PACIFIC JOURNAL OF MATHEMATICS
Vol. 177, No. 1, 1997

BERGMAN ISOMETRIES BETWEEN CONVEX DOMAINS IN $\mathbb{C}^2$ WHICH ARE POLYHEDRAL

Andrea Pagano

This article deals with the problem of analyticity of Bergman isometries. One of the most important properties of the Bergman metric of a bounded domain is that it is invariant under the action of the group of biholomorphic maps. One then can ask if all the isometries are indeed complex analytic up to an obvious complex conjugation. There are several affermative answers to this question. In the present work, we study the case of convex polyhedral domains in $\mathbb{C}^2$ and we prove that any Bergman isometry of such a domain is analytic up to a complex conjugation.

1. Introduction.

Let Ω be a bounded domain in $\mathbb{C}^n$, we call it a *polyhedral domain* if there exist k real valued functions $\rho_1, \ldots, \rho_k : \mathbb{C}^n \to \mathbb{R}$ such that

$$\Omega = \{z \in \mathbb{C}^n : \rho_1(z) < 0, \ldots, \rho_k(z) < 0\}$$

where the gradient vectors

$$\nabla\rho_{i_1}, \ldots, \nabla\rho_{i_l}$$

are linearly independent over $\mathbb{C}$ at every $p \in \mathbb{C}^n$ satisfying $\rho_{i_1}(p) = \cdots = \rho_{i_l}(p) = 0$, for all possible choices of indices $\{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$. This guarantees that the singularities of the boundary are generated only by a normal crossing.

As for a bounded symmetric domain we can use the properties of the Hessian (both real and complex) of the defining functions ρ_i to define various notions of convexity.

A point $p \in \partial\Omega$, is called *point of* (*Levi*) *pseudoconvexity* if the Levi form

$$\sum_{k=1}^{n} \frac{\partial^2 \rho_l}{\partial z_k \partial \bar{z}_k} dz_k \otimes d\bar{z}_k$$

is positive semi-definite at p. The point p is called a *strongly pseudoconvex point* if the Levi form is postive definite.

In addition to this, we say that the boundary $\partial\Omega$ is *piecewise Levi flat* if the Levi form is zero along the complex directions tangential to the surface defined by $\rho_l(z) = 0$ for any l.

As any bounded domain in $\mathbb{C}^n$ Ω can be equipped with the *Bergman metric* which is a positive definite Hermitian metric. For a detailed discussion on the properties of the Bergman kernel and Bergman metric we refer to [**1**], [**9**], Section 1.4, and to [**3**].

Perhaps the most important property of the Bergman metric is that all biholomorphic maps are isometries. From a geometric viewpoint, it is therefore natural to ask if all Bergman isometries are biholomorphic up to obvious complex conjugations.

In this article, we study the problem of complex analyticity of Bergman isometries for convex polyhedral domains.

Our main result is:

Theorem. *Let $F : \Omega \to \Omega'$ be a C^∞ Bergman isometry between two convex polyhedral domains in $\mathbb{C}^2$. Then F is complex analytic up to a combination of complex conjugations.*

Our discussion starts from a brief analysis of the first example of such domains, the bidisc Δ^2. In [**11**] Satake showed that for any symmetric domain D (reducible or not) equipped with the Bergman metric the index $[\mathrm{Iso}(D) : \mathrm{Aut}(D)]$ of the group of automorphisms $\mathrm{Aut}(D)$ in the group of isometries $\mathrm{Iso}(D)$ equals 2^n, where n is the number of irreducible components of D. In his proof, Satake uses properties of D which are related to the nature of the Lie Algebra associated to D. This result can also be seen as an application of a more general method due to Kobayashi and Nomizu ([**7**]) which also can be related to the work of Schouten and Yano ([**12**]). Under the assumption of non-degeneracy of the Ricci tensor, Kobayashi and Nomizu proved that the identity component of the group of isometries of an arbitrary Kähler manifold consists only of holomorphic maps.

