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OSCILLATORY INTEGRALS AND SCHRÖDINGER MAXIMAL OPERATORS

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In this paper we consider smooth analogues of operators studied in connection with pointwise convergence of the solution of the free Schrödinger equation to the given initial data. Such operators are interesting examples of oscillatory integral operators with degenerate phase functions, and we obtain sharp $L^2 \to L^2$ bounds.

0. Introduction.

We begin this paper by giving motivation for the objects we will study, placing them in their proper context.

Consider the initial value problem for the Schrödinger equation with no potential,

$$\begin{cases} i\partial_t u(x,t) + \Delta_x u(x,t) = 0 & (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x,0) = f(x) \in L^2(\mathbb{R}^n). \end{cases} \tag{1}$$

Then

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{f}(\xi)\, d\xi = (e^{it|\cdot|^2} \widehat{f}(\cdot))\check{}\,(x) \tag{2}$$

defines a (weak) solution of (1) such that $\lim_{t\to 0} u(x,t) = f(x)$ in the L^2 sense. When the integral in (2) is absolutely convergent the limit is a pointwise limit. However, if f is an arbitrary L^2 function the integral in (2) may not be absolutely convergent, and we must take the right hand side of (2) as the definition of $u(x,t)$. It is not self-evident that u converges pointwise to the initial data in this case, and in fact it sometimes does not. The question of what extra smoothness conditions on f will guarantee the existence pointwise a.e. of $\lim_{t\to 0} u(x,t)$ arises.

For a given $s \geq 0$ let $H^s(\mathbb{R}^n)$ denote th e L^2-Sobolev space,

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) : \|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi \right)^{1/2} < \infty \right\}.$$

In the context of L^2-Sobolev spaces the question of pointwise convergence to the initial data is completely understood when $n = 1$. It was shown by Carleson [**C**] that $\lim_{t\to 0} u(x,t) = f(x)$ whenever $f \in H^s(\mathbb{R})$, $s \geq 1/4$. Moreover Dahlberg and Kenig [**DK**] demonstrated that for all $s < 1/4$ there are functions $f \in H^s(\mathbb{R})$ such that $\overline{\lim}_{t\to 0} u(x,t) = \infty$ a.e..

The higher dimensional cases, $n \geq 2$, are not completely understood. For these cases Vega [**V**] and Sjölin [**Sj**] independently proved that the pointwise limit exists for all $f \in H^s(\mathbb{R}^n)$ provided $s > 1/2$, while there are counterexamples just as in the 1-dimensional case when $s < 1/4$. But the question of what happens when $1/4 \leq s \leq 1/2$ is in general unanswered. However, in [**B**], Bourgain shows that there is an $\epsilon > 0$ such that $f \in H^{1/2-\epsilon}(\mathbb{R}^2)$ guarantees pointwise convergence to the initial data. The value of this ϵ, although in principle calculable, is not given (although $\epsilon \ll 1/4$). The point here is not what the value of ϵ is, but that there is some improvement of the above results when $n = 2$.

The study of the pointwise behavior of $u(x,t)$ as $t \to 0$ involves the study of the corresponding maximal operator, the Schrödinger maximal operator,

$$u^*(x) = \sup_{|t|\leq 1} |u(x,t)|$$

with regard to its mapping properties—i.e., finding weak type or strong type inequalities for u^*.

The idea in [**B**] is to replace the nonlinear operator u^* by a family of linear operators. For each measurable function t(x), defined say on $\mathbb{D}^n$, the unit disk in $\mathbb{R}^n$, with the property that $|t(x)| \leq 1$, one considers the linear operator

$$f \longmapsto \int e^{i(x\cdot\xi + t(x)|\xi|^2)} \widehat{f}(\xi)\, d\xi = u(x, t(x)),$$

and shows that for some constant C, *which is independent of all such* t,

$$\|u(\,\cdot\,, t(\,\cdot\,))\|_{L^2(\mathbb{D}^n)} \leq C\, \|f\|_{H^s}.$$

In practice one looks at integral operators of the form

$$R_k f(x) = \int e^{i(x\cdot y + t(x)|y|^2)} \theta_k(y) f(y)\, dy \qquad k = 1, 2, \dots,$$

where $\{\theta_k\}_0^\infty$ is a partition of unity such that $\mathrm{supp}(\theta_k) \subset \{\, y : 2^{k-1} \leq |y| \leq 2^{k+1} \,\}$ when $k \geq 1$.

Proposition .[1] *Suppose there is a C and $s_0 \geq 0$ such that*

$$\|R_k f\|_{L^2(\mathbb{D}^n)} \leq C 2^{s_0 k}. \tag{3}$$

Then for any $s > s_0$ there is a C_s, depending on C and s, such that

$$\|u(\,\cdot\,, t(\,\cdot\,))\|_{L^2(\mathbb{D}^n)} \leq C_s \|f\|_{H^s}.$$

Thus we have reduced to the case of finding L^2 to L^2 estimates on a family of linear operators. This is a common task in harmonic analysis, and this particular one is aided by the similarites between the R_k's and a general class of operators, $\mathfrak{T}_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, of the form

$$\mathfrak{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(x,y) f(y)\, dy.$$

Such operators, called oscillatory integral operators, are usually studied when the *amplitude* $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and the *phase function* $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and one is concerned with the behavior of $\|\mathfrak{T}_\lambda\|$ as $\lambda \to \infty$.

There are two major differences, though, between $\mathfrak{T}_\lambda$ and R_k that must be considered. Firstly, since the phase function in R_k is not homogeneous, we cannot do a change of variables $y \to 2^k y$ to get into the form of $\mathfrak{T}_\lambda$. However, for the purpose of obtaining an ϵ-improvement in pointwise convergence results when $n = 2$, it is pointed out in [**B**] that it is sufficient to consider operators of the form

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t(x) - \bar{t}(y)} \right) a(x,y) f(y)\, dy, \tag{4}$$

where $a \in C_0^\infty$ and t and $\bar{t}$ are measurable functions such that $1 \leq |t(x) - \bar{t}(y)| \leq 2$, and show that there exists an $\epsilon > 0$ such that

$$\|T_\lambda f\|_2 \leq C\lambda^{-\epsilon} \|f\|_2, \qquad C \text{ independent of } t \text{ and } \bar{t}. \tag{5}$$

Such a result then implies an inequality as in (3) with $s_0 < 1/2$.

The second and more important difference is that the phase function in R_k (and T_λ) is not smooth. The main results about $\mathfrak{T}_\lambda$ in [**GS**], [**PS**] and [**H1**] involve only those cases when ϕ is smooth. Nevertheless (5) is plausible due to the following theorem, whose proof is based on ideas in [**B**].

Theorem 1. *If T_λ is as above then*

$$\|T_\lambda f\|_{L^2(\mathbb{R}^n)} \leq C\lambda^{-\frac{n-2}{4}} \|f\|_{L^2(\mathbb{R}^n)}, \tag{6}$$

[1] See [**K**] or [**B**].

where C is uniform over all measurable t and $\bar{t}$ such that $1 \leq |t(x) - \bar{t}(y)| \leq 2$.

Of course when $n = 2$ the estimate in (6) is trivial. The heart of [**B**] lies in dealing with the non-smoothness in the phase function of T_λ to get the estimate in (5), which is an ϵ improvement of (6).

In this paper we discuss operators of the form T_λ when the functions t and $\bar{t}$ are assumed to be smooth. We begin by considering a special case of T_λ when $\bar{t} \equiv 0$.

Theorem 2. *Let T_λ be as in (4) where t is a smooth function such that $t \neq 0$, and $\bar{t} \equiv 0$.*

(i) *If* $\dfrac{\nabla t(x)}{t(x)} \cdot (x - y) - 1 \neq 0$ *on* $\operatorname{supp}(a)$, *then* $\|T_\lambda f\|_2 \lesssim \lambda^{-n/2} \|f\|_2$. *Moreover the exponent of λ is sharp.*

(ii) *In general,* $\|T_\lambda f\|_2 \lesssim \lambda^{-n/2+1/4} \|f\|_2$.

(iii) *For a given amplitude function $a \not\equiv 0$, there are functions $t \in C^\infty$ such that the exponent of λ in* II *is sharp.*

This result follows from a more general theorem in [**GS**].[2] The proof given in this paper uses a similar approach to that in [**GS**], but our execution is different, and this difference allows us to consider operators, which are not oscillatory integral operators, like R_k, in another paper. And although Theorem 2 is a special case, an analogous theorem in its statement and proof is given, which is then used to prove Theorem 3—the main result of this paper.

