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A UNIQUENESS THEOREM FOR THE MINIMAL SURFACE EQUATION ON AN UNBOUNDED DOMAIN IN $\mathbb{R}^2$

Chin-Chuan Lee

In this paper, we give a variation on Nitsche's result on solutions to the minimal surface equation in sectors.

In 1965, Nitsche [1] announced the following result:

Theorem 1. *Let Ω be a sector domain of opening angle smaller than π. If $u = u(x,y) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of*

$$\begin{cases} \operatorname{div} \mathbf{T}u = 0 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

then $u \equiv 0$ in Ω. Here $\mathbf{T}u = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$.

In this paper, we give a variation of Theorem 1 as follows:

Theorem 2. *Let Ω be a sector domain of opening angle smaller than π. If $u = u(x,y) \in C^2(\Omega) \cap C^1(\overline{\Omega}\backslash\{vertex\})$ is a solution of*

$$\begin{cases} \operatorname{div} \mathbf{T}u = 0 & \textit{in } \Omega \\ \mathbf{T}u \cdot \nu = 0 & \textit{on } \partial\Omega\backslash\{vertex\} \end{cases}$$

where ν is the exterior unit normal on $\partial\Omega\backslash\{vertex\}$ then $u \equiv$ constant in Ω.

Proof. First, we notice that, although we make no assumptions at the vertex, in fact, we can show that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ as follows: Denote by m and M the lower and upper bounds for u on a fixed arc separating the vertex from infinity. Then, by Theorem 5.1 of [2] the two planes $v = m$ and $v = M$ lie respectively below and above u in the entire domain between the vertex and the arc, and thus u is bounded above and below at the vertex. It then follows directly from Simon's theorem [5], that u is C^1 to the vertex. Thus

$u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Since every solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of $\operatorname{div} \mathbf{T}u = 0$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{1+q^2}{w}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{pq}{w}\right) \\ \dfrac{\partial}{\partial x}\left(\dfrac{pq}{w}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{1+p^2}{w}\right) \end{cases}$$

where $p = \dfrac{\partial u}{\partial x}$, $q = \dfrac{\partial u}{\partial y}$, $w = \sqrt{1+p^2+q^2}$, there exist functions $F(x,y)$ and $G(x,y)$ such that

$$\begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{1+p^2}{w}, \; \dfrac{\partial F}{\partial y} = \dfrac{pq}{w} \\ \dfrac{\partial G}{\partial x} = \dfrac{pq}{w}, \qquad \dfrac{\partial G}{\partial y} = \dfrac{1+q^2}{w}. \end{cases} \tag{1}$$

If we set

$$\begin{cases} \xi = x + F(x,y) \\ \eta = y + G(x,y) \end{cases} \tag{2}$$

then ξ, η are isothermal coordinates. (c.f. [3, p. 31])

We will show that the image of $\overline{\Omega}$ under the transformations ξ, η is again a sector domain of opening angle smaller than π. ($\overline{\Omega}$ is the closure of Ω). Define a mapping $\psi : \overline{\Omega} \longrightarrow \mathbb{R}^2$ by $\psi(x,y) = (\xi, \eta)$, where ξ, η are defined by (2). Without loss of generality, we may assume Ω is symmetric with respect to the y-axis. If $\langle a, b\rangle$ is the unit tangent vector of the right side of $\partial\Omega$ (that is, $a^2 + b^2 = 1$; $a, b > 0$) and s is the arc-length function of $\partial\Omega$, then

$$\begin{cases} \dfrac{\partial \xi}{ds} = \langle \xi_x, \xi_y\rangle \cdot \langle a, b\rangle = a + \dfrac{a + p^2 a + pqb}{w} \\ \dfrac{\partial \eta}{ds} = \langle \eta_x, \eta_y\rangle \cdot \langle a, b\rangle = \dfrac{pqa + b + bq^2}{w} + b \end{cases} \quad \text{on the right side of } \partial\Omega.$$