Due to the lack of homogeneuoity and to the lack of information about the Ricci tensor, neither approach seem to work for an arbitrary polyhedral domain.

Another approach to the problem of analyticity of isometries has been devoloped by Greene and Krantz ([**2**]). They proved that all the Bergman isometry of a C^∞ strongly pseudoconvex domain are actually holomorphic up to a conjugation. To get this result, they used a theorem by Klembeck ([**6**]) which proves that the asymptotic values of the holomorphic sectional curvature of a strongly pseudoconvex domain tend to the curvature of Bergman metric of the unit ball. For a more general description of the asymptotic behavior of the curvature tensor we refer to the work of Kim and Yu ([**5**], also [4]).

For a polyhedral domain Ω we have a result by Kim ([4]) which describes the asymptotic sectional curvature of Ω in terms of the curvature of the bidisc. With the example of the bidisc in mind (see [10] for details) we proceed in our discussion.

We have to remark that we only need to prove the main theorem for convex domain with piecewise Levi flat boundary. To see this we first notice that the theorem of Greene and Krantz has been localized, and it has been improved to the case of C^2 smooth boundary by Kim and Yu ([5]). Moreover, a pseudoconvex C^2 surface must have a strongly pseudoconvex point unless is entirely Levi flat.

2. Directions of minimal sectional cuvature.

As the first example of a convex polyhedral domain with piecewise Levi flat boundary, we consider the bidisc Δ^2. This is a reducible symmetric bounded domain, therefore the result of Satake holds in this case and we have that

$$[\mathrm{Iso}(\Delta^2) : \mathrm{Aut}(\Delta^2)] = 4.$$

Using the product structure of Δ^2 and the properties of the Bergman metric, one can explicitly compute the sectional curvature of the Bergman metric of the bidisc. Let us denote by $g(\cdot,\cdot)$ the Riemannian metric of the bidisc (real part of the Bergman metric). For any pair X and Y of tangent vectors to Δ^2 at the origin, with $X = X_1 + X_2$ and $Y = Y_1 + Y_2$, X_1 and Y_1 tangent to Δ_1 (horizontal component of Δ^2) and X_2 and Y_2 tangent to Δ_2 (vertical component of Δ^2) we have:

$$K(X,Y) = -k\frac{g(X_1,X_1)g(Y_1,Y_1) + g(X_2,X_2)g(Y_2,Y_2)}{(g(X_1,X_1)+g(X_2,X_2))(g(Y_1,Y_1)+g(Y_2,Y_2))}$$

for some constant k (see [8] for details about sectional curvature in the case of complex manifolds).

We have the following:

Proposition 1. *The negative sectional curvature of the bidisc attains its minimun values along exactly two complex directions. Moreover these minimal curvature directions correspond to the Euclidean coordinate directions.*

Using this property one can also prove:

Theorem 1. *Let $F : \Delta^2 \longrightarrow \Delta^2$ be a C^∞ Bergman isometry. Then $F(z_1, 0)$ is either $\{z_1 = 0\}$ or $\{z_2 = 0\}$. The same is true for $F(0, z_2)$.*

The proofs of these results consist in a direct analysis of the curvature together with the existence of totally geodesic submanifolds along directions of the minimal value for the negative sectional curvature ([10]). Further, we can obtain a different proof of the result of Satake for the bidisc (as well as for the polydiscs Δ^n) as consequence.

For an arbitrary polyhedral domain we do not have an explicit expression for the curvature tensor, so it is not possible to extend this result directly to the general case. On the other hand, we have the following result due to Kim ([4]):

Theorem (Kim). *Let Ω be a convex polyhedral domain in $\mathbb{C}^2$ with piecewise Levi flat boundary, and let p_j be a sequence that converges to a boundary point $\hat{p}$. Then the sequence of the sectional curvature tensors of the Bergman metric of Ω at p_j converges to the curvature tensor of the bidisc at the origin as j tends to infinity.*

The first goal is to transfer the information we have at the boundary to the interior. First of all we need to set the terminology and to give some definitions.