Theorem 3. *Let T_λ be as in (4) where t and $\bar{t}$ are smooth functions such that $0 < |t(x) - \bar{t}(y)|$. Then*

(i) $\|T_\lambda f\|_2 \lesssim \lambda^{-n/2+1/2}$.

(ii) *For a given amplitude function $a \not\equiv 0$, there are t and $\bar{t}$ such that the bound in* I *is sharp.*

1. Preliminaries.

Before proving our main theorems, we catalogue a number of important lemmas used in their proofs. These lemmas are variations of standard material, so their proofs are omitted and can be found in [**K**].

The following notation is used throughout.

x, y, z and ξ will denote variables in $\mathbb{R}^n$.

$x \cdot y$ is the inner product in $\mathbb{R}^n$: $x \cdot y = \sum_1^n x_i y_i$.

[2] See also [**K**] for details.

M^t denotes the transpose of the matrix M.

$\mathrm{H}\, f$ will denote the Hessian of f.

$\widehat{f}(\xi) = \int e^{-ix\cdot\xi} a(x)\, dx$ is the Fourier transform of f.

$\check{f}(\xi) = (2\pi)^{-n} \int e^{ix\cdot\xi} f(x)\, dx$ is the inverse Fourier transform of f.

∂_j is the differential operator $\partial/\partial x_j$.

$\mathfrak{S}(\mathbb{R}^n)$ is the Schwartz class of functions on $\mathbb{R}^n$.

$B_r(p) = \{x \in \mathbb{R}^n : |x - p| < r\}$.

$\mathbb{D}^n$ denotes the unit ball in $\mathbb{R}^n$.

If $E \subset \mathbb{R}^n$ is measurable, then $|E|$ denotes the Lebesgue measure of E.

If $a(x, y)$ is a function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then denote by $\mathrm{supp}_y(a)$ the projection onto the y-coordinates of the support of a. Let $\nabla_y a(x, y)$ denote the gradient of a as a function of y with x held fixed. Similarly $\Delta_y a(x, y) = \sum \partial^2/\partial y_j^2\, a(x, y)$.

The expression $x \lesssim y$ will mean that there is a constant C, which does not depend on quantities that are otherwise to be kept track of, such that $x \leq C\, y$. Dependence on such quantities will be explicitly noted.

Given x, $y \in \mathbb{R}^n$, write $x = (x', x_n)$ and $y = (y', y_n)$, where x' and y' are in $\mathbb{R}^{n-1}$. Let $K(x, y)$ be a given bounded measurable function, which for our purposes will be assumed to have compact support, and define an operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$T f(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy,$$

whose adjoint is

$$T^* f(y) = \int_{\mathbb{R}^n} \overline{K(z, y)} f(z)\, dz.$$

If we fix x_n and y_n and let $K_{x_n y_n}(x', y') = K(x', x_n, y', y_n)$, then we get a family of *frozen operators*, $T_{x_n y_n} : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ defined by

$$T_{x_n y_n} f(x') = \int_{\mathbb{R}^{n-1}} K_{x_n y_n}(x', y') f(y')\, dy'.$$

Lemma 1.1. *Suppose there exists a measurable function $\eta(x_n, y_n)$ such that for all $g \in L^p(\mathbb{R}^{n-1})$ and for all $h \in L^p(\mathbb{R})$*

$$\|T_{x_n y_n} g\|_{L^q(\mathbb{R}^{n-1})} \leq \eta(x_n, y_n)\, \|g\|_{L^p(\mathbb{R}^{n-1})}\, ,$$

$$\left\| \int \eta(x_n, y_n) h(y_n)\, dy_n \right\|_{L^q(\mathbb{R})} \leq C\, \|h\|_{L^p(\mathbb{R})}.$$

Then for all $f \in L^p(\mathbb{R}^n)$

$$\|Tf\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} .$$

Now consider

$$TT^* f(x) = \int f(z) \left(\int K(x,y) \overline{K(z,y)}\, dy \right) dz.$$

If $T_{x_n} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{n-1})$ is the operator

$$T_{x_n} f(x') = \int_{\mathbb{R}^n} K(x', x_n, y) f(y)\, dy,$$

and $T^*_{z_n} : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^n)$

$$T^*_{z_n} f(y) = \int_{\mathbb{R}^{n-1}} \overline{K(z', z_n, y)} f(z')\, dz'$$

is its adjoint, then clearly

$$(TT^*)_{x_n z_n} = T_{x_n} T^*_{z_n},$$

$$\|T_{x_n} (T_{z_n})^* f\|_{L^2(\mathbb{R}^{n-1})} \leq \|T_{x_n}\| \, \|T^*_{z_n}\| \, \|f\|_2. \tag{1.1}$$

Lemma 1.2.

$$\|T^*_{z_n} f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^{n-1})} \left(\int_{-\infty}^{\infty} \|(T^*)_{z_n y_n}\|^2 \, dy_n \right)^{1/2} .$$

$$\|T_{x_n} f\|_{L^2(\mathbb{R}^{n-1})} \leq \|f\|_{L^2(\mathbb{R}^n)} \left(\int_{-\infty}^{\infty} \|T_{x_n y_n}\|^2 \, dy_n \right)^{1/2} .$$

The method of stationary phase is of crucial importance to our endeavors.

Theorem 1.3. *Suppose that* $a \in \mathfrak{S}(\mathbb{R}^n)$. *Then for any positive integer* k,

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{i\lambda |y|^2} a(y)\, dy \\ &\quad = \left(\frac{i\lambda}{\pi} \right)^{-n/2} \left(\sum_{j=0}^{k-1} (4i\lambda)^{-j} \Delta^j a(0)/j! + \int_{\mathbb{R}^n} r_k(i|\xi|^2/4\lambda) \check{a}(\xi)\, d\xi \right), \end{aligned} \tag{1.2}$$

where $r_k(x)$ *is the remainder of the* k*-th degree Taylor polynomial of* e^x.

This well-known result is not usually expressed in this form. We find it convenient to include a form of the remainder term in the asymptotic expansion of the left hand side of (1.2) in powers of λ. See [**H1**], [**St**].

Remark 1.4. Note that $|r_k(x)| \le |x|^k/k!$ whenever $\operatorname{Re} x \le 0$. Then an application of the Cauchy-Schwartz inequality shows that for any integer $s > n/2$,

$$\left| \int_{\mathbb{R}^n} r_k(i|\xi|^2/4\lambda)\check{a}(\xi)\,d\xi \right| \lesssim \lambda^{-k} \sum_{|\alpha|\le 2k+s} \|\mathrm{D}^\alpha a\|_2 . \tag{1.3}$$

A corollary of Theorem 1.3 is needed to prove Theorem 3, which is a variable parameter version of Theorem 1.3.

Corollary 1.5. *Suppose that a is contained in a bounded subset X of $\mathfrak{S}(\mathbb{R}^n \times \mathbb{R}^m)$. Then for any multi-index α and any $\lambda \ge 1$ there is a constant $C = C(\alpha, X)$ such that*

$$\sup_z \lambda^{n/2} \left| \mathrm{D}_z^\alpha \int_{\mathbb{R}^n} e^{i\lambda|y|^2} a(y,z)\,dy \right| \le C.$$

We continue with a couple of lemmas about about $n \times n$ matrices.

Lemma 1.6. *Let M be an $n\times n$ matrix with entries $M_{ij} = \delta_i^j + a_i b_j$, where δ_i^j is the Kronecker delta and a_i, $b_j \in \mathbb{R}$. Then*
(1) $\det M = 1 + \sum_{i=1}^n a_i b_i$,
(2) $\operatorname{rank}(M) \ge n-1$.

Lemma 1.7. *Let M be an $n \times n$ matrix of the form $M_{ij} = \delta_{i+1}^j + a_i b_j$. Then $\det(M) = a_n b_1$.*

Finally we finish this section with a standard theorem about oscillatory integral operators

$$\mathfrak{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(x,y) f(y)\,dy,$$

where the *amplitude* $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and the real-valued *phase function* $\phi \in C^\infty(X)$ with X a neighborhood of $\operatorname{supp}(a)$. Just how rapidly $\|\mathfrak{T}_\lambda\|$ decays depends on the *mixed Hessian* of ϕ, the $n\times n$ matrix $H_\phi(x,y)$ defined by

$$(H_\phi(x,y))_{i,j} = \frac{\partial^2\phi}{\partial x_i \partial y_j}(x,y).$$

The following is a sharpening of the result in [**H2**].