Since $\mathbf{T}u \cdot \nu = 0$ on $\partial\Omega$, we have $\left\langle \dfrac{p}{w}, \dfrac{q}{w}\right\rangle \cdot \langle b, -a\rangle = 0$ on the right side of $\partial\Omega$, which implies $pb = qa$ on the right side of $\partial\Omega$. Substituting this into $\dfrac{d\xi}{ds}$, $\dfrac{d\eta}{ds}$, we obtain

$$\begin{cases} \dfrac{d\xi}{ds} = a(1+w) \\ \dfrac{d\eta}{ds} = b(1+w) \end{cases} \quad \text{on the right side of } \partial\Omega.$$

Since $1 + w > 1$, ψ maps the right side of $\partial\Omega$ linearly into $\mathbb{R}^2$. For the left side of $\partial\Omega$, we have the similar result. Thus ψ maps $\overline{\Omega}$ into a sector domain

$\overline{\Omega'}$ in $\mathbb{R}^2$ and ψ is a one-one, onto map from $\partial\Omega$ to $\partial\Omega'$. Next, we show that ψ is a one-one, onto map from Ω to Ω'. Let $P_0(x_0, y_0), P_1(x_1, y_1)$ be two points in Ω, with $\psi(x_0, y_0) = (\xi_0, \eta_0)$ and $\psi(x_1, y_1) = (\xi_1, \eta_1)$. Consider $P_t(x_t, y_t) = (x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0))$, $0 \le t \le 1$. Since Ω is convex, $P_t(x_t, y_t) \in \Omega$.

By (1), there exists a function $E(x, y)$ such that

$$\frac{\partial E}{\partial x} = F \quad \text{and} \quad \frac{\partial E}{\partial y} = G.$$

Following the method of Lemma 5.1 in [3], we define a function $h : [0, 1] \longrightarrow \mathbb{R}$ by

$$h(t) = E(x_t, y_t).$$

Then

$$\begin{aligned} h'(t) &= (x_1 - x_0)\bar{p} + (y_1 - y_0)\bar{q} \\ h''(t) &= (x_1 - x_0)^2\bar{r} + 2(x_1 - x_0)(y_1 - y_0)\bar{s} + (y_1 - y_0)^2\bar{l} \end{aligned}$$

where

$$\begin{aligned} \bar{p} &= \frac{\partial E}{\partial x}(x_t, y_t) \\ \bar{q} &= \frac{\partial E}{\partial y}(x_t, y_t) \\ \bar{r} &= \frac{\partial^2 E}{\partial x^2}(x_t, y_t) \\ \bar{s} &= \frac{\partial^2 E}{\partial x \partial y}(x_t, y_t) \\ \bar{l} &= \frac{\partial^2 E}{\partial y^2}(x_t, y_t). \end{aligned}$$

So, $h''(t)$ is a quadratic form in $(x_1 - x_0)$ and $(y_1 - y_0)$, and since the matrix $\begin{pmatrix} \bar{p} & \bar{s} \\ \bar{s} & \bar{q} \end{pmatrix}$ is positive definite, we have $h''(t) > 0$. This implies $h'(t)$ is an increasing function, thus $h'(0) < h'(1)$, that is,

$$(x_1 - x_0)(\bar{p}_1 - \bar{p}_0) + (y_1 - y_0)(\bar{q}_1 - \bar{q}_0) > 0$$

where

$$\begin{aligned} \bar{p}_1 &= \bar{p}(x_1, y_1) & \bar{p}_0 &= \bar{p}(x_0, y_0) \\ \bar{q}_1 &= \bar{q}(x_1, y_1) & \bar{q}_0 &= \bar{q}(x_0, y_0). \end{aligned}$$

From (2), we have

$$(x_1 - x_0)(\xi_1 - \xi_0) + (y_1 - y_0)(\eta_1 - \eta_0) > (x_1 - x_0)^2 + (y_1 - y_0)^2.$$

By the Cauchy-Schwarz inequality, we have

$$(\xi_1 - \xi_0)^2 + (\eta_1 - \eta_0)^2 > (x_1 - x_0)^2 + (y_1 - y_0)^2. \tag{3}$$

Thus, if $(x_0, y_0) \neq (x_1, y_1)$, then $(\xi_0, \eta_0) \neq (\xi_1, \eta_1)$, so ψ is a one-one map from Ω to Ω'.