Following [4], we fix a sequence of points $\{p_j\}$ which converges to a boundary point. For any sequence of pair of tangent vectors X_j, Y_j we have:

$$\lim_{j\to\infty} K^{\Omega}_{p_j}(X_{p_j}, Y_{p_j}) = \lim_{j\to\infty} K^{\Delta^2}_{o}(G_j X_{p_j}, G_j Y_{p_j})$$

for some 2×2 matrices G_j, and some point $q \in \Delta^2$, where the matrices G_j depend on the sequence p_j and on the nature of the boundary point $\hat{p}$.

From now on we identify all the tangent spaces with $\mathbb{C}^2$ equipped with standard complex structure J.

We then consider the space $Gr_{\mathbb{R}}(2,4)$ of the real two-dimensional planes in $\mathbb{C}^2$. This is a compact Riemannian manifold; it therefore makes sense to speak about the convergence of a sequences in $Gr_{\mathbb{R}}(2,4)$. Consider the elements of the tangent space of $\mathbb{C}^2$ at $\hat{p}$ given as the limit in the Grassmanian of the tangent vectors at each p_j, i.e.

$$X_{\hat{p}} =: \lim_{j\to\infty} X_{p_j}.$$

Let $X_{\hat{p}}$ and $Y_{\hat{p}}$ be two elements of $T_{\hat{p}}(\Omega)$ obtained by this procedure; we define the sectional curvature at $\hat{p}$ along the section $\langle X_{\hat{p}}, Y_{\hat{p}}\rangle$ as

$$K^{\Omega}_{\hat{p}}(X_{\hat{p}}, Y_{\hat{p}}) = \lim_{j\to\infty} K^{\Delta^2}_{o}(G_j X_{p_j}, G_j Y_{p_j}).$$

We have the following

Proposition 2. *Let $\{p_j\}$ be a sequence of points converging to $\hat{p}$. For any $\epsilon > 0$ there exists j_o such that at each p_j with $j \geq j_o$ there exists a basis $\{Z_{p_j}, W_{p_j}\}$ of $T^{\mathbb{C}}_{p_j}(\Omega)$ such that:*

(1) *if we denote by K the minimum value for the sectional curvature $K^{\Omega}_{\hat{p}}$ we have*

$$K^{\Omega}_{p_j}(Z_{p_j}, JZ_{p_j}) - K \leq \epsilon$$

and

$$K^{\Omega}_{p_j}(W_{p_j}, JW_{p_j}) - K \leq \epsilon$$

(2) *if we denote by $Z_{\hat{p}} = \lim_{j\to\infty} Z_{p_j}$ and by $W_{\hat{p}} = \lim_{j\to\infty} W_{p_j}$ we have that $\{Z_{\hat{p}}, W_{\hat{p}}\}$ is a basis for $T^{\mathbb{C}}_{\hat{p}}(\Omega)$.*

Proof. Along the sequence p_j we construct two sequences of complex tangent vectors Z_{p_j}, W_{p_j} such that

$$\lim_{j\to\infty} G_j Z_{p_j} = \frac{\partial}{\partial z}$$

and

$$\lim_{j\to\infty} G_j W_{p_j} = \frac{\partial}{\partial w},$$

where (z, w) is the euclidean coordinate system of Δ^2 at the origin. Since $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial w}\}$ is an orthonormal basis we can assume that Z_{p_j}, W_{p_j} are $\mathbb{C}$-linearly independent.

To prove (1) it suffices to notice that the sectional curvature is continuous with respect to the point, and by Proposition 1 both $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ are directions of minimum value for the sectional curvature of the bidisc.