Theorem 1.8. *Suppose that H_ϕ is non-singular on* $\operatorname{supp}(a)$ *and that the following quantities are uniformly bounded on* $\operatorname{supp}(a)$*:*
(i) $\|H_\phi^{-1}(x,y)\|$

(ii) $\|\nabla_y \mathrm{D}_x^\alpha \phi\|_{L^\infty(X)}$ *for all* α *with* $|\alpha| = 2$

(iii) $\left\|\nabla_x \mathrm{D}_y^\alpha \phi\right\|_{L^\infty(X)}$ *for all* α *with* $|\alpha| \leq n+2$.

Then if $M = \max\{1, |\operatorname{supp}_x(a)|\}$ *and*

(1.4)
$$M_a = \|a\|_\infty \left(M\,|\operatorname{supp}_y(a)| \left\{ \sum_{|\alpha|\leq n+1} \sup_{xyz} |D_y^\alpha a(x,y)\overline{a(z,y)}| \right\}^{\frac{n}{n+1}} \right)^{1/2},$$

then

$$\|\mathfrak{T}_\lambda f\|_2 \leq C M_a \lambda^{-n/2} \|f\|_2\,, \tag{1.5}$$

where C *is bounded.*

Remark 1.9. When bounding $\|H_\phi^{-1}\|$ uniformly from above it is convenient to use the classical theorem for the inverse of a matrix, $H_\phi^{-1} = (\det H_\phi)^{-1} \operatorname{adj} H_\phi$. Then we need only bound $\det H_\phi$ uniformly from below and the entries of H_ϕ uniformly from above.

2. Main Theorem: A Special Case.

We let $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a fixed given function, and let $t \in C^\infty(\mathbb{R}^n)$ be such that $0 \neq t(x)$ on $\operatorname{supp}_x(a)$. We shall consider oscillatory integral operators T_λ with phase function $\phi(x,y) = \dfrac{|x-y|^2}{t(x)}$ and catalogue the behavior of $\|T_\lambda\|$ below.

Remark 2.1. Since $\operatorname{supp}_x(a)$ is compact and $0 \neq t(x)$ $\forall x \in \operatorname{supp}_x(a)$, there is a constant $c > 0$ such that $c \leq |t(x)|$ $\forall x \in \operatorname{supp}_x(a)$.

The best possible case is when H_ϕ is non-degenerate, and this situation is characterized in Theorem 2.2 below.

Theorem 2.2. *Suppose* $1 - \dfrac{\nabla t(x)}{t(x)} \cdot (x-y) \neq 0$ *on* $\operatorname{supp}(a)$*. Then for all* f $\|T_\lambda f\|_2 \leq C\lambda^{-n/2}\|f\|_2$*, and the exponent of* λ *is sharp. Moreover if we fix* $c_1 > 0$*, then* $\exists\, c_2$ *(which depends on* c_1 *and* $\operatorname{supp}(a)$*) such that the constant* C *above is uniform over the set* [3]

$$\begin{aligned}\Sigma &= \Sigma(c_1, c_2) \\ &= \{ t \in C^\infty : c_1 \leq |t(x)|,\ |\nabla t(x)| \leq c_2,\ \textit{and}\ \|\mathrm{H}\, t(x)\| \leq c_2\, \forall x \in \operatorname{supp}(a)\}.\end{aligned}$$

[3] Here $\|\mathrm{H}\, t\|$ denotes the matrix norm of the $n \times n$ matrix H.

Proof. It is easy to calculate that

$$\frac{\partial^2 \phi}{\partial x_i \partial y_j}(x,y) = \frac{-2}{t(x)}\left(\delta_{ij} - (x_j - y_j)\frac{\partial_i t(x)}{t(x)}\right). \tag{2.1}$$

Then we see that H_ϕ is of the form described in Theorem 1.8, and we conclude that

$$\det H_\phi(x,y) = \left(\frac{-2}{t(x)}\right)^n \left(1 - \frac{\nabla t(x)}{t(x)} \cdot (x-y)\right). \tag{2.2}$$

The first part of the theorem follows now form Theorem 1.8; the second part will follow after a careful examination of the hypotheses in (i), (ii) and (iii) of Theorem 1.8 as they relate to ϕ.

First note that if c_2 is chosen to be small enough, and $|\nabla t| \leq c_2$, then

$$\left|\frac{\nabla t(x)}{t(x)} \cdot (x-y)\right| \leq c_1^{-1} |\nabla t(x)|\, 2\,\mathrm{diam}(\mathrm{supp}(a)) \leq 1/2.$$

So if we assume that $t \in \Sigma$ for this choice of c_2 (and c_1), then $|\det H_\phi| \geq c_1^{-n} 2^{n-1}$, and, given (2.1) and Remark1.9, $\|H_\phi^{-1}\|$ is uniformly bounded on $\mathrm{supp}(a)$ and over all $t \in \Sigma$. We claim also that $\left\|\nabla_x \mathrm{D}_y^\alpha\, \phi\right\|_{L^\infty}$ is uniformly bounded for all α with $|\alpha| \leq n+2$. In fact $|\alpha| \leq 2$ will suffice as all higher order derivatives vanish. The claim is evident from the form of ϕ as

$$\frac{\partial}{\partial x_j}\mathrm{D}_y^\alpha\, \phi(x,y) = -\frac{\partial_j t(x)}{(t(x))^2} D_y^\alpha(|x-y|^2),$$

and $t \in \Sigma$. Finally we check that $\|\nabla_y \mathrm{D}_x^\alpha\, \phi\|_{L^\infty}$ is also uniformly bounded when $\alpha = 2$ since

$$\frac{\partial^3 \phi(x,y)}{\partial x_i \partial x_j \partial y_k} = \frac{-2}{t(x)^2}\left(\partial_j t(x)\delta_i^k - \delta_j^k \partial_i t(x) - (x_k - y_k) + (x_k - y_k)\frac{\partial_i t(x)\partial_j t(x)}{t(x)}\right).$$

The hypotheses of Theorem 1.8 being satisfied, the theorem is proven. □

The more interesting case is when $\det H_\phi = 0$. Lemma 1.6 readily gives the estimate $\|T_\lambda f\| \lesssim \lambda^{-n/2+1/2}\, \|f\|_2$ in this case. But before proving a stronger estimate, a few remarks are in order.

Remark 2.3. If we cover $\mathrm{supp}_x(a)$ with balls of radius δ and take a partition of unity subordinate to these balls, we may assume that $\mathrm{diam}(\mathrm{supp}_x(a)) < \delta$ without any loss of generality if we provide that δ does not depend on λ. Then δ is chosen to be as small as necessary to assist in technical matters.

Remark 2.4. (2.2) says that we may assume (after a partition of unity) that on the support of a, $1 \lesssim |\nabla t(x)|$. Otherwise H_ϕ is non-singular, and we may again appeal to Theorem 1.8.

Remark 2.5. If A is a rotation then the change of variables $(x, y) \to (Ax, Ay)$ "preserves" ϕ in the sense that $\phi(Ax, Ay) = \frac{|x-y|^2}{t \circ A}$ is of the same form as $\phi(x, y)$, the form of phase function presently under consideration, with t replaced by $t \circ A$. Similarly if A is a translation, ϕ is again "preserved" under the operation $\phi(x, y) \to \phi(Ax, Ay)$. Since such transformations are measure preserving, the norm of the oscillatory integral operator with phase function $\phi(x, y)$ and amplitude $a(x, y)$ is the same as the one with phase $\phi(Ax, Ay)$ and amplitude $a(Ax, Ay)$.

Theorem 2.6. *Let ϕ be as above. Then in general*

$$\|T_\lambda f\|_2 \leq C\lambda^{-n/2+1/4} \|f\|_2 .$$

Proof. Let $x \in \operatorname{supp}_x(a)$ be given. By Remark 2.4 we may find a rotation A_x such that $\nabla t(x)A_x$ is parallel to the n-th unit vector e_n in $\mathbb{R}^n$. Let $z = A_x^{-1}x$ and $t_{A_x} = t \circ A_x$. Then $\nabla t_{A_x}(z)$ is parallel to e_n, and so we may assume without loss of generality that $\nabla t_{A_x}(z) = e_n$. Furthermore it is clear that $|\nabla t_{A_x}| \gtrsim 1$ on $\operatorname{supp}(a)$. Then there is a neighborhood U_x of z, a neighborhood V_x of x and a diffeomorphism $\rho_x : V_x \to U_x$ such that $t_{A_x} \circ \rho_x(w) = w_n$ for all $w \in V_x$. Moreover $\mathrm{D}\rho_x(z) = \mathrm{I}$, and we may assume that $\operatorname{diam}(U_x)$ and $\operatorname{diam}(V_x)$ are as small as necessary. Let $\widetilde{U}_x = A(U_x)$, and take a finite subcover $\{\widetilde{U}_{x_i}\}_{i=1}^m$ of $\operatorname{supp}_x(a)$. By Remark 2.3 we may assume that $\operatorname{supp}_x(a) \subset \widetilde{U}_{x_1}$ for example. Since A_{x_1} is a rotation, by Remark 2.5 we may assume therefore that there is a ball $B_\delta(x_0)$ and a diffeomorphism $\rho\colon B_\delta(x_0) \to \operatorname{supp}_x(a)$ such that $t \circ \rho(x) = x_n$, and $\mathrm{D}\rho(x_0) = \mathrm{I}$.

