and Hodge theoretic connectivity*, Inv. Math., **111** (1993), 349-373.

Received February 16, 1995 and revised October 31, 1995.

School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005
INDIA
E-mail address: inamdar@tifrvax.tifr.res.in