The linear independence of $\{Z_{\hat{p}}, W_{\hat{p}}\}$ follows from the scaling technique used to define the matrices G_j are constructed. For all the details about scaling we refer to the work of Kim ([**4**], [**5**]). □

From the preceding Proposition, at each point $p_j \in \Omega$ close to the boundary, we can find at least two $\mathbb{C}$-linearly independent directions Z_{p_j} and W_{p_j}, which give almost minimum values for the sectional curvature. In this way we are only able to give an estimate to the values of the sectional curvature; our goal is to be able to control directions which give almost minimal values of the sectional curvature.

Let Z be a complex tangent vector, the complex plane determine by the linear span of Z and JZ is denoted by $\mathcal{P}(Z)$. Let $\mathcal{X}$ and $\mathcal{Y}$ be a pair of tangent planes. We denote by $\gamma(\mathcal{X}, \mathcal{Y})$ the angle between them. We consider a basis $\{Z_p, W_p\}$ of $T^{\mathbb{C}}_p(\Omega)$; for any c we define:

$$\alpha(Z_p, c) = \max\Big\{\gamma(\mathcal{P}(Z_p), \mathcal{Y}) : |K^{\Omega}_p(\mathcal{Y}) - K| \\ \leq c|K^{\Omega}_p(\mathcal{P}(Z_p)) - K|;\ \gamma(\mathcal{Y}, \mathcal{P}(Z_p)) \leq \frac{\pi}{4}\Big\}$$

and

$$\alpha(W_p, c) = \max\Big\{\gamma(\mathcal{X}, \mathcal{P}(W_p)) : |K_p^\Omega(\mathcal{X}) - K_p| \\ \leq c|K_p^\Omega(\mathcal{P}(W_p)) - K|;\ \gamma(\mathcal{X}, \mathcal{P}(W_p)) \leq \frac{\pi}{4}\Big\}$$

where K denotes the minimum of the sectional curvature at $\hat{p}$ and $K_p^\Omega(\mathcal{X})$ is the sectional curvature at p along the section X.

Lemma. *For any $c > 0$ we can find a sequence of bases $\{Z_{p_j}, W_{p_j}\}$ of $T_{p_j}^{\mathbb{C}}(\Omega)$ such that*

$$\lim_{j\to\infty} \alpha(Z_{p_j}, c) = 0 \quad \textit{and} \quad \lim_{j\to\infty} \alpha(Z_{p_j}, c) = 0.$$

Proof. Let $\{Z_{p_j}, W_{p_j}\}$ be a basis for $T_{p_j}^{\mathbb{C}}(\Omega)$ as in Proposition 2. We have that

$$\lim_{j\to\infty} Z_{p_j} = Z_{\hat{p}} \quad \text{and} \quad \lim_{j\to\infty} W_{p_j} = W_{\hat{p}}.$$

We now consider a sequence of sections $\mathcal{Y}_{p_j}$ such that $K_{p_j}^\Omega(\mathcal{Y}_{p_j})$ is almost minimal and $\gamma(\mathcal{Y}_{p_j}, \mathcal{P}(W_{p_j})) \geq \frac{\pi}{4}$. If we take the limit $\mathcal{Y}_{\hat{p}}$ for $j \to \infty$ of $\mathcal{Y}_{p_j}$, by Proposition 1 it must coincide with $\mathcal{P}(Z_{\hat{p}})$ which implies that $\alpha(Z_{p_j}, c) \to 0$. The same it true for $\alpha(W_{p_j}, c)$. □

At this point we need to introduce a new object. Let $\{Z_p, W_p\}$ be a basis for $T_p^{\mathbb{C}}(\Omega)$. By a *slate* we mean the set

$$\langle Z_p \rangle_c = \left\{\mathcal{X} \in T_p(\Omega) : |K_p^\Omega(\mathcal{X}) - K_p| \leq c|K_p^\Omega(\mathcal{P}(Z_p)) - K_p|\right\}.$$