Given this, it suffices after a change of variables to consider the operator

$$\widetilde{T}_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda\left(\frac{|\rho(x)-y|^2}{x_n}\right)} a(\rho(x), y) f(y)\, dy.$$

If $S_\lambda = \widetilde{T}_\lambda \widetilde{T}_\lambda^*$, then S_λ is an integral operator with kernel

$$K_\rho(x, z) = \int_{\mathbb{R}^n} \exp\left(i\lambda\left\{\frac{|\rho(x)-y|^2}{x_n} - \frac{|\rho(z)-y|^2}{z_n}\right\}\right) a(\rho(x), y)\, \overline{a(\rho(z), y)}\, dy.$$

We must show that

$$\|S_\lambda f\|_2 \lesssim \lambda^{-n+1/2} \|f\|_2 . \tag{2.3}$$

Let $\psi \in C_0^\infty(B_1(0))$ be such that $\psi \equiv 1$ on $B_{1/2}(0)$, and let $\widetilde{\psi} = 1 - \psi$. Then

$$\begin{aligned} S_\lambda f(x) &= \int_{\mathbb{R}^n} f(z) K_\rho(x,z) \psi\left(\frac{x_n - z_n}{\epsilon}\right) dz \\ &\quad + \int_{\mathbb{R}^n} f(z) K_\rho(x,z) \widetilde{\psi}\left(\frac{x_n - z_n}{\epsilon}\right) dz \\ &= S_\lambda^1 f(x) + S_\lambda^2 f(x), \end{aligned} \tag{2.4}$$

where ϵ is to be chosen.

First consider S_λ^1 and its corresponding frozen operators

$$\begin{aligned} (S_\lambda^1)_{x_n z_n} f(x') &= \psi\left(\frac{x_n - z_n}{\epsilon}\right) \int_{\mathbb{R}^{n-1}} f(z') (K_\rho)_{x_n z_n}(x', z')\, dz' \\ &= \psi\left(\frac{x_n - z_n}{\epsilon}\right) \left(\widetilde{T}_\lambda \widetilde{T}_\lambda^*\right)_{x_n z_n} f(x'). \end{aligned} \tag{2.5}$$

We consider $\left(\widetilde{T}_\lambda\right)_{x_n y_n}$ for fixed x_n and y_n. The $(n-1) \times (n-1)$ matrix

$$\left(\frac{\partial^2}{\partial x_i \partial y_j} |\rho(x) - y|^2\right)_{i,j=1}^{n-1} = \left(-2\mathrm{D}_i \rho_j(x)\right)_{i,j=1}^{n-1}$$

is the mixed Hessian for $\left(\widetilde{T}_\lambda\right)_{x_n y_n}$. As noted above, when $x = x_0$ this is -2 times the $(n-1) \times (n-1)$ identity matrix. So in a small neighborhood of x_0 the determinant of the above matrix does not vanish (see Remark 2.3). Hence

$$\left\| \left(\widetilde{T}_\lambda\right)_{x_n y_n} \right\| \lesssim \lambda^{-(n-1)/2} \chi(x_n, y_n) \|f\|_2 \,,$$

where χ is compactly supported. Then this along with (1.1) and Lemma 1.2 implies that

$$\left\|(S_\lambda^1)_{x_n y_n} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{-n+1} \chi'(x_n, z_n)\, \psi\left(\frac{x_n - z_n}{\epsilon}\right) \|f\|_{L^2(\mathbb{R}^{n-1})} \,,$$

where χ' is also compactly supported. By Schur's Lemma and Lemma 1.1 we have

$$\left\|S_\lambda^1 f\right\|_2 \lesssim \lambda^{-n+1} \epsilon\, \|f\|_2 \,. \tag{2.6}$$

Now consider S_λ^2 and its corresponding frozen operators $(S_\lambda^2)_{x_n z_n}$ for fixed x_n and z_n. Note that

$$\begin{aligned} &\frac{|\rho(x) - y|^2}{x_n} - \frac{|\rho(z) - y|^2}{z_n} \\ &\quad = \left(\frac{1}{x_n} - \frac{1}{z_n}\right) |y - F(x,z)|^2 - \frac{|\rho(x', x_n) - \rho(z', z_n)|^2}{x_n - z_n}, \end{aligned} \tag{2.7}$$

where

$$F(x,z)=\frac{z_n\rho(x)-x_n\rho(z)}{x_n-z_n}. \tag{2.8}$$

Let

$$\begin{gathered} A(x,z,y)=a(\rho(x),y)\,\overline{a(\rho(z),y)},\\ \mu=\lambda\left(\frac{1}{x_n}-\frac{1}{z_n}\right). \end{gathered} \tag{2.9}$$

Then

$$\begin{aligned} K_\rho(x',z')_{x_nz_n}=\widetilde{\psi}\left(\frac{x_n-z_n}{\epsilon}\right)\exp\left(i\frac{\lambda}{z_n-x_n}|\rho(x)-\rho(z)|^2\right)\\ \cdot\int_{\mathbb{R}^n}e^{i\mu|y|^2}A(x,z,y+F(x,z))\,dy \end{aligned} \tag{2.10}$$

is the kernel of $(S_\lambda^2)_{x_nz_n}$. By Theorem 1.3, for fixed N to be chosen,

$$\begin{aligned} &\int_{\mathbb{R}^n}e^{i\mu|y|^2}A(x,z,y+F(x,z))\,dy\\ &=\left(\frac{i\mu}{\pi}\right)^{-n/2}\left(\sum_{j=0}^{N-1}(4i\mu)^{-j}\Delta_y^jA(x,z,F(x,z))/j!\right.\\ &\qquad\left.+\int_{\mathbb{R}^n}r_N(i|\xi|^2/4\mu)e^{-i\xi\cdot F(x,z)}\check{A}(x,z,\xi)\,d\xi\right), \end{aligned} \tag{2.11}$$

where $\check{A}$ denotes the inverse Fourier transform in the last variable. In view of (2.10) and (2.11), $(S_\lambda^2)_{x_nz_n}$ is a sum of oscillatory integral operators

$$\left(\frac{i}{\pi}\right)^{-n/2}\widetilde{\psi}\left(\frac{x_n-z_n}{\epsilon}\right)\sum_{j=0}^{N}\frac{\mu^{-n/2-j}}{(4i)j!}R_{\lambda'}^j,\qquad \lambda'=\frac{\lambda}{x_n-z_n}, \tag{2.12}$$

where

$$\begin{aligned} R_{\lambda'}^jf(x')&=\int_{\mathbb{R}^n}e^{i\lambda'|\rho(x)-\rho(z)|^2}(\Delta_y^jA)(x,z,F(x,z))f(z')\,dz'\\ &\qquad\qquad j=1,\dots,N-1\\ R_{\lambda'}^Nf(x')&=\int_{\mathbb{R}^n}e^{i\lambda'|\rho(x)-\rho(z)|^2}\mu^N\\ &\qquad\cdot\left(\int_{\mathbb{R}^n}r_N(i|\xi|^2/4\mu)e^{-i\xi\cdot F(x,z)}\check{A}(x,z,\xi)\,d\xi\right)f(z')\,dz'. \end{aligned} \tag{2.13}$$

Each $R^j_{\lambda'}$ has phase function of the form $|\rho(x', x_n) - \rho(z', z_n)|^2$ for fixed x_n and z_n. Look at the mixed Hessian:

$$\left(\frac{\partial^2}{\partial x_i \partial z_j} |\rho(x', x_n) - \rho(z', z_n)|^2\right)_{i,j=1}^{n-1} = \left(-2\sum_k \partial_i \rho_k(x) \partial_j \rho_k(z)\right)_{i,j=1}^{n-1}.$$

Since $\mathrm{D}\rho(x_0) = \mathrm{I}$,

$$\det\left(\frac{\partial^2}{\partial x_i \partial z_j} |\rho(x', x_n) - \rho(z', z_n)|^2\right)_{i,j=1}^{n-1} = (-2)^{n-1} \qquad \text{when } x = z = x_0.$$