To show ψ is an onto map, let $\psi(\Omega) = K \subset \Omega'$. Because $\dfrac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$, ψ is locally one-one. Thus every neighborhood of $(\xi_0, \eta_0) \in K$ belongs to K, that is, K is relatively open to Ω'. On the other hand, if $(\xi_0, \eta_0) \in \Omega'$ is the limit of sequence of points $(\xi_i, \eta_i) \in K$, we will show $(\xi_0, \eta_0) \in K$. But $\xi_i = x_i + F(x, y)$, $\eta_i = y_i + G(x, y)$, and bounded (because $\lim_{i\to\infty}(\xi_i, \eta_i) = (\xi_0, \eta_0)$). By (3), we see that (x_i, y_i) is also bounded. So, there exists a convergent subsequence (x_{n_i}, y_{n_i}) of (x_i, y_i) such that $\lim_{n\to\infty}(x_{n_i}, y_{n_i}) = (x_0, y_0)$. Now, since ψ is continuous on $\overline{\Omega}$, we obtain

$$\begin{aligned} \xi_0 &= \lim_{i\to\infty} \xi_{n_i} = \lim_{i\to\infty} (x_{n_i} + F(x_{n_i}, y_{n_i})) \\ &= x_0 + F(x_0, y_0) \\ \text{and} \qquad \eta_0 &= \lim_{i\to\infty} \eta_{n_i} = \lim_{i\to\infty} (y_{n_i} + G(x_{n_i}, y_{n_i})) \\ &= y_0 + G(x_0, y_0). \end{aligned}$$

Because ψ is a one-one, onto map from $\partial\Omega$ to $\partial\Omega'$, and ψ is continuous on $\overline{\Omega}$ and $(\xi_0, \eta_0) \in \Omega'$, it follows that $(x_0, y_0) \in \Omega$. Since $\psi(\Omega) = K$ and $\xi_0 = x_0 + F(x_0, y_0)$, $\eta_0 = y_0 + G(x_0, y_0)$, we have $(\xi_0, \eta_0) \in K$. Thus K is relatively closed to Ω'. Therefore, $\psi(\Omega) = K \neq \phi$ is relatively open and closed to Ω'. On the other hand, $\psi(\Omega) = K$ is connected in Ω', which implies $\psi(\Omega) = K = \Omega'$, so ψ is an onto map from Ω to Ω'.

We have shown that the image of Ω under the transformations ξ, η is again a sector domain Ω' of opening angle smaller than π.

Because $\operatorname{div} \mathbf{T}u = 0$, there exists a function $v(x, y)$ such that

$$\frac{\partial v}{\partial x} = \frac{q}{w} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{-p}{w}.$$

Therefore, $|\nabla v| < 1$, which implies $v(x, y) < \sqrt{x^2 + y^2}$. Now, by (3), we have $v(\xi, \eta) < \sqrt{\xi^2 + \eta^2}$, that is, $v = O(r)$, where $r = \sqrt{\xi^2 + \eta^2}$.

Also

$$\begin{aligned} \Delta_{(\xi,\eta)} v &= 0 \quad \text{in } \Omega', \\ v\big|_{\partial\Omega'} &= \text{constant} \end{aligned}$$

where $\Delta_{(\xi,\eta)}$ is the Laplacian with respect to (ξ, η).

Thus, using the Phragmén-Lindelöf theorem for harmonic function on a sector domain (c.f. [4, p. 94]), we have

$$v(\xi, \eta) \equiv \text{constant} \quad \text{in } \Omega'$$

this implies $\tilde{u}(\xi, \eta) \equiv$ constant in Ω', so $u(x, y) \equiv$ constant in Ω. □

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National Taiwan University
Taipei, Taiwan, R.O.C.
E-mail address: cclee@math.ntu.edu.tw