Analogously we have

$$\langle W_p \rangle_c = \left\{\mathcal{Y} \in T_p(\Omega) : |K_p^\Omega(\mathcal{Y}) - K_p| \leq c|K_p^\Omega(\mathcal{P}(W_p)) - K_p|\right\}.$$

From the Lemma and the definition of slates we have the following

Proposition 3. *Let Z_{p_j}, W_{p_j} as in (1) of Proposition 1. As j tends to infinity, we have*

$$\langle Z_{p_j} \rangle_c \quad \textit{tends to} \quad Z_{\hat{p}}$$

and

$$\langle W_{p_j} \rangle_c \quad \textit{tends to} \quad W_{\hat{p}}$$

for any choice of c.

This Proposition enables us to make the slates as thin as we need and also it makes possible to separate directions of minimal values for the sectional curvature whenever we approach the boundary. Moreover we can assume that the sequence of basis defined in Proposition 2, can be obtained by parallel transport along a smooth curve connecting the points of a suitable subsequence of $\{p_j\}$.

3. Analyticity of Bergman isometries.

We are now ready to show that any C^∞ Bergman isometry

$$F : \Omega \to \Omega'$$

between two convex polyhedral domains is an analytic map (up to conjugation).

Let X, Y be two elements of the real tangent space of Ω at a point. We denote by $\mathcal{P}(X, Y)$ the plane determined by the linear span of X and Y. The plane $\mathcal{P}(X, Y)$ is complex if it is invariant under the action of the complex structure J, and this is equivalent to choosing a basis $\{X, Y\}$ for $\mathcal{P}(X, Y)$ such that $X = JY$.

We first prove the following

Proposition 4. *Let $F : \Omega \to \Omega'$ be a C^∞ Bergman isometry, and let $p \in \Omega$ be a point close to the boundary. Then there exists a basis*

$$\{Z_p, W_p\}$$

of $T^{\mathbb{C}}_p(\Omega)$ such that

$$F_*(\mathcal{P}(Z_p, JZ_p)) \quad \textit{and} \quad F_*(\mathcal{P}(W_p, JW_p))$$

are complex planes.

Proof. We consider the sequence of basis $\{Z_{p_j}, W_{p_j}\}$ defined in Proposition 2.

Let U be a small neighborhood of p_j and let $p \in U$. We join p to p_j by a smooth curve $\mathcal{C} : \gamma(t),\ 0 \le t \le 1$, with $\gamma(0) = p_j$ and $\gamma(1) = p$, and we consider the parallel transport of the slates $\langle Z_{p_j} \rangle$ and $\langle W_{p_j} \rangle$ along $\mathcal{C}$. In such a way we define a new pair of slates at each point along the curve $\langle Z_{p_j} \rangle_t$ and $\langle Z_{p_j} \rangle_t$. Since we are in a neighborhood of p_j we have:

$$|K^{\Omega}_p(\mathcal{P}(Z_{p_j})_t) - K| \le \epsilon$$

and

$$|K^{\Omega}_p(\mathcal{P}(W_{p_j})_t) - K| \le \epsilon$$

for any t and in particular for $t = 1$. We can then assume that $\langle Z_{p_j}\rangle_1 = \langle (Z_{p_j})_1\rangle_c$ as well as $\langle W_{p_j}\rangle_1 = \langle (W_{p_j})_1\rangle_c$ for some constant c.

According to our notation we denote $Z_p = (Z_{p_j})_1$ and $W_p = (W_{p_j})_1$. This a basis for $T_p^{\mathbb{C}}(\Omega)$. We want to prove that $F_*\mathcal{P} =: \mathcal{P}(F_*Z_p, F_*JZ_p)$ is a complex plane.