So we may assume that this mixed Hessian is non-degenerate (see Remark 2.3). Now for $j = 1, \dots, N-1$, $R^j_{\lambda'}$ has amplitude $\Delta^j_y A(x, z, F(x, z))$. Since

$$|\,\mathrm{D}^\alpha_{z'} F(x, z)| \lesssim |x_n - z_n|^{-1}, \tag{2.14}$$

we see that

$$\sup_{x' z'} \left|\mathrm{D}^\alpha_{z'} \left(\Delta^j_y A(x, z, F(x, z))\, \overline{\Delta^j_y A}(w, z, F(w, z))\right)\right| \lesssim |x_n - z_n|^{-|\alpha|}. \tag{2.15}$$

To apply Theorem 1.8 we must calculate, for fixed x_n and z_n, the volume of $\mathrm{supp}_{z'}(\Delta^j_y A(x, z, F(x, z)))$. Note that by the properties of $\mathrm{supp}(A)$ we must have that $|F(x, z)| \lesssim 1$ — i.e., $|x_n \rho(z) - z_n \rho(x)| \lesssim |x_n - z_n|$. This says that for fixed x, $\rho(z)$ is in the ball of radius $\dfrac{1}{x_n}|x_n - z_n|$ centered at $\dfrac{z_n}{x_n}\rho(x)$. Since $|x_n|$ is bounded from below (see Remark 2.1) and ρ is a diffeomorphism, z lies in a set of diameter $\sim |x_n - z_n|$. So

$$|\,\mathrm{supp}_{z'}\, \Delta^j_y A(x, z, F(x, z))| \sim |x_n - z_n|^{n-1} \quad \text{for } \; j = 0, \dots, N-1. \tag{2.16}$$

Putting (2.15) and (2.16) into (1.4) $\left(\text{recalling that } \lambda' = \dfrac{\lambda}{(x_n - z_n)}\right)$ gives that for $j = 1, \dots, N-1$,

$$\left\|R^j_{\lambda'} f\right\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{\frac{-(n-1)}{2}} |x_n - z_n|^{\frac{(n-1)}{2}} \chi(x_n, z_n)\, \|f\|_{L^2(\mathbb{R}^{n-1})}, \tag{2.17}$$

where χ has compact support.

Coming to $R^N_{\lambda'}$, it has amplitude, call it $A_N(x, z)$, equal to

$$\mu^N \int_{\mathbb{R}^n} r_N(i|\xi|^2/4\mu) e^{-i\xi \cdot F(x,z)} \check{A}(x, z, \xi)\, d\xi.$$

This means that in view of the Remark 1.4 and (2.14)

$$\sup_{z'}\left|\mathrm{D}^{\alpha}_{z'}A_N(x,z)\,\overline{A_N(w,z)}\right| \lesssim |x_n - z_n|^{-|\alpha|}. \tag{2.18}$$

Then (2.18) implies that

$$\|R^N_{\lambda'}f\|_{L^2(\mathbb{R}^{n-1})} \lesssim \lambda^{\frac{-(n-1)}{2}}\chi(x_n,z_n)\,\|f\|_{L^2(\mathbb{R}^{n-1})}. \tag{2.19}$$

Looking at (2.12), (2.17) and (2.19) we see that

$$\|(S^2_\lambda)_{x_nz_n}f\|_{L^2(\mathbb{R}^{n-1})} \lesssim \widetilde{\psi}\left(\frac{x_n-z_n}{\epsilon}\right)\chi(x_n,z_n)$$

$$\cdot\Big(\lambda^{-n+1/2}|x_n-z_n|^{-1/2} \tag{2.20}$$

$$+\sum_{j=1}^{N-1}\lambda^{-n+1/2-j}|x_n-z_n|^{-1/2-j} \tag{2.20$'$}$$

$$+\lambda^{-n+1/2-N}|x_n-z_n|^{-n/2-N}\Big)\,\|f\|_{L^2(\mathbb{R}^{n-1})} \tag{2.20$''$}$$

where χ is compactly supported. Now we apply Lemma 1.1 to obtain

$$\|S^2_\lambda f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

$$\cdot\Big(\lambda^{-n+1/2} \tag{2.21}$$

$$+\sum_{j=1}^{N-1}\lambda^{-n+1/2-j}\epsilon^{1/2-j} \tag{2.21$'$}$$

$$+\lambda^{-n+1/2-N}\epsilon^{-n/2-N+1}\Big). \tag{2.21$''$}$$

In consideration of (2.6) and (2.21′′) set

$$\epsilon = \lambda^{-\frac{2N+1}{n+2N}}. \tag{2.22}$$

In this case

$$\lambda^{-n+1}\epsilon = \lambda^{-n+1/2-N}\epsilon^{-n/2-N+1} = \lambda^{-n+1/2}\lambda^{\frac{n-2(N+2)}{2(n+2N)}}.$$

Also, it is easy to check, in consideration of (2.21′), that

$$\lambda^{-n/2+1/2-j}\epsilon^{1/2-j} \leq \lambda^{-n+1/2}$$

for this choice of ϵ. Evidently (2.21) is the main term in the estimate of $\|S_\lambda\|$ provided that $n \leq 2(N+2)$. We choose N as so and then let ϵ be as in (2.22) for this choice of N. Now (2.3) is demonstrated, and the proof of Theorem 2.6 is complete. □

Given the general nature of Theorem 2.6, it is natural to ask whether or not the result is sharp. We shall find in the next theorem that we may not always be able to improve the exponent of λ in Theorem 2.6.

Theorem 2.7. *For a given amplitude function $a \not\equiv 0$, there are $t \in C^\infty$ such that the exponent of λ in Theorem 2.6 is sharp.*

Proof. By assumption there is a point (x_0, y_0) such that $a(x_0, y_0) \neq 0$. After perhaps a translation and a rotation—in view of Remark 2.5 — we may assume that $x_0 = e_n$, the n-th unit vector in $\mathbb{R}^n$, and $y_0 = 0$. Then let $t(x) = x_n$, and note that we may assume $x_n \neq 0$ on supp(a). Let $f_\epsilon(y) = f(y')\tilde{f}(y_n)$, where $0 \leq f \in C_0^\infty(\mathbb{R}^{n-1})$, $f \equiv 1$ on supp(a) and $\tilde{f}(y_n) = e^{i2\lambda y_n}\chi_{[-\epsilon,\epsilon]}(y_n)$. Then

$$\begin{aligned}
e^{-i\lambda x_n} T_\lambda f(x) &= \int_{\mathbb{R}^{n-1}} e^{\frac{\lambda}{x_n}|x'-y'|^2} f(y') \int_{-\epsilon}^{\epsilon} e^{i\lambda \frac{y_n^2}{x_n}} a(x, y', y_n)\, dy_n\, dy' \\
&= \left(\frac{i\lambda}{x_n \pi}\right)^{-(n-1)/2} 2\epsilon f(x') \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} a(x, x', y_n)\, dy_n \\
&\quad + \left(\frac{i\lambda}{x_n \pi}\right)^{-(n-1)/2} f(x') \int_{-\epsilon}^{\epsilon} \left(e^{i\lambda \frac{y_n^2}{x_n}} - 1\right) a(x, x', y_n)\, dy_n \\
&\quad + \left(\frac{i\lambda}{x_n \pi}\right)^{-(n-1)/2} 2\epsilon \int_{\mathbb{R}^{n-1}} r_1(ix_n|\xi'|^2/4\lambda) \\
&\qquad \cdot \left(f(x' + \cdot\,) \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{i\lambda \frac{y_n^2}{x_n}} a(x, \cdot + x', y_n)\, dy_n\right)\widehat{\ }(\xi')\, d\xi' \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}\,.
\end{aligned}$$

Now it is easily seen that

$$|\,\mathrm{II}\,| \lesssim \lambda^{-n/2+3/2}\epsilon^3,$$

and by (1.3)

$$\begin{aligned}
|\,\mathrm{III}\,| &\lesssim \epsilon \lambda^{-n/2-1/2} \sum_{|\alpha| \leq 2+n/2} \left\| \mathrm{D}_{y'}^{\alpha} f(x' + \cdot\,) \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} e^{i\lambda \frac{y_n^2}{x_n}} a(x, \cdot\,, y_n)\, dy_n \right\|_{L^2(\mathbb{R}^{n-1})} \\
&\lesssim \epsilon \lambda^{-n/2-1/2}.
\end{aligned}$$

Supposing that ϵ is small we have that

$$|\,\mathrm{I}\,| \gtrsim \epsilon \lambda^{-n/2+1/2} \left| \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} a(x, x', y_n)\, dy_n \right| \gtrsim \epsilon \lambda^{-n/2+1/2}$$

on a set of positive measure in x-space. If we let $\epsilon = c\lambda^{-1/2}$ where c is a small constant independent of λ (i.e. let $\epsilon\lambda^{-n/2+1/2} = \epsilon^3\lambda^{-n/2+3/2}$), then clearly

$$\|T'_\lambda f_\epsilon\|_2 \gtrsim \lambda^{-n/2} \quad \text{and} \quad \|f_\epsilon\|_2 = \lambda^{-1/4}.$$

So

$$\frac{\|T'_\lambda f_\epsilon\|_2}{\|f_\epsilon\|_2} \gtrsim \lambda^{-n/2+1/4}$$

as desired. □

Remark 2.8. Using a result of Pan and Sogge [**PS**] it is a routine matter to show that if $Ht(x)(x-y)\cdot(x-y) \neq 0$ on $\operatorname{supp}(a)$ where $Ht(x)$ denotes the Hessian of t at x, then $\|T'_\lambda f\|_2 \leq C\lambda^{-n/2+1/6}\|f\|_2$.