Suppose that $F_*\mathcal{P}$ is not complex. Then there exists a constant $\alpha > 0$ such that the angle between $F_*\mathcal{P}$ and any complex plane is at least α. On the other hand, we know that F is an isometry. Therefore there exists c' such that

$$|K^{\Omega'}_{F(p)}(F_*\mathcal{P}) - K| \leq c'|K^{\Omega}_{F(p)}(\mathcal{P}(Z_{F(p)})) - K| \tag{1}$$

or

$$|K^{\Omega'}_{F(p)}(F_*\mathcal{P}) - K| \leq c'|K^{\Omega}_{F(p)}(\mathcal{P}(W_{F(p)})) - K|. \tag{2}$$

This implies that F_*Z_p belongs to either one of the slates $\langle Z_{F(p)}\rangle_{c'}$ or $\langle W_{F(p)}\rangle_{c'}$. We consider a parallel transport of these two slates along the smooth curve $F(\mathcal{C})$, and we obtain a sequence of pair of slates $\left(\langle Z_{F(p)}\rangle_{c'}\right)_t$ and $\left(\langle W_{F(p)}\rangle_{c'}\right)_t$. By the Proposition 3, both $\langle Z_{F(p_j)}\rangle_{c'} = \left(\langle Z_{F(p)}\rangle_{c'}\right)_1$ and $\langle W_{F(p_j)}\rangle_{c'} = \left(\langle W_{F(p)}\rangle_{c'}\right)_1$ can be taken to have a sufficiently small aperture. Thus, the angle $\gamma(F_*\mathcal{P}, \mathcal{P}(Z_{F(p)})) \leq \alpha$, for any choice of α. This implies that the angle between any vector X on $\mathcal{P}(F_*Z_p, F_*JZ_p)$ and any vector X' on $\mathcal{P}(Z_{F(p)})$ cannot be bounded away from zero. This is a contradiction.

We can use exactly the same argument to show that $\mathcal{P}(F_*W_p, F_*JW_p)$ is a complex plane. □

Theorem 2. *Let Ω and Ω' be two convex polyhedral domains with piecewise Levi flat boundary. Then F is a complex analytic map (up to conjugation).*

Proof. Let $p \in \Omega$ and let $\mathcal{C}$ be a smooth curve joining p with a boundary point $\hat{p}$. When the curve is approaching the boundary, we have in particular a sequence of points p_j which tends to $\hat{p}$. Along this sequence we define a basis $\{Z_{p_j}, W_{p_j}\}$ as in Proposition 2. Then we move this basis to the point p, by the parallel transport along $\mathcal{C}$, in order to obtain a basis $\{Z_p, W_p\}$ at p. The first result to achieve is to prove that $\mathcal{P}(F_*Z_p, F_*JZ_p)$ and $\mathcal{P}(F_*W_p, F_*JW_p)$ are complex planes. To see this one may refer to the argument used in the proof of Proposition 4 and also notice that the parallel transport preserves the complex stucture (see [**8**]).

We now consider the standard complex structure

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We know that the complex structure acts on complex planes as a rotation by an angle $\frac{\pi}{2}$. We denote by J_1 the first 2×2 block of J, i.e. $J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, so that we have

$$J_1 F_*(Z_p) - F_* J_1(Z_p) = \begin{cases} 0 \\ 2J_1 F_*(Z_p) \end{cases} \tag{3}$$

and

$$J_1 F_*(W_p) - F_* J_1(W_p) = \begin{cases} 0 \\ 2J_1 F_*(W_p) \end{cases} . \tag{4}$$

Let us now consider the following linear transformations in $\mathbb{C}^2$:

$$L_p^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad L_p^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$
$$L_p^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad L_p^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can always reduce the analysis of the effect of any L_p on J to the analysis of the effect of the 2×2 matrix $L_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on J_1.

More precisely, we can obtain any L_p^n , $n = 1, 2, 3, 4$ as a direct sum of the matrix L_1 and the identity matrix. We have

$$\begin{array}{ll} L_p^1 = Id \oplus Id & L_p^2 = L_1 \oplus Id \\ L_p^3 = Id \oplus L_1 & L_p^4 = L_1 \oplus L_1 \end{array}$$

so that $\{L_p^1, L_p^2, L_p^3, L_p^4\}$ is a group of four elements, which is then homomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Using an appropriate action of L_1 on J_1 we can assume that, at some point p

$$JF_* - F_* J = 0 \tag{5}$$

for both Z_p and W_p (basis for the tangent space at p).

Since the operator $JF_* - F_*J$ is a linear operator on $T_p(\Omega)$ we can extend, by linearity, (5) to the whole tangent space, so that

$$JF_* - F_* J \equiv 0 \tag{6}$$

on $T_p(\Omega)$.

Therefore at each point $p \in \Omega$ we have that each Bergman isometry is holomorphic, up to the action of the appropriate L_p^n , $n = 1, 2, 3, 4$.

It only remains to show that $JF_* - F_*J \equiv 0$ on $T\Omega$. We have that at each point q

$$\|JF_* - F_*J\| = \begin{cases} 0 \\ 2 \end{cases} \tag{7}$$

on the basis given by $\{Z_q, W_q\}$, where $\|\cdot\|$ is the norm on the tangent bundle. By continuity $\|JF_* - F_*J\| \equiv 0$. Hence we have that F is holomorphic after the action of the appropriate L^n , $n = 1, 2, 3, 4$. □

Let Ω be a bounded domain, we denote by $\mathrm{Iso}(\Omega)$ the group of all Bergman isometries of Ω and by $\mathrm{Aut}(\Omega)$ the group of biholomorphisms. As a consequence of the main theorem we have

Corollary. *Let Ω be a convex polyhedral domain in $\mathbb{C}^2$. Then*

$$[\mathrm{Iso}(\Omega) : \mathrm{Aut}(\Omega)] \leq 4.$$

References

[1] S. Bergman, *The kernel function and conformal mapping*, Amer. Math. Soc. Providence, RI, 1950.

[2] R. Greene and S.G. Krantz, *Deformation of complex structures, estimate for $\bar\partial$ equation, and stability of the Bergman kernel*, Adv. Math., **43** (1982), 1-86.

[3] R. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Math., No. 699, Springer-Verlag, Berlin, 1979.

[4] K-T. Kim, *Boundary behavior of the Bergman Curvature and applications*, in *Geometry and Topology. Proceedings of Workshop in Pure Mathematics*, Kangwon National University, 1992.

[5] K-T. Kim and J. Yu, *Boundary Behavior of the Bergman Curvature in Strictly pseudoconvex Polyhedral Domains*, preprint, 1994.

[6] P. Klembeck, *Kähler metric of negative curvature, the Bergman metric near the boundary and the Kobayashi metric on smooth bounded strictly pseudoconvex sets*, Indiana U. Math., **27** (1978), 275-282.

[7] S. Kobayashi and K. Nomizu, *On automorphisms of Kählerian structures*, Nagoya J. Math., **11** (1957), 115-124.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, II, Interscience, New York, 1969.

[9] S.G. Krantz, *Function Theory of Several Complex Variables, Second Edition*, Wodsworth & Brooks/Cole, Pacific Grove, CA, 1992.

[10] A. Pagano, *Analyticity of Bergman Isometries*, Ph.D. Thesis, Brown University Providence, RI USA, 1994.

[11] I. Satake, *Algebraic Structure of Symmetric Domains*, Princeton University Press, Princeton NJ, 1980.

[12] J.A. Schouten and K. Yano, *On pseudo-Kählerian spaces admitting a continuous groups of motions*, Indigationes Math., **17** (1955), 565-560.

Received April 21, 1995. This article is part of my Ph.D. dissertation at Brown University. I would like to thank my advisor Kang-Tae Kim for having introduced me to the subject and for having given many valuable guidelines throughout this work.

Brown University
Providence, RI 02912, USA

Current Address:
Università di Roma "la Sapienzo"
Roma 00185, Italy