To conclude this section we consider a variant of the operator previously considered. We let $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be a given fixed amplitude function, but we let t denote any smooth function. Now we shall study the oscillatory integral operator T'_λ, with phase function $\phi(x,y) = -2\,x\cdot y + t(x)|y|^2$. Its purpose is to provide a means of understanding T_λ in its full generality. The phase function involved is more suited to this task from a technical standpoint. We could have just as easily applied Theorem 2.6 to the more general case of T_λ, but the proofs of Theorems 2.9, 2.10 and 2.11 follow closely those of their counterparts, Theorems 2.2, 2.6 and 2.7, and we get these results almost for free. The proofs are therefore omitted and can be found in [**K**].

Theorem 2.9. *If* $1 - \nabla t(x)\cdot y \neq 0$ *on* $\operatorname{supp}(a)$ *then* $\|T'_\lambda f\|_2 \leq C\lambda^{-n/2}\|f\|_2$, *and the exponent of* λ *is sharp. Moreover* $\exists\, c$ *such that the constant* C *is uniform over the set*

$$\Sigma = \Sigma(c) = \{\, t \in C^\infty \,:\, |\nabla t(x)|,\, \|\,\mathrm{H}\, t\| \leq c\ \forall x \in \operatorname{supp}(a)\,\}.$$

Theorem 2.10. *Let* ϕ *be as above. Then in general*

$$\|T'_\lambda f\|_2 \leq C\lambda^{-n/2+1/4}\|f\|_2\,.$$

Theorem 2.11. *For a given amplitude function* $a \not\equiv 0$, *there are* $t \in C^\infty$ *such that the exponent of* λ *in Theorem 2.10 is sharp.*

3. Main Theorem.

We now turn our attention to the family of operators T_λ with phase function of the form $\phi(x,y) = \frac{|x-y|^2}{t(x)-\bar{t}(y)}$ where t and $\bar{t}$ are smooth functions such that $0 < |t(x)-\bar{t}(y)|$. In what follows we will always take a to be a fixed amplitude function with compact, connected support. Hence we may assume without loss of generality that there is a constant $c > 0$ such that $c \le t(x) - \bar{t}(y)$ on $\operatorname{supp}(a)$. This will be our general setup through this section.

We shall find it to be possible that $\operatorname{rank} H_\phi = n-2$, and therefore the ideas in Section 2 do not carry over into the analysis of $\|T_\lambda\|$. Instead, we would like to consider T_λ as a composition of operators whose factors are known to us, these factors already having been studied in the previous section. We may then use the estimate in Theorem 2.10 to get sharp results for T_λ. In actual fact, though, we will not realize T_λ directly as a composition, but nearly so. The composition of operators we consider has the same phase function as T_λ but has a different amplitude. The transition to T_λ in Theorem 3.3 from Theorem 2.10 is facilitated by Lemma 3.2, which allows us to compare oscillatory integral operators with the same phase function but different amplitudes.

We begin with the following result about the singularities of H_ϕ.

Proposition 3.1. *Let ϕ be as above. Then* $\operatorname{rank} H_\phi(x,y) \ge n-2$, *and moreover we have that* $\operatorname{rank} H_\phi(x,y) = n-2$ *if and only if*

(i) $1 - \dfrac{\nabla t(x)\cdot(x-y)}{t(x)-\bar{t}(y)} = 0.$

(ii) $1 - \dfrac{\nabla \bar{t}(y)\cdot(x-y)}{t(x)-\bar{t}(y)} = 0.$

(iii) $\nabla t(x)\cdot\nabla\bar{t}(y) = 0.$

Proof. We begin by noting that

$$\frac{\partial^2\phi(x,y)}{\partial x_i\partial y_j} = \frac{-2}{t(x)-\bar{t}(y)}\Bigg(\delta_i^j - \frac{1}{t(x)-\bar{t}(y)} \cdot\left(\frac{\partial}{\partial y_j}\bar{t}(y)(x_i-y_i) + \frac{\partial}{\partial x_i}t(y)(x_j-y_j) - |x-y|^2\frac{\partial_i t(x)\partial_j\bar{t}(y)}{t(x)-\bar{t}(y)}\right)\Bigg). \tag{3.1}$$

Suppose first that $\nabla\bar{t}(y) = 0$. Then from (3.1),

$$(H_\phi(x,y))_{ij} = \frac{-2}{t(x)-\bar{t}(y)}\left(\delta_i^j - \frac{(x_j-y_j)}{t(x)-\bar{t}(y)}\partial_i t(x)\right),$$

and such a matrix has rank $\ge n-1$ by Lemma 1.6. So we may assume that $\nabla\bar{t}(y) \ne 0$. In fact we may assume that $\nabla\bar{t}(y) \parallel e_n$. For let A be a rotation of

$\mathbb{R}^n$ such that $\nabla\bar{t}(y)A \parallel e_n$. Consider $\phi_A(z,w) = \phi(Az, Aw) = \dfrac{|z-w|^2}{t_A(z) - \bar{t}_A(w)}$, $t_A = t \circ A$, $\bar{t}_A = \bar{t} \circ A$. Then $H_{\phi_A} = A^t H_\phi(Az, Aw)A$. For given x and y, let $z = A^{-1}x$, $w = A^{-1}y$. Then clearly $\operatorname{rank} H_\phi(x,y) = \operatorname{rank} H_{\phi_A}(z,w)$. Moreover, it is routine to check that

(i′) $1 - \dfrac{\nabla t(x)\cdot(x-y)}{t(x)-\bar{t}(y)} = 1 - \dfrac{\nabla t_A(z)\cdot(z-w)}{t_A(z)-\bar{t}_A(w)}$.

(ii′) $1 - \dfrac{\nabla \bar{t}(x)\cdot(x-y)}{t(x)-\bar{t}(y)} = 1 - \dfrac{\nabla \bar{t}_A(w)\cdot(z-w)}{t_A(z)-\bar{t}_A(w)}$.

(iii′) $\nabla t(x)\cdot\nabla\bar{t}(y) = \nabla t_A(z)\cdot\nabla\bar{t}_A(w)$.

Since $\nabla\bar{t}(y) \parallel e_n$, we have that the $(n-1)\times(n-1)$ submatrix H'_ϕ of H_ϕ formed by deleting the last row and last column is of the form

$$(H'_\phi(x,y))_{ij} = \frac{-2}{t(x)-\bar{t}(y)}(\delta_i^j - (x_j - y_j)\partial_j t(x)) \quad 1 \le i \le n-1,\ 1 \le j \le n-1.$$

This submatrix has rank $\ge n-2$ (by Lemma 1.6); hence so does H_ϕ.

Suppose that (i), (ii) and (iii) are satisfied. Since $\nabla\bar{t}(y) \parallel e_n$ and (iii) holds, we have that $\partial_n t(x) = 0$. So by (ii),

$$H_\phi(x,y) = \left(\begin{array}{ccc|c} & & & * \\ & H'_\phi(x,y) & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & 0 \end{array}\right).$$

Since (i) (or (ii)) holds, $x - y \neq 0$. Clearly,

$$(0, 0, \ldots, x_n - y_n)H_\phi(x,y) = 0,$$

while by (i)

$$(x' - y', 0)H_\phi(x,y) = 0.$$

So $\operatorname{rank} H_\phi(x,y) = n-2$.

Now suppose that $\operatorname{rank} H_\phi(x,y) = n-2$ (assuming again that $\nabla\bar{t}(y) \neq 0$). In particular, by Lemma 1.6, H'_ϕ has rank $n-2$, and $\nabla_{x'} t(x)\cdot(x'-y') = t(x) - \bar{t}(y) \neq 0$. So we know that $x' - y' \neq 0$. Now it is clear that (i), (ii) and (iii) hold if we know that $\partial_n t(x) = 0$. The claim is that indeed $\partial_n t(x) = 0$, for suppose not. Consider the $(n-1)\times(n-1)$ submatrix of H_ϕ given by deleting the first row and n-th column. It is of the type described in Lemma 1.7. Thus $(x_1 - y_1)\partial_n t(x) = 0$, and hence $(x_1 - y_1) = 0$. Now delete the second row and n-th column from H_ϕ. After switching two columns, we may again apply Lemma 1.7 to obtain that $x_2 - y_2 = 0$, and continuing in this way we find that $x' - y' = 0$ which is a contradiction. □

It is easy now to construct t and $\bar{t}$ such that (i), (ii) and (iii) of Proposition 3.1 are satisfied, and we give a simple example to demonstrate this fact. Let $t(x) = x_1$ and $\bar{t}(y) = y_n$; then (i)–(iii) are satisfied on the set $\{ (x,y) : x_1 = x_n,\, y_1 = y_n \}$. Suffice it to say that there are many t and $\bar{t}$ such that $\operatorname{rank}(H_\phi) = n-2$, and we reserve a more detailed discussion of this for a future publication preferring to move on to the main technical lemma of this chapter.

Lemma 3.2. *Let K_1, $K_2 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and let*

$$T_j f(x) = \int_{\mathbb{R}^n} f(y) K_j(x,y)\, dy \qquad j = 1,2.$$

Suppose that there is an open set $U \subset \mathbb{R}^n$ such that

$$\operatorname{supp}(K_2) \subset U \quad \text{and} \quad K_1 \neq 0 \text{ on } \bar{U}. \tag{3.2}$$

If there are constants C_1, C_2 and $C_3 > 0$ such that

$$\begin{aligned} &\|K_2/K_1\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} \le C_1, \\ &\Big\| \sum_{|\alpha|=s} \mathrm{D}^\alpha (K_2/K_1) \Big\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \le C_2 \qquad \text{for some } \ s > n, \\ &\operatorname{supp}(K_2) \subset B_{C_3}(0) \times B_{C_3}(0). \end{aligned} \tag{3.3}$$

Then

$$\|T_2\| \le C\, \|T_1\|,$$

where $C = C(C_1, C_2, C_3)$ is bounded once C_1, C_2 and C_3 are bounded.

The proof of this lemma is simple: we write K_2 as the product of K_1 and an absolutely convergent sum of compactly supported tensors via an application of the Fourier transform in the two variables x, y. Then T_2 is the absolutely convergent sum of operators each one being obtained from T_1 by composition with unitary operators. A full proof may be found in [**K**], and I am indebted to T. Wolff for pointing out the existence of such a lemma.

If we let $K_1(x,y) = e^{i\lambda\phi(x,y)} b(x,y)$ and $K_2(x,y) = e^{i\lambda\phi(x,y)} a(x,y)$, the conditions (3.2) and (3.3) in this case translate as

$$\exists U \subset \mathbb{R}^n \text{ open, } \operatorname{supp}(a) \subset U \quad \text{and} \quad b \neq 0 \text{ on } \bar{U}, \tag{3.2$'$}$$

$$\|a/b\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} \leq C_1,$$

$$\left\| \sum_{|\alpha|=s} \mathrm{D}^\alpha (a/b) \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq C_2 \qquad \text{for some } \ s > n, \tag{3.3$'$}$$

$$\operatorname{supp}(a) \subset B_{C_3}(0) \times B_{C_3}(0).$$

From Proposition 3.1 and Theorem 1.8 we readily obtain the result $\|T_\lambda f\|_2 \leq C\lambda^{-(n-2)/2} \|f\|_2$, and although it is possible to find functions which satisfy conditions (i), (ii) and (iii) of Proposition 3.1 at a point, or on even larger varieties, it is not possible that H_ϕ should be so singular that such an estimate is sharp. This is the content of our main theorem.

Theorem 3.3. *Let T_λ be the oscillatory integral operator with phase function ϕ as above. Then*

$$\|T_\lambda f\|_2 \leq C\lambda^{-(n-1)/2} \|f\|_2 .$$

Proof. Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Suppose with out loss of generality (after perhaps a dilation) that $\operatorname{supp}(a) \subset B_1 \times B_1$. Assume also (after a change of the parameter λ) that $1 \leq t(x) - \bar{t}(y)$. Choose $\psi \in C_0^\infty(B_5)$ with $\psi \equiv 1$ on B_4. Consider the following operators:

$$S^1_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda(-2x\cdot y + t(x)|y|^2)} \psi(x)\psi(y) f(y)\, dy, \tag{3.4}$$

$$S^2_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda(-2x\cdot y + \bar{t}(x)|y|^2)} \psi(x)\psi(y) f(y)\, dy. \tag{3.5}$$

We know that

$$\|S^j_\lambda\| \lesssim \lambda^{-n/2+1/4} \qquad j = 1, 2.$$

Thus

$$\|\lambda^{n/2} S^1_\lambda (S^2_\lambda)^*\| \lesssim \lambda^{-n/2+1/2}.$$

Now

$$\lambda^{n/2} S^1_\lambda (S^2_\lambda)^* f(x) = \int_{\mathbb{R}^n} e^{i\lambda \frac{|x-z|^2}{t(x)-\bar{t}(z)}} b_\lambda(x,z) f(z)\, dz,$$

where

$$b_\lambda(x,z) = \lambda^{n/2} \frac{\psi(x)\psi(z)}{(t(x)-\bar{t}(z))^{n/2}} \int_{\mathbb{R}^n} e^{i\lambda|y|^2} \psi^2 \cdot \left(\frac{y}{(t(x)-\bar{t}(z))^{1/2}} + \frac{x-z}{t(x)-\bar{t}(z)} \right) dy. \tag{3.6}$$

We wish to apply Lemma 3.2 to finish the proof; this amounts to checking (3.2′) and (3.3′) for a and b_λ. Let $U = B_2 \times B_2$, and suppose that $(x,z) \in \bar{U}$ — i.e.$|x| \le 2,\ |z| \le 2$. We apply Theorem 1.3 to the integral in (3.6) to obtain that

$$b_\lambda(x,z) = \pi^{\frac{n}{2}} e^{-i\frac{n}{4}\pi} \frac{\psi(x)\psi(z)}{(t(x)-\bar{t}(z))^{n/2}} \left(\psi^2\left(\frac{x-z}{t(x)-\bar{t}(z)}\right) + E(x,z) \right),$$

where $E(x,z)$ is the first-order remainder in (1.2). Now since

$$\left| \frac{x-z}{t(x)-\bar{t}(z)} \right| \le 4 \qquad \text{when} \ \ (x,z) \in \bar{U},$$

then

$$\psi^2\left(\frac{x-z}{t(x)-\bar{t}(z)}\right) = 1 \qquad \text{when} \ \ (x,z) \in \bar{U}.$$

Moreover by 1.3,

$$|E(x,z)| \le C\lambda^{-1} \sum_{|\alpha| \le n/2+2} \left\| \mathrm{D}^\alpha \psi^2 \right\|_2 \le 1/2 \quad \text{when} \ \ \lambda \gg 1.$$

So for large λ (depending only on ϕ), $|b_\lambda(x,z)| \ge 1/2$ on $\bar{U}$. We also see from Corollary 1.5 that

$$\left\| \mathrm{D}^\alpha b_\lambda \right\|_\infty \lesssim 1 \qquad \text{for} \ \ |\alpha| \le n+1$$

then (3.2′) and (3.3′) are satisfied with C_1, C_2 and $C_3 \lesssim 1$. □

Again, we may not make an improvement in Theorem 3.3 as evidenced by the follwing.

Theorem 3.4. *For a given amplitude function $a \not\equiv 0$, we may find t and $\bar{t}$ such that the exponent of λ in Theorem 3.3 is sharp.*

Proof. For some $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, $a(\tilde{x},\tilde{y}) \ne 0$. Following Remark 2.5 we may assume that $\tilde{y} = 0$ and $\tilde{x} = (0, \cdots, 0, \tilde{x}_n)$ for some $\tilde{x}_n > 0$, and furthermore we may reduce the size of supp(a) so that $c^{-1} \leq x_n - y_n \leq c$ for some fixed constant $c > 0$. Finally we may assume that $a(x, y) \geq 0$.

Let $t(x) = x_n$ and $\bar{t}(y) = y_n$, and choose $\tilde{f}$ to be a C_0^∞ function which is unity on $\text{supp}_y(a)$ and such that $\|\tilde{f}\|_2 = 1$. Noting that

$$\frac{|x-y|^2}{x_n - y_n} = \frac{|x' - y'|^2}{x_n - y_n} + x_n - y_n,$$

take $f(y) = e^{iy_n}\tilde{f}(y)$. Then by (1.2) and (1.3)

$$\begin{aligned} |T_\lambda f(x)| &= \left| \int \int e^{i\frac{\lambda}{x_n - y_n}|x'-y'|^2} \, dy' \, dy_n \right| \\ &= \left| \int \left(\frac{i\lambda}{\pi}\right)^{-\frac{n-1}{2}} a(x, (x', y_n)) \, dy_n + \mathcal{O}(\lambda^{-(n-3)/2}) \right|. \end{aligned}$$

Thus $\|T_\lambda f\|_2 \gtrsim \lambda^{-n/2+1/2}$ as desired. □

We finish this section by noting that the estimate in Theorem 3.3 may be improved when we have a better bound on one of the "factors" S_λ^1 or S_λ^2 of T_λ. If for example t and $\bar{t}$ satisfy the hypothesis of Theorem 2.9 then in fact $\|T_\lambda\| \lesssim \lambda^{-n/2}$. More precisely we have the following theorem whose proof is simply that of Theorem 3.3.

Theorem 3.5. *Let t, $\bar{t}$ and ϕ be as above. For a given amplitude a with support contained in $B_1 \times B_1$, let*

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \exp^{i\lambda \frac{|x-y|^2}{t(x) - \bar{t}(y)}} \, a(x, y) f(y) \, dy.$$

If S_λ^1 and S_λ^2 in (3.4) and (3.5) are such that

$$\|S_\lambda^1\| \lesssim \lambda^{-p} \quad \text{and} \quad \|S_\lambda^2\| \lesssim \lambda^{-q},$$

then

$$\|T_\lambda f\|_2 \lesssim \lambda^{-p-q+n/2} \|f\|_2 .$$

4. Appendix.

We finish this paper with a few concluding remarks.

In the introduction we stated the following theorem.

Theorem 4.1. *Let*

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t(x)-\bar{t}(y)}\right) a(x,y) f(y)\, dy,$$

where $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *and* t *and* $\bar{t}$ *are measurable functions defined on* $\operatorname{supp}_x(a)$ *and* $\operatorname{supp}_y(a)$ *respectively such that* $1 \le |t(x) - \bar{t}(y)| \le 2$. *Then*

$$\|T_\lambda f\|_2 \le C\lambda^{-\frac{n-2}{4}} \|f\|_2 . \tag{4.1}$$

Although this theorem does not appear in [B], some of the ideas in the proof may be found there. It is interesting to note that in spite of the non-smoothness of the phase function, $\|T_\lambda\| \to 0$ as $\lambda \to \infty$ when $n > 2$.

Proof. For ϵ to be chosen, let

$$\begin{aligned} U_j &= \{\, x \in \operatorname{supp}_x(a) \,:\, j\epsilon \le t(x) < (j+1)\epsilon \,\} \\ \bar{U}_k &= \{\, x \in \operatorname{supp}_y(a) \,:\, k\epsilon \le \bar{t}(y) < (k+1)\epsilon \,\}. \end{aligned}$$

Letting $\chi_j(x)$ and $\bar{\chi}_k(y)$ be the characteristic functions of U_j and $\bar{U}_k$ respectively, we see that

$$\begin{aligned} T_\lambda f(x) &= \sum_{j\,k} \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t(x)-\bar{t}(y)}\right) \chi_j(x) a(x,y) \bar{\chi}_k(y) f(y)\, dy \\ &= \sum_{j\,k} \int_{\mathbb{R}^n} \exp\left(i\lambda \frac{|x-y|^2}{t_j-\bar{t}_k}\right) \chi_j(x) a(x,y) \bar{\chi}_k(y) f(y)\, dy \\ &\quad + \int_{\mathbb{R}^n} \sum_{j\,k} \left(\exp\left(i\lambda \frac{|x-y|^2}{t(x)-\bar{t}(y)}\right) \right. \\ &\qquad \left. - \exp\left(i\lambda \frac{|x-y|^2}{t_j-\bar{t}_k}\right)\right) \chi_j(x) a(x,y) \bar{\chi}_k(y) f(y)\, dy \\ &= T_{\lambda 1} f(x) + T_{\lambda 2} f(x), \end{aligned}$$

where $t_j \in [j\epsilon,\ (j+1)\epsilon)$ and $\bar{t}_k \in [k\epsilon,\ (k+1)\epsilon)$. When $x \in U_j$ and $y \in\ \bar{U}_k$ it is clear that

$$\left| \exp\left(i\lambda \frac{|x-y|^2}{t(x)-\bar{t}(y)}\right) - \exp\left(i\lambda \frac{|x-y|^2}{t_j-\bar{t}_k}\right) \right| \lesssim \lambda\epsilon.$$

Then by Schur's Lemma

$$\|T_{\lambda 1} f\|_2 \lesssim \lambda\epsilon \|f\|_2 . \tag{4.2}$$

Now we estimate $\|T_{\lambda 2}\|$ by duality. Let $g \in L^2$ be such that $\|g\|_2 = 1$. First notice that the number of indices j or k is comparable to ϵ^{-1}, and let $T_{j\,k}$ be the oscillatory integral operator with phase function $\exp\left(i\lambda\frac{|x-y|^2}{t_j - \bar{t}_k}\right)$ and amplitude $\chi_j(x)a(x,y)\bar{\chi}_k(y)$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_{\lambda 2} f(x) g(x)\,dx\right| \\
&= \left|\sum_{j\,k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(i\lambda\frac{|x-y|^2}{t_j-\bar{t}_k}\right)g(x)\chi_j(x)a(x,y)\bar{\chi}_k(y)f(y)\,dx\,dy\right| \\
&\leq \sum_{j\,k}\|g\bar{\chi}_k\|_2\,\|T_{j\,k}(f\chi_j)\|_2 \leq \sum_j\left(\sum_k\|g\bar{\chi}_k\|_2^2\right)^{1/2}\left(\sum_k\|T_{j\,k}(f\chi_j)\|_2^2\right)^{1/2} \\
&\lesssim \|g\|_2\,\|f\|_2\,\lambda^{-n/2}\epsilon^{-1}.
\end{aligned}$$

So

$$\|T_{\lambda 2} f\|_2 \lesssim \lambda^{-n/2}\epsilon^{-1}\,\|f\|_2\,. \tag{4.3}$$

Choosing $\epsilon = \lambda^{-n/4-1/2}$ in (4.2) and (4.3) yields (4.1). □

Further study involves relaxing the smoothness assumption on t and $\bar{t}$. One would like to drop all smoothness assumptions in order to prove pointwise convergence results in higher dimensions. The scheme in [**B**] can certainly be carried out in three space dimensions. In higher dimensions it seems more likely that one will have to study directly the operator R_k mentioned in the introduction, without recourse to T_λ. Preliminary results in this direction can be found in [**K**].

References

[B] J. Bourgain, *A Remark on Schrödinger Operators*, Isr. J. of Math., **77** (1992), 1-16.

[C] L.Carleson, *Some Analytical Problems Related to Stastical Mechanics*, Euclidean Harmonic Analysis, Lecture Notes in Math., Springer-Verlag, Berlin and New York, **779** (1979), 5-45.

[DK] B. Dahlberg and C. Kenig, *A Note on the Almost Everywhere Convergence of Solutions of the Schrödinger Equation*, Harmonic Analysis, Lecture Notes in Math., Springer-Verlag, Berlin, **908** (1982), 205-209.

[GS] A. Greenleaf and A. Seeger, *Fourier Integral Operators With Fold Singularities*, preprint.

[H1] L. Hörmander, *The Analysis of Linear Partial Differential Operators* I, Springer-Verlag, Grund. der math. Wiss, Berlin and New York, **256** (1990).

[H2] ———, *Oscillatory Integrals and Multipliers on FL^p*, Ark. Mat., **11** (1971), 1-11.

[K] L. Kolasa, *Oscillatory Integral Operators Related to Pointwise Convergence of Schrödinger Operators*, Ph.D. Thesis California Institute of Technology, 1994.

[PS] Y. Pan and C. Sogge, *Oscillatory Integrals Associated to Folding Canonical Relations*, Colloquium Mathematicum, **60** (1990), 413-419.

[Sj] P. Sjölin, *Regularity of Solutions to the Schrödinger Equation*, Duke Math. J., **55** (1987), 669-715.

[St] E.M. Stein, *Harmonic Analysis*, Princeton Univ. Press, Princeton N.J., 1993.

[V] L. Vega, *Schrödinger Equations: Pointwise Convergence to the Initial Data*, Proc. Am. Math. Soc., **102** (1988), 874-878.

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